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Monotone Drawings of Graphs

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Abstract

We study a new standard for visualizing graphs: A monotone drawing is a straight-line drawing such that, for every pair of vertices, there exists a path that monotonically increases with respect to some direction. We show algorithms for constructing monotone planar drawings of trees and biconnected planar graphs, we study the interplay between monotonicity, planarity, and convexity, and we outline a number of open problems and future research directions.

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1 Introduction

A traveler that consults a road map to find a route from a site u to a site v would like to easily spot at least one path connecting u and v. Such a task is harder if each path from u to v on the map has legs moving away from v. Travelers rotate maps to better perceive their content. Hence, even if in the original orientation of the map all the paths from u to v have annoying back and forth legs, the traveler might be happy to find at least one orientation where a path from u to v smoothly flows from left to right.

Leaving the road map metaphora for the Graph Drawing terminology, we say that a path P in a straight-line drawing of a graph is *monotone* if there exists a line l such that the orthogonal projections of the vertices of P on l appear along l in the order induced by P. A straight-line drawing of a graph is *monotone* if it contains at least one monotone path for each pair of vertices. Having at disposal a monotone drawing (map), a user (traveler) can find for each pair of vertices u and v a rotation of the drawing such that there exists a path from u to v always increasing in the x-coordinate.

Upward drawings [8, 11] are related to monotone drawings, as in an upward drawing every directed path is monotone. Notice that, in this case, such paths are monotone with respect to the same (vertical) line, while in a monotone drawing each monotone path, in general, is monotone with respect to a different line. Even more related to monotone drawings are *greedy drawings* [14, 13, 1]. Namely, in a greedy drawing, between any two vertices a path exists such that the Euclidean distance from an intermediate vertex to the destination decreases at every step, while, in a monotone drawing, between any two vertices a path and a line l exist such that the Euclidean distance from the projection of an intermediate vertex on l to the projection of the destination on l decreases at every step.

Monotone drawings have a strict correlation with an important problem in Computational Geometry: Arkin, Connelly, and Mitchell [2] studied how to find monotone trajectories connecting two given points in the plane avoiding convex obstacles. As a corollary of their study, it is possible to conclude that every planar convex drawing is monotone. Hence, the graphs admitting a convex drawing [5] have a planar monotone drawing. Such a class of graphs is a super-class (sub-class) of the triconnected (biconnected) planar graphs.

In this paper we first deal with trees (Sect. 4). We prove several properties relating the monotonicity of a tree drawing to its planarity and "convexity" [4]. Moreover, we show two algorithms for constructing monotone planar grid drawings of trees. The first one constructs drawings lying on a grid of size $O(n^{1.6}) \times O(n^{1.6})$. The second one has a better area requirement, namely $O(n^3)$, but a worse $\Omega(n)$ aspect ratio.

The existence of monotone drawings of trees allows us to construct a monotone drawing of any graph G by drawing any of its spanning trees, by adding the edges not belonging to the spanning tree, and by slightly perturbing the position of the vertices in order to avoid possible overlappings between edges and vertices. However, in the obtained monotone drawing there could be crossings between edges that cannot be avoided by a slight perturbation of the vertices, even if G is a planar graph. Motivated by this and since every triconnected planar graph admits a planar monotone drawing, we devise an algorithm to construct planar monotone drawings of biconnected planar graphs (Sect. 5). Such an algorithm exploits the SPQR-tree decomposition [9] of a biconnected planar graph.

We conclude the paper discussing several open problems (Sect. 6).

2 Definitions and Preliminaries

A *straight-line drawing* of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a segment connecting its endpoints. A drawing is *planar* if the segments representing its edges do not cross but, possibly, at common endpoints. A graph is *planar* if it admits a planar drawing. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face*. A *strictly convex drawing* (resp. a *(non-strictly) convex drawing*) is a straight-line planar drawing in which each face is delimited by a strictly (resp. non-strictly) convex polygon.

We denote by $P(v_1, v_m)$ a path between vertices v_1 and v_m. A graph G is *connected* if every pair of vertices is connected by a path and is *biconnected* (resp. *triconnected*) if removing any vertex (resp. any two vertices) leaves G connected.

A *subdivision* of G is the graph obtained by replacing each edge of G with a path. A *subdivision of a drawing* Γ of G is a drawing Γ' of a subdivision G' of G such that, for every edge (u, v) of G that has been replaced by a path $P(u, v)$ in G', u and v are drawn at the same point in Γ and in Γ', and all the vertices of $P(u, v)$ lie on the segment between u and v.

2.1 Monotone Drawings

Let p be a point in the plane and l a half-line starting at p. The *slope* of l, denoted by $slope(l)$, is the angle spanned by a counter-clockwise rotation that brings a horizontal half-line starting at p and directed towards increasing x-coordinates to coincide with l. We consider slopes that are equivalent modulo 2π as the same slope (e.g., $\frac{3}{2}\pi$ is regarded as the same slope as $-\frac{\pi}{2}$).

Let Γ be a drawing of a graph G and let (u, v) be an edge of G. The half-line starting at u and passing through v, denoted by $d(u, v)$, is the *direction* of (u, v). The *slope* of edge (u, v), denoted by $slope(u, v)$, is the slope of $d(u, v)$. The direction and the slope of an edge (u, v) are depicted in Fig. 1(a). Observe that $slope(u, v) = slope(v, u) - \pi$. When comparing directions and their slopes, we assume that they are applied at the origin of the axes.

An edge (u, v) is *monotone* with respect to a half-line l if it has a "positive projection" on l, i.e., if $slope(l) - \frac{\pi}{2} < slope(u, v) < slope(l) + \frac{\pi}{2}$. A path $P(u_1, u_n) = (u_1, \ldots, u_n)$ is *monotone with respect to a half-line* l if (u_i, u_{i+1}) is monotone with respect to l, for each $i = 1, \ldots, n-1$; $P(u_1, u_n)$ is a *monotone* path if there exists a half-line l such that $P(u_1, u_n)$ is monotone with respect to l. Fig. 1(b) shows a path that is monotone with respect to a half-line. Observe that, as shown in the figure, if a path $P(u_1, u_n) = (u_1, \dots, u_n)$ is monotone with respect to l, then the orthogonal projections on l of $u_1, \dots, u_n$ appear in this order along l. A drawing Γ of a graph G is *monotone* if, for each pair of vertices u and v in G, there exists a monotone path $P(u, v)$ in Γ. Observe that monotonicity implies connectivity.

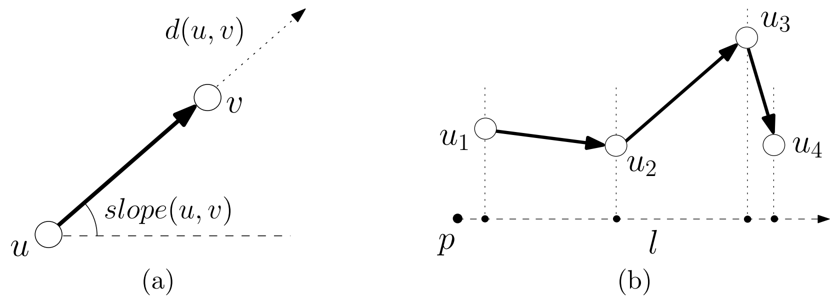


Figure 1: (a) The direction $d(u, v)$ of an edge (u, v) and its slope $slope(u, v)$. (b) A path $P(u_1, u_4)$ that is monotone with respect to a half-line l.

2.2 The Stern-Brocot Tree

The *Stern-Brocot tree* [15, 3] is an infinite tree whose nodes are in bijective mapping with the irreducible positive rational numbers. The Stern-Brocot tree $\mathcal{SB}$ has two nodes $0/1$ and $1/0$ that are connected to the same node $1/1$, where $1/1$ is the right child of $0/1$ and the left child of $1/0$. An ordered binary tree rooted at $1/1$ is then constructed as follows (see Fig. 2). Consider a node y/x of the tree. The left child of y/x is the node $(y + y')/(x + x')$, where y'/x' is the ancestor of y/x that is closest to y/x (in terms of graph-theoretic distance in $\mathcal{SB}$) and that has y/x in its right subtree. The right child of y/x is the node $(y + y'')/(x + x'')$, where y''/x'' is the ancestor of y/x that is closest to y/x and that has y/x in its left subtree. Not considering nodes $0/1$ and $1/0$, the *first level* of $\mathcal{SB}$ is composed of node $1/1$. The *i-th level* of $\mathcal{SB}$ is composed of the children of the nodes of the $(i-1)$-th level of $\mathcal{SB}$. The following property of the Stern-Brocot tree is well-known and easy to observe:

Property 1 *The sum of the numerators of the elements of the i-th level of $\mathcal{SB}$ is 3^{i-1} and the sum of the denominators of the elements of the i-th level of $\mathcal{SB}$ is 3^{i-1}.*

2.3 The SPQR-Tree Decomposition

To decompose a biconnected graph into its triconnected components, we use the *SPQR-tree*, a data structure introduced by Di Battista and Tamassia [9, 10].

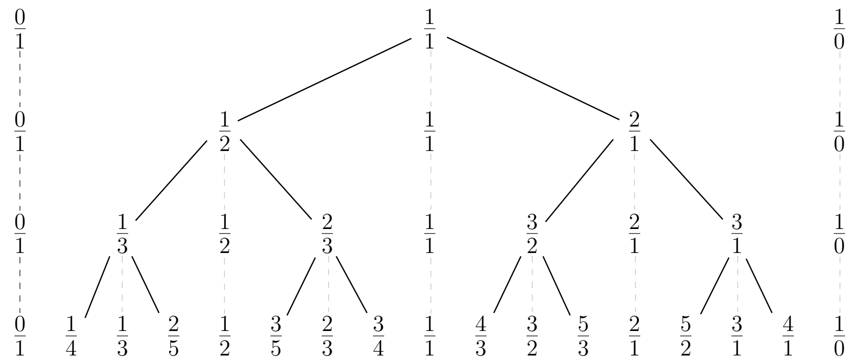


Figure 2: Construction of the Stern-Brocot tree.

A graph is *st-biconnectible* if adding edge (s,t) to it yields a biconnected graph. Let G be an st-biconnectible graph. A *separation pair* of G is a pair of vertices whose removal disconnects the graph. A *split pair* of G is either a separation pair or a pair of adjacent vertices. A *maximal split component* of G with respect to a split pair $\{u,v\}$ (or, simply, a maximal split component of $\{u,v\}$) is either an edge (u,v) or a maximal subgraph G' of G such that G' contains u and v, and $\{u,v\}$ is not a split pair of G'. A vertex $w \neq u,v$ belongs to exactly one maximal split component of $\{u,v\}$. We say that the union of any number of maximal split components of $\{u,v\}$ is a *split component* of $\{u,v\}$.

In the paper, we assume that any SPQR-tree of a graph G is rooted at one edge of G, called *reference edge*.

The rooted SPQR-tree $\mathcal{T}$ of a biconnected graph G, with respect to a reference edge e, describes a recursive decomposition of G induced by its split pairs. The nodes of $\mathcal{T}$ are of four types: S, P, Q, and R. Their connections are called *arcs*, in order to distinguish them from the edges of G.

Each node μ of $\mathcal{T}$ has an associated st-biconnectible multigraph, called the *skeleton* of μ and denoted by $skel(\mu)$. Skeleton $skel(\mu)$ shows how the children of μ, represented by "virtual edges", are arranged into μ. The virtual edge in $skel(\mu)$ associated with a child ν, is called the *virtual edge of ν in $skel(\mu)$*.

For each virtual edge e_i of $skel(\mu)$, recursively replace e_i with the skeleton $skel(\mu_i)$ of its corresponding child μ_i. The subgraph of G that is obtained in this way is the *pertinent graph* of μ and is denoted by $pert(\mu)$.

Given a biconnected graph G and a reference edge $e = (u', v')$, tree $\mathcal{T}$ is recursively defined as follows. At each step, a split component G^*, a pair of vertices $\{u,v\}$, and a node ν in $\mathcal{T}$ are given. A node μ corresponding to G^* is introduced in $\mathcal{T}$ and attached to its parent ν. Vertices u and v are the *poles* of μ and denoted by $u(\mu)$ and $v(\mu)$, respectively. The decomposition possibly recurs on some split components of G^*. At the beginning of the decomposition $G^* = G - \{e\}$, $\{u,v\} = \{u', v'\}$, and ν is a Q-node corresponding to e.

Base Case: If G^* consists of exactly one edge between u and v, then μ is a Q-node whose skeleton is G^* itself.

Parallel Case: If G^* is composed of at least two maximal split components $G_1, \dots, G_k$ ($k \geq 2$) of G with respect to $\{u, v\}$, then μ is a P-node. Graph $skel(\mu)$ consists of k parallel virtual edges between u and v, denoted by $e_1, \dots, e_k$ and corresponding to $G_1, \dots, G_k$, respectively. The decomposition recurs on $G_1, \dots, G_k$, with $\{u, v\}$ as pair of vertices for every graph, and with μ as parent node.

Series Case: If G^* is composed of exactly one maximal split component of G with respect to $\{u, v\}$ and if G^* has cutvertices $c_1, \dots, c_{k-1}$ ($k \geq 2$), appearing in this order on a path from u to v, then μ is an S-node. Graph $skel(\mu)$ is the path $e_1, \dots, e_k$, where virtual edge e_i connects c_{i-1} with c_i ($i = 2, \dots, k-1$), e_1 connects u with c_1, and e_k connects c_{k-1} with v. The decomposition recurs on the split components corresponding to each of $e_1, e_2, \dots, e_{k-1}, e_k$ with μ as parent node, and with $\{u, c_1\}, \{c_1, c_2\}, \dots, \{c_{k-2}, c_{k-1}\}, \{c_{k-1}, v\}$ as pair of vertices, respectively.

Rigid Case: If none of the above cases applies, the purpose of the decomposition step is that of partitioning G^* into the minimum number of split components and recurring on each of them. We need some further definition. Given a maximal split component G' of a split pair $\{s, t\}$ of G^*, a vertex $w \in G'$ *properly belongs* to G' if $w \neq s, t$. Given a split pair $\{s, t\}$ of G^*, a maximal split component G' of $\{s, t\}$ is *internal* if neither u nor v (the poles of G^*) properly belongs to G', *external* otherwise. A *maximal split pair* $\{s, t\}$ of G^* is a split pair of G^* that is not contained in an internal maximal split component of any other split pair $\{s', t'\}$ of G^*. Let $\{u_1, v_1\}, \dots, \{u_k, v_k\}$ be the maximal split pairs of G^* ($k \geq 1$) and, for $i = 1, \dots, k$, let G_i be the union of all the internal maximal split components of $\{u_i, v_i\}$. Observe that each vertex of G^* either properly belongs to exactly one G_i or belongs to some maximal split pair $\{u_i, v_i\}$. Node μ is an R-node. Graph $skel(\mu)$ is the graph obtained from G^* by replacing each subgraph G_i with the virtual edge e_i between u_i and v_i. The decomposition recurs on each G_i with μ as parent node and with $\{u_i, v_i\}$ as pair of vertices.

For each node μ of $\mathcal{T}$, the construction of $skel(\mu)$ is completed by adding a virtual edge (u, v) representing the rest of the graph.

The SPQR-tree $\mathcal{T}$ of a graph G with n vertices and m edges has m Q-nodes and $O(n)$ S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of $\mathcal{T}$ is $O(n)$. Finally, SPQR-trees can be constructed and handled efficiently. Namely, given a biconnected planar graph G, the SPQR-tree $\mathcal{T}$ of G can be computed in linear time [9, 10, 12].

3 Properties of Monotone Drawings

In this section we give some basic properties of monotone drawings.

Property 2 *Any sub-path of a monotone path is monotone.*

Property 3 *A path $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ is monotone if and only if it contains two edges e_1 and e_2 such that the closed wedge centered at the origin of the axes, delimited by the two half-lines $d(e_1)$ and $d(e_2)$, and having an angle smaller than π, contains all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n-1$.*

Refer to Fig. 3(a)–(b). Edges e_1 and e_2 as in Property 3 are the *extremal edges* of $P(u_1, u_n)$. The closed wedge delimited by $d(e_1)$ and $d(e_2)$ and containing all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n-1$, is the *range* of $P(u_1, u_n)$ and is denoted by $range(P(u_1, u_n))$, while the closed wedge delimited by $d(e_1) - \pi$ and $d(e_2) - \pi$, and not containing $d(e_1)$ and $d(e_2)$, is the *opposite range* of $P(u_1, u_n)$ and is denoted by $opp(P(u_1, u_n))$.

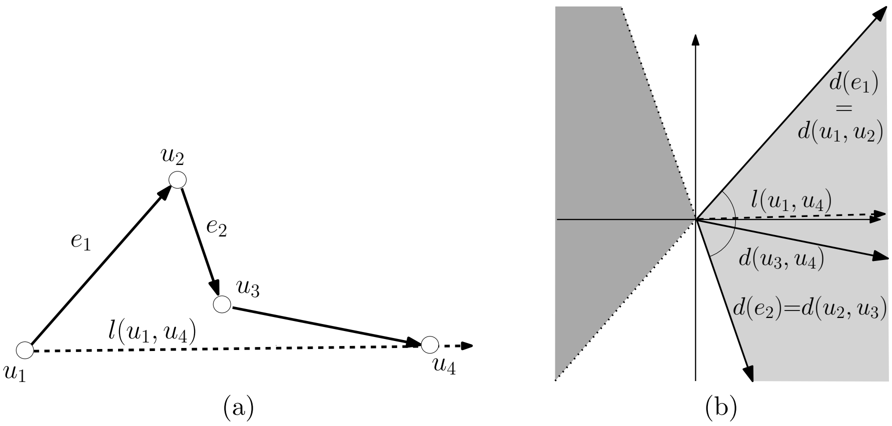


Figure 3: (a) A monotone path $P(u_1, u_4)$ with extremal edges e_1 and e_2. (b) The light-gray shaded region represents the range of $P(u_1, u_4)$ defined by $d(e_1)$ and $d(e_2)$ and the dark-gray shaded region represents the opposite range of $P(u_1, u_4)$. Observe that $range(P(u_1, u_4))$ contains the half-line $l(u_1, u_4)$ from u_1 through u_4 (Property 4).

Property 4 *The range of a monotone path $P(u_1, u_n)$ contains the half-line from u_1 through u_n.*

Proof: Draw the closed wedge $range(P(u_1, u_n))$ centered at u_1. Observe that a sequence of edges whose directions are $d(u_i, u_{i+1})$, with $i = 1, \ldots, n-1$, allow to reach from u_1 only points contained in $range(P(u_1, u_n))$. Hence, u_n and the half line from u_1 through u_n are contained in $range(P(u_1, u_n))$. □

Lemma 1 *Let $P(u_1, u_n) = (u_1, \ldots, u_n)$ be a monotone path and let (u_n, u_{n+1}) be an edge. Then, path $P(u_1, u_{n+1}) = (u_1, u_2, \ldots, u_n, u_{n+1})$ is monotone if and only if $d(u_n, u_{n+1})$ is not contained in $opp(P(u_1, u_n))$. Further, if $P(u_1, u_{n+1})$ is monotone, then $range(P(u_1, u_n)) \subseteq range(P(u_1, u_{n+1}))$.*

Proof: Denote by e_1 and e_2 the extremal edges of $P(u_1, u_n)$.

If $d(u_n, u_{n+1})$ is in $opp(P(u_1, u_n))$, then no wedge having an angle smaller than π contains all of $d(e_1)$, $d(e_2)$, and $d(u_n, u_{n+1})$. By Property 3, $P(u_1, u_{n+1})$ is not monotone. See Fig. 4.

If $d(u_n, u_{n+1})$ is in $range(P(u_1, u_n))$, then $P(u_1, u_{n+1})$ is monotone by Property 3; further, we have that $range(P(u_1, u_{n+1})) = range(P(u_1, u_n))$. See Fig. 5.

If $d(u_n, u_{n+1})$ is contained in the smallest wedge delimited by $d(e_1)$ and by $d(e_2) - \pi$ (the case in which $d(u_n, u_{n+1})$ is contained in the smallest wedge delimited by $d(e_2)$ and by $d(e_1) - \pi$ being symmetric), then $d(u_n, u_{n+1})$ and $d(e_2)$ delimit a wedge having an angle smaller than π and all the half-lines $d(u_i, u_{i+1})$, for $i = 1, \ldots, n$, are contained in such a wedge (hence such a wedge contains $range(P(u_1, u_n))$; by Property 3, $P(u_1, u_{n+1})$ is monotone. Further, we have $range(P(u_1, u_n)) \subset range(P(u_1, u_{n+1}))$. See Fig. 6. □

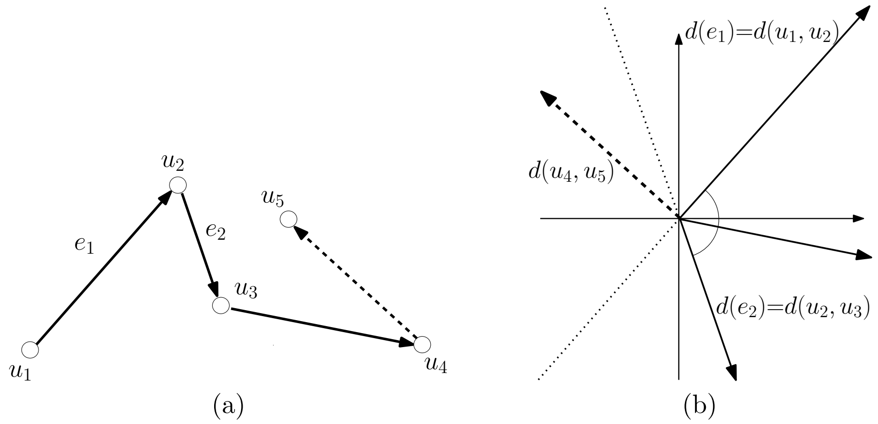


Figure 4: (a) A path $P(u_1, u_4)$ and an edge (u_4, u_5) whose direction is in $opp(P(u_1, u_4))$. (b) There exists no wedge having an angle smaller than π containing all of $d(e_1)$, $d(e_2)$, and $d(u_4, u_5)$.

Corollary 1 *Let $P(u_1, u_n) = (u_1, \ldots, u_n)$ and $P(u_n, u_{n+k}) = (u_n, \ldots, u_{n+k})$ be monotone paths. Then, path $P(u_1, u_{n+k}) = (u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+k})$ is monotone if and only if $range(P(u_1, u_n)) \cap opp(P(u_n, u_{n+k})) = \emptyset$. Further, if $P(u_1, u_{n+k})$ is monotone, then $range(P(u_1, u_n)) \cup range(P(u_n, u_{n+k})) \subseteq range(P(u_1, u_{n+k}))$.*

The following properties relate monotonicity to planarity and convexity.

Property 5 *A monotone path is planar.*

Proof: Rotate the path so that it is monotone with respect to the x-axis. Then, the x-coordinates of the vertices are strictly increasing along the path. This implies that the path cannot cross itself. □

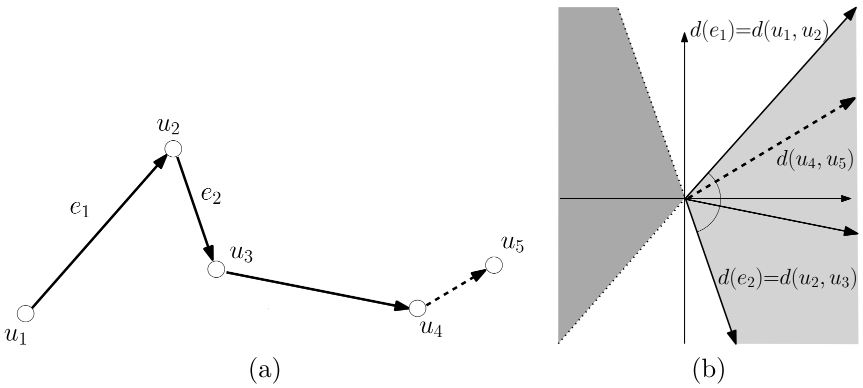


Figure 5: (a) A path $P(u_1, u_4)$ and an edge (u_4, u_5) whose direction is in $range(P(u_1, u_4))$. (b) $range(P(u_1, u_4)) = range(P(u_1, u_5))$.

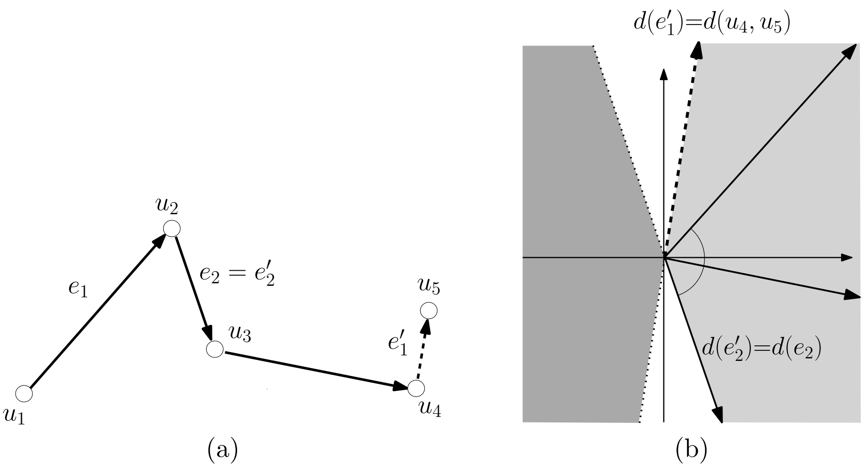


Figure 6: (a) A path $P(u_1, u_4)$ and an edge (u_4, u_5) whose direction is contained in the smallest wedge delimited by $d(e_1)$ and by $d(e_2) - \pi$ (b) Extremal edges $e_1' = (u_4, u_5)$ and $e_2' = e_2$ delimit $range(P(u_1, u_5)) \supset range(P(u_1, u_4))$.

Lemma 2 *[2] Any strictly convex drawing of a planar graph is monotone.*

When proving that every biconnected planar graph admits a planar monotone drawing, we will need to construct non-strictly convex drawings of graphs. Observe that any graph containing a degree-2 vertex does not admit a strictly convex drawing, while it might admit a non-strictly convex drawing. While not every non-strictly convex drawing is monotone, we can relate non-strict convexity and monotonicity:

Lemma 3 *Any non-strictly convex drawing of a graph such that each maximal set of parallel edges induces a collinear path is monotone.*

Proof: In order to prove the lemma, we directly apply to a non-strictly-convex drawing of a graph a technique which is similar to the one used in [2] to prove the existence of a monotone path between any two points of the plane in the presence of convex obstacles.

First observe that, given a (not necessarily strictly) convex drawing Γ of a graph G and a direction d which is not orthogonal to any edge of Γ, orienting the edges of G in Γ according to d yields an upward drawing Γ_d of the directed graph G_d with a single source s_d and a single sink t_d (see Fig. 7(a)). Since any directed path in Γ_d is a monotone path with respect to d, for each two vertices u and v, we aim at finding a direction d^* such that there is a directed path from u to v in Γ_{d^*}.

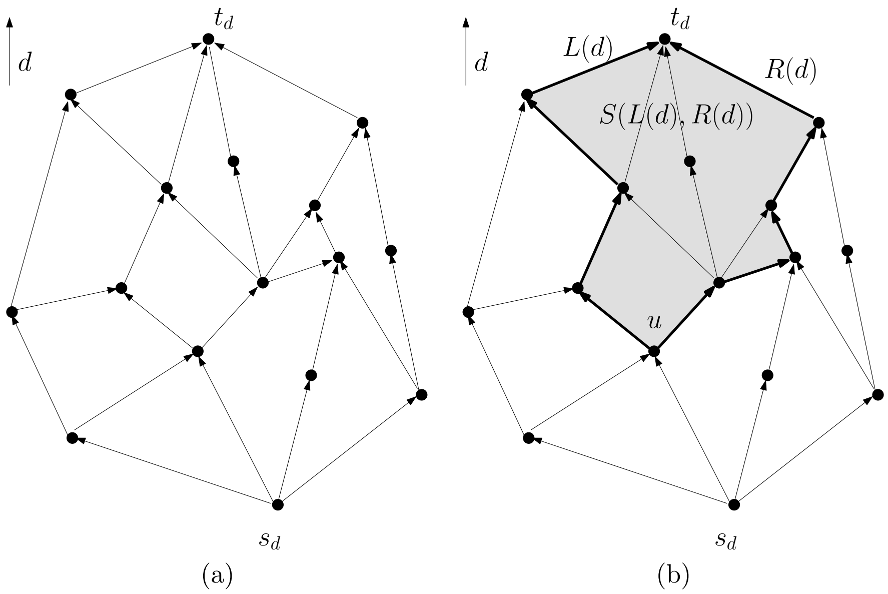


Figure 7: (a) The directed graph G_d induced by direction d on a convex drawing of a graph. (b) Paths $L(d)$ and $R(d)$ and the region $S(L(d), R(d))$.

Given a direction d and the corresponding upward drawing Γ_d, the *leftmost path* (resp. *rightmost path*), denoted by $L(d)$ (resp. $R(d)$), is the directed path of Γ_d that starts from u, traverses Γ_d by taking the last (resp. first) exiting edge of each vertex in the counter-clockwise direction, and terminates on t_d (see Fig. 7(b) for an example). Since Γ is convex, such paths always exist. Further, the *slice of* $L(d)$ *and* $R(d)$, denoted by $S(L(d), R(d))$, is the closed region enclosed by the clockwise cycle composed of $L(d)$, traversed forward, and of $R(d)$, traversed backward. Note that a vertex v is reachable from u by a path monotone with respect to d if and only if v is inside $S(L(d), R(d))$.

We show how to find the direction d^* such that there is a directed path from u to v in Γ_{d^*}. Start by choosing any direction d. Consider the corresponding upward drawing Γ_d. If v is inside $S(L(d), R(d))$, then there exists a monotone path between u and v. Otherwise, continuously rotate counter-clockwise d until a direction d' is found that produces an upward drawing $\Gamma_{d'} \neq \Gamma_d$. Again, if v is inside $S(L(d'), R(d'))$, a monotone path is found. Otherwise, carry on the rotation process until a direction d^* is found such that $S(L(d^*), R(d^*))$ contains v. To prove that such a direction d^* always exists, it suffices to show that, for any two (counter-clockwise) consecutive directions d' and d'', $S(L(d'), R(d')) \cup S(L(d''), R(d''))$ contains all the vertices of the closed region $S^+(L(d''), R(d'))$ delimited by $L(d'')$, by $R(d')$, and by the path delimiting the outer face of G clockwise connecting $t_{d''}$ to $t_{d'}$. Observe that such a path could be composed of a single vertex, in the case in which $t_{d''} = t_{d'}$.

First observe that, as each maximal set of parallel edges induces a collinear path in Γ, $G_{d''}$ differs from $G_{d'}$ in the fact that the direction of a single edge or a single collinear path is reversed.

Suppose that only one among $L(d')$ and $R(d')$ changes, that is, that either $L(d'') = L(d')$ or $R(d'') = R(d')$. Observe that this implies that $t_{d''} = t_{d'}$. In both cases, the statement is trivially true.

Suppose that both $L(d')$ and $R(d')$ change into $L(d'')$ and $R(d'')$, respectively. Observe that, since $G_{d''}$ differs from $G_{d'}$ in the fact that the direction of a single edge or of a single collinear path is reversed, this can only happen if $L(d')$ and $R(d')$ have a common subpath and such a subpath contains at least one of the reversed edges.

In this case, a closed region Δ of $S^+(L(d''), R(d'))$ which is not contained in $S(L(d'), R(d'))$ or in $S(L(d''), R(d''))$ might exist. We show that Δ does not contain any internal vertex. This implies that $S(L(d'), R(d')) \cup S(L(d''), R(d''))$ contains all the vertices of $S^+(L(d''), R(d'))$.

If Δ is not a degenerate region, in which case the statement is trivial, then there exists a vertex w belonging to both $R(d'')$ and $L(d')$ such that, denoting by w_r and by w_l the neighbors of w in $R(d'')$ and in $L(d')$, respectively, that are farther from u than w, we have that w_r follows w_l in the counter-clockwise order of the edges around w (See Fig. 8).

Since $L(d')$ contains (w, w_l), in $G_{d'}$ edge (w, w_l) is directed from w to w_l and edge (w, w_r) is directed from w_r to w. On the other hand, since $R(d'')$ contains (w, w_r), in $G_{d''}$ edge (w, w_l) is directed from w_l to w and edge (w, w_r) is directed from w to w_r. Hence, (w, w_l) and (w, w_r) belong to a collinear path $P(w_l, w_r)$,

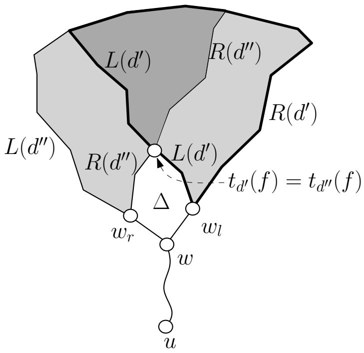


Figure 8: Illustration for the case in which $L(d')$ and $R(d')$ change into $L(d'')$ and $R(d'')$, respectively. The boundary of slice $S(L'_d, R'_d)$ is fat, while the boundary of $S(L''_d, R''_d)$ is normal.

and such a path contains all and only the edges whose direction changes when passing from d' to d''. Also, (w, w_r) directly precedes (w, w_l) in the clockwise order of the edges around w, as otherwise any edge exiting w between (w, w_r) and (w, w_l) would have been contained in $R(d'')$ instead of (w, w_r). Hence, (w, w_r) and (w, w_l) are on the same face f. Also, $L(d')$ traverses the right border of face f until the sink $t_{d'}(f)$ of f in $G_{d'}$ is reached. Analogously, $R(d'')$ traverses the left border of f until the sink $t_{d''}(f)$ of f in $G_{d''}$ is reached.

We prove that $t_{d'}(f) = t_{d''}(f)$. This implies that region Δ coincides with face f and hence does not contain internal vertices.

If $t_{d'}(f)$ does not belong to $P(w_l, w_r)$ then, since all and only the edges of $P(w_l, w_r)$ change direction when passing from d' to d'', vertex $t_{d'}(f)$ is also a sink in $G_{d''}$. By convexity, $t_{d'}(f) = t_{d''}(f)$.

Otherwise, $t_{d'}(f)$ belongs to $P(w_l, w_r)$. We show that this case does not occur under the current assumptions.

Suppose, for a contradiction, that $t_{d'}(f)$ belongs to $P(w_l, w_r)$.

If the source $s_{d'}(f)$ of f in $G_{d'}$ belongs to $P(w_l, w_r)$, then the edges of the path between $s_{d'}(f)$ and $t_{d'}(f)$ delimiting f and different from $P(w_l, w_r)$ are directed in the same way in $G_{d'}$ and in $G_{d''}$, hence the cycle delimiting f is a directed cycle in $G_{d''}$, a contradiction.

If $s_{d'}(f)$ does not belong to $P(w_l, w_r)$, then rotate the coordinate axes in such a way that $P(w_l, w_r)$ lies on the positive x-axis. Then, when applied to the origin of the axes, the direction d'' is in the second quadrant, arbitrarily close to the positive y-axis, and by convexity, the direction of the half-line from $s_{d'}(f)$ through $t_{d'}(f)$ lies in the fourth quadrant. Thus, such two directions create an angle larger than $\frac{\pi}{2}$. By Property 4, the drawing is not monotone, a contradiction. □

Still on the relationship between convexity and monotonicity, we have:

Lemma 4 *Consider a strictly convex drawing* Γ *of a graph* G. *Let* u, v, *and* w *be three consecutive vertices incident to the outer face of* Γ. *Let* d *be any half-line that is not orthogonal to any edge in* Γ *and that splits the angle* $\widehat{uvw}$ *into two angles smaller than* $\frac{\pi}{2}$. *Then, for each vertex* t *of* G, *there exists a path from* v *to* t *in* Γ *that is monotone with respect to* d.

Proof: Orient each edge of G according to d, hence obtaining a directed graph G'. Observe that, as no edge is orthogonal to d, there is no ambiguity in the orientation of the edges. Since d splits $\widehat{uvw}$ into two angles smaller than $\frac{\pi}{2}$, all the edges incident to v exit v in G'. Further, since Γ is convex, every vertex different from v has at least one entering edge. It follows that v is the unique source of G'. Therefore, for each vertex t of G there exists a path from v to t in G', and hence a path from v to t in Γ that is monotone with respect to d. $\square$

Next, we provide a powerful tool for "transforming" monotone drawings.

Lemma 5 *An affine transformation applied to a monotone drawing yields a monotone drawing.*

Proof: Consider a path $P(u_1, u_n) = (u_1, u_2, \ldots, u_n)$ that is monotone with respect to an oriented line l. Suppose, without loss of generality, that l does not intersect $P(u_1, u_n)$. Construct a graph G whose vertices are $u_1, u_2, \ldots, u_n$ and their projections $v_1, v_2, \ldots, v_n$ on l, and whose edges are (u_i, u_{i+1}), (v_i, v_{i+1}), for $i = 1, \ldots, n-1$, and (u_i, v_i), for $i = 1, \ldots, n$. Observe that G is planar and that edges (u_i, v_i), for $i = 1, \ldots, n$, are parallel. The affine transformation preserves the planarity of G, the parallelism of edges (u_i, v_i), for $i = 1, \ldots, n$, and the collinearity of vertices v_i, for $i = 1, \ldots, n$. In particular, vertices $v_1, \ldots, v_n$ appear in this order on the line passing through them. Therefore, they appear in the same order when projected on any line perpendicular to (u_i, v_i), for $i = 1, \ldots, n$, and the statement follows. $\square$

4 Monotone Drawings of Trees

In this section we study monotone drawings of trees.

We present two properties concerning monotone drawings of trees. The first one is about the relationship between monotonicity and planarity. Such a property directly descends from the fact that every monotone path is planar (by Property 5) and that in a tree there exists exactly one path between every pair of vertices.

Property 6 *Every monotone drawing of a tree is planar.*

The second property relates monotonicity and convexity. A *convex drawing* of a tree T [4] is a straight-line planar drawing such that replacing each edge between an internal vertex u and a leaf v with a half-line starting at u through v yields a partition of the plane into convex unbounded polygons. A convex drawing of a tree might not be monotone, because of the presence of two parallel edges. However, we will show that if such two edges do not exist, then any convex drawing is monotone. Define a *strictly convex drawing* of a tree T as a straight-line planar drawing such that each maximal set of parallel edges induces a collinear path and such that replacing every edge of T between an internal vertex u and a leaf v with a half-line starting at u through v yields a partition of the plane into convex unbounded polygons. We have the following:

Property 7 *Every strictly convex drawing of a tree is monotone.*

Proof: In order to prove the property we need to introduce some definitions. Consider a tree T. Suppose that T is rooted at some node r and is embedded in the plane. For any node $x \neq r$, define the *clockwise path* $C(x)$ as the path $(w_1, w_2, \ldots, w_y)$ such that $w_1 = x$, w_{i+1} is the child of w_i such that edge (w_i, w_{i+1}) immediately follows the edge from w_i to its parent in the counterclockwise order of the edges incident to w_i, and w_y is a leaf. The *counter-clockwise path* $CC(x)$ of any node $x \neq r$ is defined analogously. See Fig. 9. Consider a node x with parent y in T and consider two edges $e_1 = (u_a, u_b)$

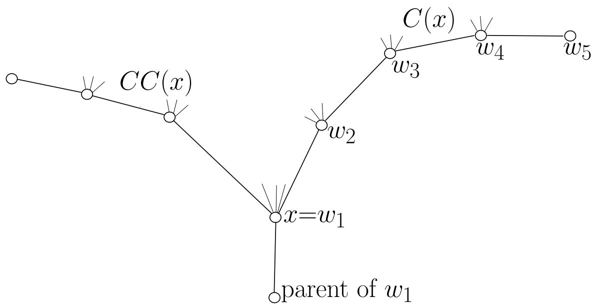


Figure 9: The clockwise path $C(x)$ and the counter-clockwise path $CC(x)$ of a node x in a tree T.

and $e_2 = (v_a, v_b)$ in the subtree $T(x)$ of T rooted at x, where u_a and v_a are the parents of u_b and v_b, respectively. Then we say that e_1 *clockwise precedes* (*clockwise follows*) e_2 if the edges (x, y), e_1, and e_2 come in this clockwise order (resp. in this counter-clockwise order) if they are translated so that x, u_a, and v_a are at the same point.

We have the following claim, whose statement is illustrated in Fig. 10:

Claim 1 *The slope of every edge in a subtree $T(x)$ of T rooted at a node $x \neq r$ clockwise follows the slope of the last edge of $CC(x)$ and clockwise precedes the slope of the last edge of $C(x)$. Moreover, denote by $(x, x_1), (x, x_2), \ldots, (x, x_k)$ the clockwise order of the edges incident to x starting at the edge (x, y), where y is the parent of x, and denote by $T(x_i)$ the subtree of $T(x)$ rooted at x_i. Then, the slope of each edge in $T(x_i) \cup (x, x_i)$ clockwise strictly precedes the slope of each edge in $T(x_{i+1}) \cup (x, x_{i+1})$, for each $1 \leq i \leq k-1$.*

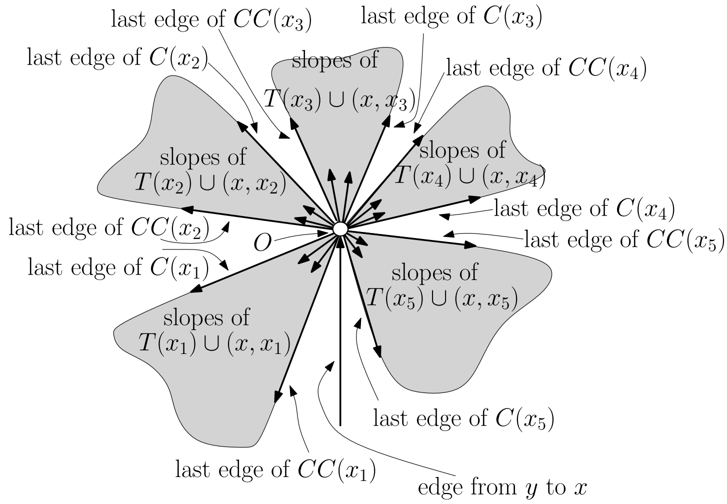


Figure 10: Illustration for the statement of Claim 1, with $k = 5$. The slopes of all the edges in $T(x)$ are shown as applied at the same point O. The slope of the edge from y to x is directed to O.

Proof: We prove the statement by induction on the number N of edges in the longest path from x to a leaf. If $N = 1$, then the statement is trivially true. Suppose that the statement holds for a certain N. We prove that it holds for $N + 1$.

We start by proving the second part of the statement. Observe that, by definition of strictly convex drawing, the path $(x, x_i) \cup C(x_i)$ (whose last edge is elongated to infinity) and the path $(x, x_{i+1}) \cup CC(x_{i+1})$ (whose last edge is elongated to infinity) form a convex unbounded polygon. Then, the slope of each edge in $(x, x_i) \cup C(x_i)$ clockwise strictly precedes the slope of each edge in $(x, x_{i+1}) \cup C(x_{i+1})$. By induction, the slope of every edge in $T(x_i)$ clockwise strictly precedes the slope of the last edge of $C(x_i)$ and the slope of every edge in $T(x_{i+1})$ clockwise strictly follows the slope of the last edge of $CC(x_{i+1})$. Hence, the slope of every edge in $T(x_i)$ clockwise strictly precedes the slope of every edge in $T(x_{i+1})$. Moreover, the slope of (x, x_i) clockwise precedes the slope of the last edge of $C(x_i)$, since the path $(x, x_i) \cup C(x_i)$ (whose last edge is elongated to infinity) and the path $(x, x_{i+1}) \cup CC(x_{i+1})$ (whose last edge is elongated to infinity) form a convex unbounded polygon. Hence, the slope of (x, x_i) clockwise strictly precedes the slope of every edge in $T(x_{i+1})$. Analogously, the slope of (x, x_{i+1}) clockwise strictly follows the slope of every edge in $T(x_i)$. Finally, the slope of (x, x_i) clockwise strictly precedes the slope of (x, x_{i+1}), thus proving the second part of the statement.

Concerning the first part of the statement, since the slope of each edge in $T(x_i) \cup (x, x_i)$ clockwise strictly precedes the slope of each edge in $T(x_{i+1}) \cup (x, x_{i+1})$, for each $1 \leq i \leq k - 1$, it follows that the slope of every edge in

$T(x) \setminus \{T(x_1), (x, x_1)\}$ clockwise follows the slope of every edge in $T(x_1)$ and the slope of every edge in $T(x) \setminus \{T(x_k), (x, x_k)\}$ counter-clockwise follows the slope of every edge in $T(x_k)$. By induction, the slope of every edge in $T(x_1)$ clockwise strictly follows the slope of the last edge in $CC(x_1)$. Finally, by definition of strictly-convex drawing, path $(x, x_1) \cup CC(x_1)$ is part of the boundary of a convex polygon, hence the slope of (x, x_1) clockwise follows the slope of the last edge of $CC(x_1)$. Hence, the slope of every edge in $T(x)$ clockwise follows the slope of the last edge of $CC(x_1)$ which is also the last edge of $CC(x)$. Analogously, the slope of every edge in $T(x)$ clockwise precedes the slope of the last edge of $C(x_k)$ which is also the last edge of $C(x)$. □

We now prove the property. Suppose, for a contradiction, that there exists a strictly convex drawing Γ of a tree T that is not monotone. Then, there exists a path $P(u, v) = (u, z_1, z_2, \ldots, z_k, v)$ connecting two vertices u and v of T that is not monotone. Suppose w.l.o.g. that $P(u, v)$ is a minimal non-monotone path, that is, every subpath of $P(u, v)$ is monotone. Root T at u and orient all the edges of T away from u. Then, edges (u, z_1), (z_k, v), and an edge (z_i, z_{i+1}), for some $1 \leq i \leq k - 1$, are such that the angle smaller than π defined by the directions of any two of them does not include the direction of the third edge, when such directions are applied at the origin. Assume, w.l.o.g. up to a reflection of the whole drawing, that $d(u, z_1)$, $d(z_i, z_{i+1})$, and $d(z_k, v)$ come in this counter-clockwise order around the origin.

Consider edge (z_i, z_{i+1}) and consider the edge e^* following (z_i, z_{i+1}) in the clockwise order of the edges incident to z_i. Notice that e^* could connect z_i either to one of its children or to its parent. In any case, the angle described by a clockwise rotation bringing $d(z_i, z_{i+1})$ to coincide with $d(e^*)$ is smaller than π, as otherwise the polygon having (z_i, z_{i+1}) and e^* as consecutive sides would not be strictly convex. Since, by Claim 1, every edge in $T(z_{i+1})$, including (z_k, v), clockwise precedes e^*, the angle described by a clockwise rotation bringing $d(z_i, z_{i+1})$ to coincide with $d(z_k, v)$ is smaller than π. Since, by assumption, $d(u, z_1)$ is between $d(z_i, z_{i+1})$ and $d(z_k, v)$ in the clockwise order around the origin, $d(u, z_1)$ is contained in a wedge delimited by $d(z_i, z_{i+1})$ and $d(z_k, v)$ having an angle smaller than π, a contradiction. □

A simple modification of the algorithm presented in [4] constructs strictly convex drawings of trees. Hence, monotone drawings exist for all trees.

However, in order to obtain monotone drawings of trees on a grid with polynomial area, we introduce *slope-disjoint drawings* of trees and show that they are monotone. Then, we provide two algorithms for constructing slope-disjoint drawings of trees in $O(n^{1.6}) \times O(n^{1.6})$ area and $O(n^2) \times O(n)$ area, respectively.

Let T be a tree rooted at a node r. Denote by $T(u)$ the subtree of T rooted at a node u. A *slope-disjoint* drawing of T (see Fig. 11) is such that:

(P1) For every node $u \in T$, there exist two angles $\alpha_1(u)$ and $\alpha_2(u)$, with $0 < \alpha_1(u) < \alpha_2(u) < \pi$, such that, for every edge e that is either in $T(u)$ or that connects u with its parent, it holds that $\alpha_1(u) < \mathit{slope}(e) < \alpha_2(u)$;

(P2) for every two nodes $u, v \in T$ with v child of u, it holds that $\alpha_1(u) < \alpha_1(v) < \alpha_2(v) < \alpha_2(u)$;

(P3) for every two nodes v_1, v_2 with the same parent, it holds that either $\alpha_1(v_1) < \alpha_2(v_1) < \alpha_1(v_2) < \alpha_2(v_2)$ or $\alpha_1(v_2) < \alpha_2(v_2) < \alpha_1(v_1) < \alpha_2(v_1)$.

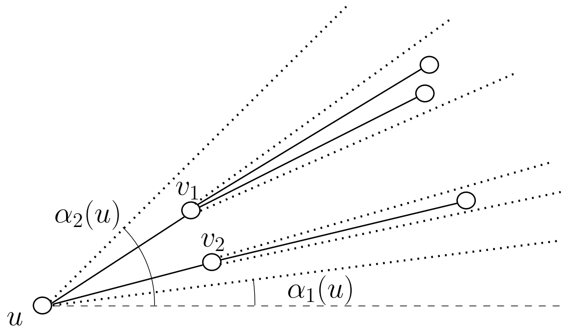


Figure 11: A slope-disjoint drawing of a tree.

We have the following:

Theorem 1 *Every slope-disjoint drawing of a tree is monotone.*

Proof: Let T be a tree and let Γ be a slope-disjoint drawing of T. We show that, for every two vertices $u, v \in T$, a monotone path between u and v exists in Γ. Observe that, by Property 2, it is sufficient to study only the cases when u and v are leaves of T.

Let w be the lowest common ancestor of u and v in T, and let u' and v' be the children of w in T such that $u \in T(u')$ and $v \in T(v')$. Path $P(u, v)$ is composed of path $P(u, w)$ and of path $P(w, v)$. First observe that, by Property P1 applied to vertex u', for every edge $e \in P(u, w)$, it holds $-\pi < \alpha_1(u') - \pi < slope(e) < \alpha_2(u') - \pi < 0$. Hence, $P(u, w)$ is monotone with respect to a half-line with slope $-\pi/2$ and it holds $\alpha_1(u') < slope(l) < \alpha_2(u')$ for each half-line l contained in $opp(P(u, w))$. Analogously, $P(w, v)$ is monotone with respect to a half-line with slope $\pi/2$ and it holds $\alpha_1(v') < slope(l) < \alpha_2(v')$ for each half-line l contained in $range(P(w, v))$. Further, since u' and v' are children of the same node w, by Property P3 we have $\alpha_1(u') < \alpha_2(u') < \alpha_1(v') < \alpha_2(v')$ (the case in which $\alpha_1(v') < \alpha_2(v') < \alpha_1(u') < \alpha_2(u')$ being symmetric). Hence, $opp(P(u, w)) \cap range(P(w, v)) = \emptyset$. By Corollary 1, $P(u, v)$ is monotone. $\square$

By Theorem 1, as long as the slopes of the edges in a drawing of a tree T guarantee the slope-disjoint property, one can *arbitrarily* assign lengths to such edges always obtaining a monotone drawing of T.

In the following we present two algorithms for constructing slope-disjoint drawings of trees. In both algorithms, we individuate a suitable set of elements of the Stern-Brocot tree $\mathcal{SB}$. Each of such elements is then used as a slope of an edge of T in the drawing.

Algorithm BFS-based: Consider the first $\lceil \log_2(n) \rceil$ levels of the Stern-Brocot tree $\mathcal{SB}$. Such levels contain a total number of at least $n-1$ elements y/x of $\mathcal{SB}$. Order such elements by increasing value of the ratio y/x and consider the first $n-1$ elements in such an order S, say $s_1 = y_1/x_1, s_2 = y_2/x_2, \dots, s_{n-1} = y_{n-1}/x_{n-1}$.

Each subtree of T is assigned a subset of such elements that are consecutive in S as follows. First, consider the subtrees of r, say $T_1(r), T_2(r), \dots, T_{k(r)}(r)$ and assign to each subtree $T_i(r)$ the $|T_i(r)|$ elements of S from the $(1+\sum_{j=1}^{i-1} |T_j(r)|)$-th to the $(\sum_{j=1}^{i} |T_j(r)|)$-th. Then, for every node u of T, suppose that a subsequence $S(u) = s_a, s_{a+1}, \dots, s_b$ of S has been assigned to $T(u)$, where $|T(u)| = b-a+1$. Consider the subtrees $T_1(u), T_2(u), \dots, T_{k(u)}(u)$ of u and assign to $T_i(u)$ the $|T_i(u)|$ elements of $S(u)$ from the $(1+\sum_{j=1}^{i-1} |T_j(u)|)$-th to the $(\sum_{j=1}^{i} |T_j(u)|)$-th.

Now we illustrate how to exploit the above described assignment to construct a grid drawing of T. Place r at $(0,0)$. For each node u of T, suppose that a sequence $S(u) = s_a, s_{a+1}, \dots, s_b$ of S has been assigned to $T(u)$ and suppose that the parent $p(u)$ of u has been already placed at grid point $(p_x(u), p_y(u))$. Place u at grid point $(p_x(u) + x_b, p_y(u) + y_b)$, where $s_b = y_b/x_b$. See Fig. 12 for an example. We have the following:

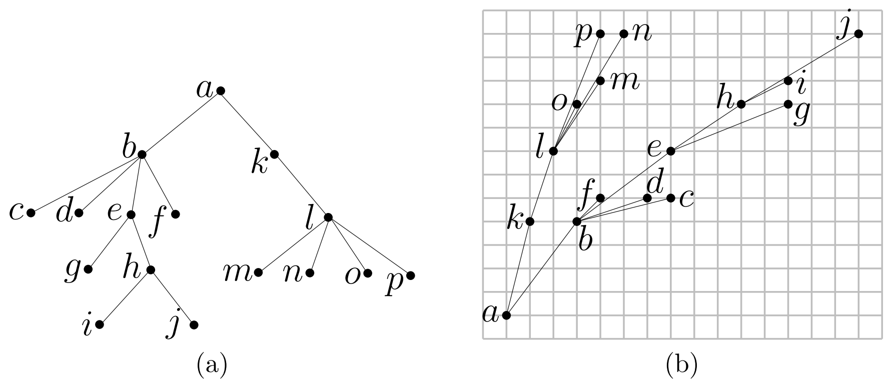


Figure 12: (a) A tree T. (b) The drawing of T constructed by Algorithm BFS-based.

Theorem 2 *Let T be a tree. Then, Algorithm BFS-based constructs a monotone drawing of T on a grid of area $O(n^{1.6}) \times O(n^{1.6})$.*

Proof: We prove that the drawing Γ constructed by Algorithm BFS-based is slope-disjoint and hence, by Theorem 1, monotone. In order to do this, we describe how to choose values $\alpha_1(u)$ and $\alpha_2(u)$ for every node $u \in T$. Recall that a sequence $S(u)$ is associated to each node $u \in T$ in such a way that the edge connecting u to its parent and all the edges in $T(u)$ have slopes in $S(u)$. Choose as $\alpha_1(r)$ any angle larger than $0°$ and smaller than the smallest angle $\arctan(\frac{y}{x})$, for all elements y/x in $S(r)$. Analogously, choose as $\alpha_2(r)$ any angle smaller than $90°$ and larger than the largest angle $\arctan(\frac{y}{x})$, for all elements y/x in $S(r)$. Now suppose that values $\alpha_1(u)$ and $\alpha_2(u)$ have been set for a node $u \in T$ and not for a child v of u. Choose as $\alpha_1(v)$ any angle: (i) smaller than the smallest angle $\arctan(\frac{y}{x})$, for all elements y/x in $S(v)$, (ii) larger than $\alpha_1(u)$, (iii) larger than the largest angle $\arctan(\frac{y_1}{x_1})$, for all elements y_1/x_1 not in $S(v)$ such that $y_1/x_1 < y_2/x_2$, for some element y_2/x_2 in $S(v)$, and (iv) larger than the largest value $\alpha_2(z)$, where z is a child of u such that $\alpha_2(z)$ has been already set and $y_1/x_1 < y_2/x_2$, for some element y_2/x_2 in $S(v)$ and some element y_1/x_1 in $S(z)$. Set $\alpha_2(v)$ in a symmetric way. It is easy to see that Properties P1–P3 are satisfied by the assignment of values $\alpha_1(u)$ and $\alpha_2(u)$ for every node $u \in T$.

It remains to show that Γ lies on a grid with area $O(n^{1,6}) \times O(n^{1,6})$. Consider the node z of T which has greatest x-coordinate and consider the path $P(z,r)$ from z to r. An edge (u,v) of such a path, where u is the parent of v, has x-extension equal to x_1, where y_1/x_1 is an element of $\mathcal{SB}$. Hence, the x-extension of $P(z,r)$ and the x-extension of Γ are bounded by the sum of all the x-coordinates of the elements of $\mathcal{SB}$. By Property 1 and by $\sum_{j=0}^{i-1} 3^j < 3^i$, the x-extension of Γ is $O(3^{\log_2 n}) = O(n^{1/\log_3 2}) = O(n^{1.6})$. Analogously, the y-extension of Γ is $O(n^{1.6})$ and the theorem follows. □

Algorithm DFS-based: Consider the sequence S composed of the first $n-1$ elements $1/1, 2/1, \ldots, n-1/1$ of the rightmost path of $\mathcal{SB}$. Assign sub-sequences of S to the subtrees of T and construct a grid drawing in the same way as in Algorithm BFS-based. We have the following.

Theorem 3 *Let T be a tree. Then, Algorithm DFS-based constructs a monotone drawing of T on a grid of area $O(n^2) \times O(n)$.*

Proof: The proof that the drawing Γ constructed by Algorithm DFS-based is slope-disjoint and hence monotone is the same as for Algorithm BFS-based.

We show that Γ lies on a grid with area $O(n^2) \times O(n)$. Every edge has x-extension equal to 1, hence the width of Γ is $O(n)$. The y-extension of an edge is at most $n-1$, hence the height of Γ is $O(n^2)$. □

As a further consequence of Theorem 1, we have the following:

Corollary 2 *Every (even non-planar) graph admits a monotone drawing.*

Namely, for any graph G, construct a monotone drawing of a spanning tree T of G with vertices in general position. Draw the other edges of G as segments, obtaining a straight-line drawing of G in which, for any pair of vertices, there exists a monotone path (the one whose edges belong to T). Such a drawing is a monotone drawing of G.

5 Planar Monotone Drawings of Biconnected Graphs

First, we restate, using the terminology of this paper, the well-known result of Chiba and Nishizeki [6, 5].

Lemma 6 *[5] Let G be a biconnected planar graph with a given planar embedding such that each split pair $\{u, v\}$ is incident to the outer face and each maximal split component of $\{u, v\}$ has at least one edge incident to the outer face except, possibly, for edge (u, v). Then, G admits a strictly convex drawing with the given embedding in which the outer face is drawn as an arbitrary strictly convex polygon.*

Second, we introduce some further definition concerning monotonicity.

Let Γ be a monotone drawing, d any direction, and k a positive value. A *directional-scale*, denoted by $\mathcal{DS}(d, k)$, is an affine transformation defined as follows. Rotate Γ and d by an angle δ until d is orthogonal to the x-axis. Scale Γ by $(1, k)$ (i.e., multiply its y-coordinates by k). Rotate back the obtained drawing by an angle $-\delta$.

Lemma 7 *Let Γ be a monotone drawing and let d be a direction such that no edge in Γ is orthogonal to d. For any $\alpha > 0$ there exists a directional-scale $\mathcal{DS}(d - \frac{\pi}{2}, k(\Gamma, \alpha))$ that transforms Γ into a monotone drawing in which the slope of any edge is between $d - \alpha$ and $d + \alpha$.*

Proof: First observe that, when Γ and d are rotated of an angle δ until $d - \frac{\pi}{2}$ is orthogonal to the x-axis, when Γ is scaled, and when Γ and d are rotated back of an angle δ, by Lemma 5, Γ remains monotone.

Consider any edge $(v, w) \in \Gamma$ such that the projection of v on d appears before the projection of w on d. Let β be the smaller angle created by the elongation of (v, w) and d. As (v, w) is not orthogonal to d, it holds $\beta < \frac{\pi}{2}$. Further, $\tan(\beta) = \frac{\Delta_y}{\Delta_x}$, where $\Delta_y = y(w) - y(v)$ and $\Delta_x = x(w) - x(v)$ when Γ and d have been rotated in such a way that $d - \frac{\pi}{2}$ is orthogonal to the x-axis.

Observe that scaling Γ by $(1, k)$, with $k < 1$, reduces Δ_y by a factor of $\frac{1}{k}$, while it keeps Δ_x unchanged. Hence, $\tan(\beta)$ is reduced by a factor of $\frac{1}{k}$, which implies that β is reduced by a factor linear in $\frac{1}{k}$. As $k \to 0$ implies $\beta \to 0$, it follows that for every $\alpha > 0$ there exists a $k > 0$ such that $\beta < \alpha$. □

A path monotone with respect to a direction d is (α, d)*-monotone* if, for each edge e, $d - \alpha < slope(e) < d + \alpha$. Observe that $\alpha < \frac{\pi}{2}$. A path from a vertex u to a vertex v is an (α, d_1, d_2)*-path* if it is a composition of an (α, d_1)-monotone path from u to a vertex w and of an (α, d_2)-monotone path from w to v.

Finally, we describe an algorithm to construct a planar monotone drawing of any biconnected graph G. Such an algorithm is based on a bottom-up visit of the SPQR-tree $\mathcal{T}$ of G, rooted at any edge e. At each step of the visit, a monotone drawing of the pertinent graph of a node μ of $\mathcal{T}$ is constructed inside a particular polygon, called *boomerang* and described in the following, that is completely reserved to it, in such a way to satisfy certain properties that allow the construction of an analogous drawing for the parent node of μ.

Namely, let μ be a node of $\mathcal{T}$. The *boomerang* of μ, denoted by $boom(\mu)$, is a quadrilateral $(p_N(\mu),\ p_E(\mu),\ p_S(\mu),\ p_W(\mu))$ such that point $p_W(\mu)$ is inside triangle $\triangle(p_N(\mu), p_S(\mu), p_E(\mu))$, angle $p_W(\mu)\widehat{p_S(\mu)}p_E(\mu)$ is equal to angle $p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$, angle $p_W(\mu)\widehat{p_S(\mu)}p_N(\mu)$ is equal to angle $p_W(\mu)\widehat{p_N(\mu)}p_S(\mu)$, and $p_W(\mu)\widehat{p_S(\mu)}p_N(\mu) + 2p_W(\mu)\widehat{p_S(\mu)}p_E(\mu) < \frac{\pi}{2}$ (see Fig. 13).

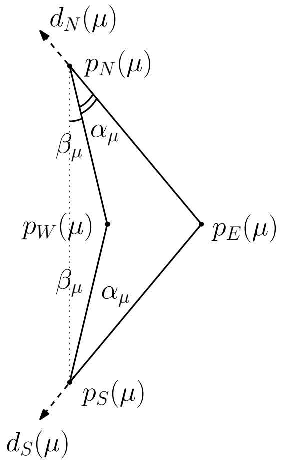


Figure 13: A boomerang.

We prove that G admits a planar monotone drawing by means of an inductive algorithm which, given a node μ of $\mathcal{T}$ with poles u and v, and a boomerang $boom(\mu)$, constructs a drawing Γ_μ of $pert(\mu)$ satisfying the following properties.

Let $d_N(\mu)$ be the half-line from $p_E(\mu)$ passing through $p_N(\mu)$, let $d_S(\mu)$ be the half-line from $p_E(\mu)$ through $p_S(\mu)$, let α_μ be $p_W(\mu)\widehat{p_S(\mu)}p_E(\mu) = p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$, and let β_μ be $p_W(\mu)\widehat{p_S(\mu)}p_N(\mu) = p_W(\mu)\widehat{p_N(\mu)}p_S(\mu)$ (see Fig. 13).

(A) Γ_μ is monotone;

(B) Γ_μ is planar and, with the possible exception of edge (u, v), it is contained inside $boom(\mu)$, with u drawn on $p_N(\mu)$ and v on $p_S(\mu)$;

(C) each vertex $w \in pert(\mu)$ belongs to a $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path from u to v.

We first prove a lemma stating the monotonicity of any $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path connecting the poles of a node of $\mathcal{T}$.

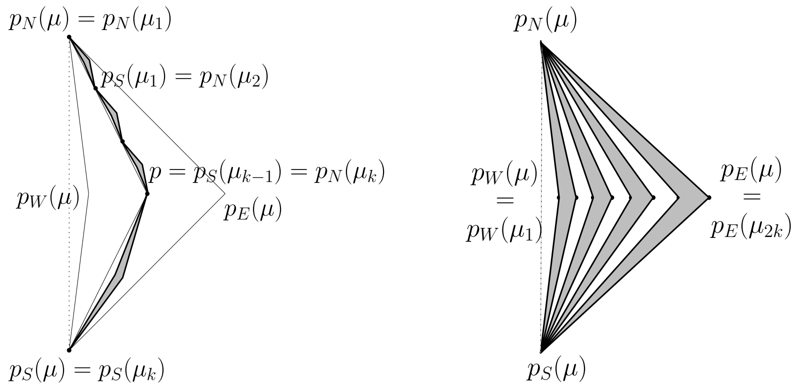
Lemma 8 *Let μ be a node of $\mathcal{T}$ with poles u and v. Every $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path from u to v is monotone with respect to the half-line from u through v.*

Proof: The proof is based on the fact that $\beta_\mu + 2\alpha_\mu < \frac{\pi}{2}$. Namely, rotate the coordinate axes in such a way that both u and v lie on the y-axis. As $\beta_\mu + 2\alpha_\mu < \frac{\pi}{2}$, the direction of any edge of the $(\alpha_\mu, -d_N(\mu))$-monotone path is contained in the fourth quadrant, while the direction of any edge of the $(\alpha_\mu, d_S(\mu))$-monotone path is contained in the third quadrant. It follows that the range of any $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path $P(u, v)$ from u to v contains the negative y-axis, that coincides with the direction defined by the half-line from u through v. This implies that $P(u, v)$ is monotone with respect to such a half-line. $\square$

Then, we describe how to compose the inductively constructed drawings $\Gamma_{\mu_1}, \dots, \Gamma_{\mu_k}$ of the children $\mu_1, \dots, \mu_k$ of μ in $\mathcal{T}$, with poles $(u_1, v_1), \dots, (u_k, v_k)$, in order to construct a drawing Γ_μ of $pert(\mu)$ satisfying Properties A–C.

If μ is a Q-node, then draw an edge between $p_N(\mu)$ and $p_S(\mu)$.

If μ is an S-node (see Fig. 14(a)), then let p be the intersection point between segment $\overline{p_W(\mu)p_E(\mu)}$ and the bisector line of $p_W(\mu)\widehat{p_N(\mu)}p_E(\mu)$. Consider k equidistant points $p_1, \dots, p_k$ on segment $\overline{p_N(\mu)p}$ such that $p_1 = p_N(\mu)$ and $p_k = p$. For each μ_i, with $i = 1, \dots, k-1$, consider a boomerang $boom(\mu_i) = (p_N(\mu_i),\ p_E(\mu_i),\ p_S(\mu_i),\ p_W(\mu_i))$ such that $p_N(\mu_i) = p_i$, $p_S(\mu_i) = p_{i+1}$, and $p_E(\mu_i)$ and $p_W(\mu_i)$ determine $\beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$. Apply the inductive algorithm to μ_i and $boom(\mu_i)$. Also, consider a boomerang $boom(\mu_k) = (p_N(\mu_k),\ p_E(\mu_k),\ p_S(\mu_k),\ p_W(\mu_k))$ such that $p_N(\mu_k) = p$, $p_S(\mu_k) = p_S(\mu)$, and $p_E(\mu_k)$ and $p_W(\mu_k)$ determine $\beta_{\mu_k} + 2\alpha_{\mu_k} < \frac{\alpha_\mu}{2}$. Apply the inductive algorithm to μ_k and $boom(\mu_k)$.



(a) (b)

Figure 14: The construction rules for an S-node (a) and for a P-node (b).

If μ is a P-node (see Fig. 14(b)), then consider $2k$ points $p_1, \dots, p_{2k}$ on segment $\overline{p_W(\mu)p_E(\mu)}$ such that $p_1 = p_W(\mu)$, $p_{2k} = p_E(\mu)$, and $p_i\widehat{p_N(\mu)}p_{i+1} = \frac{\alpha_\mu}{2k-1}$, for each $i = 1, \dots, 2k-1$. For each μ_i, with $i = 1, \dots, k$, consider a boomerang $boom(\mu_i) = (p_N(\mu_i),\ p_E(\mu_i),\ p_S(\mu_i),\ p_W(\mu_i))$ such that $p_N(\mu_i) =$

$p_N(\mu)$, $p_S(\mu_i) = p_S(\mu)$, $p_W(\mu_i) = p_{2i-1}$, and $p_E(\mu_i) = p_{2i}$. Apply the inductive algorithm to μ_i and $boom(\mu_i)$.

If μ is an R-node, then consider the graph G' obtained by removing v and its incident edges from $skel(\mu)$. Since $skel(\mu)$ is triconnected, G' is a biconnected graph whose possible split pairs are incident to the outer face. Further, each of such split pairs separates at most three maximal split components, and in this case one of them is an edge. By Lemma 6, G' admits a convex drawing whose outer face is represented by any strictly convex polygon. Consider a strictly convex polygon C with one vertex placed on $p_N(\mu)$, one vertex placed on $p_E(\mu)$, and $m-2$ vertices placed inside $boom(\mu)$ so that they are visible from $p_S(\mu)$ inside $boom(\mu)$ and so that the internal angle incident to $p_E(\mu)$ is smaller than $\frac{\pi}{2}$ (see Fig. 15(a)).

Construct a convex drawing $\Gamma(G')$ of G' such that the vertices of the outer face of G' are placed on the vertices of C, with u placed on $p_N(\mu)$. By Lemma 2, $\Gamma(G')$ is monotone. Slightly perturb the position of the vertices of $\Gamma(G')$ so that no two parallel edges exist and no edge is orthogonal to $d_N(\mu)$. Then, apply a directional-scale $\mathcal{DS}(d_N(\mu) - \frac{\pi}{2}, k(\Gamma(G'), \frac{\alpha_\mu}{2}))$ to $\Gamma(G')$. By Lemma 7, for every edge $e \in G'$, $slope(-d_N(\mu)) - \frac{\alpha_\mu}{2} < slope(e) < slope(-d_N(\mu)) + \frac{\alpha_\mu}{2}$. Further, by Lemma 4 and by the fact that the internal angle of C incident to $p_N(\mu)$ is smaller than $\frac{\alpha_\mu}{2} < \frac{\pi}{2}$, for every vertex $w \in G'$, a $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path exists from u to w.

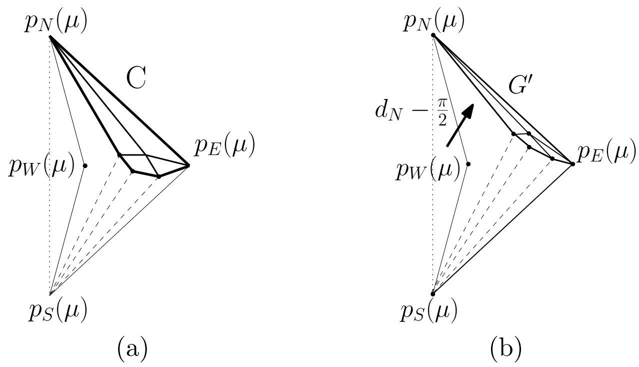


Figure 15: Two phases of the construction for an R-node: (a) definition of the strictly convex polygon C and (b) the directional-scale applied to G'.

Let $\Gamma(skel(\mu))$ be the drawing of $skel(\mu)$ obtained from $\Gamma(G')$ by placing v on $p_S(\mu)$ and drawing its incident edges as straight-line segments (see Fig. 15(b)). We have the following:

Claim 2 $\Gamma(skel(\mu))$ *is monotone.*

Proof: Every two vertices different from v are connected in $\Gamma(skel(\mu))$ by the same monotone path as in $\Gamma(G')$. Also, for every vertex $w \in skel(\mu)$, consider a path $P(u,v)$ from u to v that is a composition of a $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path from u to a vertex w' adjacent to v passing through w and of the $(\alpha_\mu, d_S(\mu))$-monotone path composed only of edge (w', v). Observe that $P(u,v)$ is an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. Since, by Lemma 8, $P(u,v)$ is monotone, the subpath of $P(u,v)$ between w and v is monotone, by Property 2. $\square$

Consider a drawing $\Gamma'(skel(\mu))$ of a subdivision of $skel(\mu)$ obtained as a subdivision of $\Gamma(skel(\mu))$. We have the following:

Claim 3 *$\Gamma'(skel(\mu))$ is monotone.*

Proof: Consider a graph G_* obtained by adding edge (u,v) to $skel(\mu)$ and consider a drawing Γ_* of G_* obtained by adding a straight-line edge between u and v in $\Gamma(skel(\mu))$. Note that Γ_* is convex and has no two parallel edges. Hence, any subdivision Γ'_* of Γ_* is monotone, by Lemma 3. We prove that for each pair of vertices in Γ'_* there exists a monotone path not containing (u,v). Observe that this implies the claim, as every path in Γ'_* not containing (u,v) is also a path in $\Gamma'(skel(\mu))$.

Consider any two vertices a and b both different from u and v. Consider any path $P(a,b)$ from a to b containing edge (u,v) and consider the two vertices a' and b' (possibly with $a' = a$ and $b' = b$) such that edges (a',u) and (v,b') belong to $P(a,b)$. Rotate the coordinate axes in such a way that u and v lie on the y-axis. Since u and v are the highest and the lowest point of Γ'_*, respectively, we have that $\widehat{a'uv} + \widehat{uvb'} < \pi$. Hence, when applied to the origin, $d(u,v)$ coincides with the negative y-axis, that is, it separates the third and the fourth quadrant. Moreover, $d(a',u)$ lies in the second quadrant and $d(v,b')$ lies in the first quadrant. Thus, there exists no wedge with an angle smaller than π that contains all of $d(a',u)$, $d(u,v)$, and $d(v,b')$, which implies that $P(a,b)$ is not monotone. We conclude that any monotone path between a and b, which exists by the fact that Γ'_* is monotone, does not contain (u,v).

Consider any two vertices such that one of the vertices is a pole, say u, and the other is a vertex w (possibly $w = v$). If w is also a vertex of $skel(\mu)$, then a monotone path between u and w exists by the fact that $\Gamma(skel(\mu))$ is monotone. Otherwise, if w is a subdivision vertex, then let (w_1, w_2) be the edge of $skel(\mu)$ whose subdivision vertex is w. Assume, without loss of generality, that the projection of w_1 on $d_N(\mu)$ appears after the projection of w_2 on $d_N(\mu)$. Consider the path $P(u,v)$ from u to v that is a composition of the monotone path $P(u,w_1)$ between u and w_1, of the monotone path (w_1, w, w_2), and of the monotone path from w_2 to v. As $d_N(\mu) - \alpha_\mu < slope(w_1, w) < d_N(\mu) + \alpha_\mu$, $P(u,v)$ is monotone, and hence the subpath of $P(u,v)$ between u and w is monotone, by Property 2. $\square$

Consider the pair of vertices x, y belonging to the subdivision of $skel(\mu)$ such that the range $range(P(x,y))$ of the monotone path $P(x,y)$ between them in $\Gamma'(skel(\mu))$ creates the largest angle $\angle(x,y)$ among all the pairs of vertices. Let $\gamma = \pi - \angle(x,y)$. Let δ be the smallest angle between two adjacent edges in $\Gamma(skel(\mu))$. For each μ_i, with $i = 1, \ldots, k$, let $p_N(\mu_i)$ and $p_S(\mu_i)$ be the points where u_i and v_i have been drawn in $\Gamma(skel(\mu))$, respectively. Consider a boomerang $boom(\mu_i) = (p_N(\mu_i), p_E(\mu_i), p_S(\mu_i), p_W(\mu_i))$ such that $p_E(\mu_i)$ and $p_W(\mu_i)$ determine $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\}$. For each μ_p such that either $p_N(\mu_p)$ and $p_S(\mu_p)$ lie on the vertices of C or $p_S(\mu_p) = p_S(\mu)$, choose points $p_W(\mu_p)$ and $p_E(\mu_p)$ inside $boom(\mu)$. Then, apply the inductive algorithm to μ_i, with poles u_i and v_i, and $boom(\mu_i)$ (see Fig. 16).

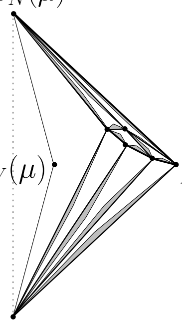


Figure 16: Inductive step when μ is an R-node.

In the following we prove that the above described algorithm constructs a planar monotone drawing of every biconnected planar graph.

Theorem 4 *Let G be an n-vertex biconnected planar graph. Then, it is possible to compute a planar straight-line monotone drawing of G in $O(n)$ time.*

Proof: Let $\mathcal{T}$ be the SPQR-decomposition of G, rooted at any Q-node μ_e corresponding to an edge e. Let $boom(\mu_e) = (p_N(\mu_e), p_E(\mu_e), p_S(\mu_e), p_W(\mu_e))$ be a boomerang such that:

- $x(p_N(\mu_e)) = x(p_S(\mu_e)) < x(p_W(\mu_e)) < x(p_E(\mu_e))$,
- $y(p_S(\mu_e)) < y(p_W(\mu_e)) = y(p_E(\mu_e)) < y(p_N(\mu_e))$, and
- $\beta_{\mu_e} + 2\alpha_{\mu_e} < \frac{\pi}{2}$.

Apply the inductive algorithm described above to μ_e and $boom(\mu_e)$. We prove that the resulting drawing is monotone by showing that at each step of the induction the constructed drawing satisfies Properties A–C. This is trivial if μ is a Q-node. Otherwise, μ is an S-node, a P-node, or an R-node and the statement is proved by the following claims:

Claim 4 *If μ is an S-node, Γ_μ satisfies Properties A, B and C.*

Proof: We prove that Γ_μ satisfies Property A. Refer to Fig. 17(a). Let $\mu_1, \ldots, \mu_k$ be the child nodes of μ. Consider any two vertices $w', w'' \in pert(\mu)$

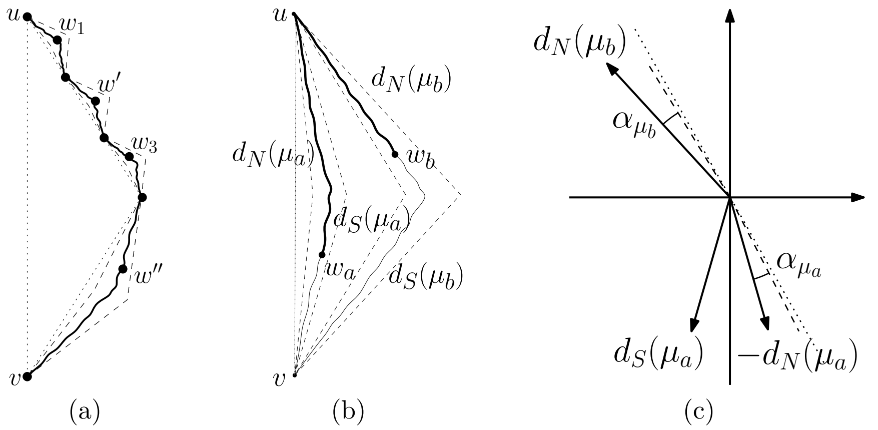


Figure 17: Γ_μ satisfies Property A (a) if μ is an S-node and (b)–(c) if μ is a P-node.

and the nodes μ_a and μ_b, with $1 \leq a, b \leq k$, such that $w' \in \mathit{pert}(\mu_a)$ and $w'' \in \mathit{pert}(\mu_b)$. If $a = b$, then a monotone path between w' and w'' exists by induction. Otherwise, for each μ_i, with $i = 1, \dots, k$, consider a vertex w_i, where $w_a = w'$ and $w_b = w''$, and a $(\alpha_{\mu_i}, -d_N(\mu_i), d_S(\mu_i))$-path $P(u_i, v_i)$ from u_i to v_i containing w_i. Observe that such paths exist since, for each μ_i, Γ_{μ_i} satisfies Property C. Consider a path $P(u_i, v_i)$ with $1 \leq i \leq k-1$. Since $\beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$, and since $p_N(\mu_i)$ and $p_S(\mu_i)$ lie on the bisector line of α_μ, for each edge $e \in P(u_i, v_i)$, it holds $\mathit{slope}(e) < \beta_\mu + \frac{\alpha_\mu}{2} + \beta_{\mu_i} + 2\alpha_{\mu_i} < \beta_\mu + \alpha_\mu < \beta_\mu + 2\alpha_\mu = -d_N(\mu) + \alpha_\mu$, and $\mathit{slope}(e) > \beta_\mu + \frac{\alpha_\mu}{2} - (\beta_{\mu_i} + 2\alpha_{\mu_i}) > \beta_\mu = -d_N(\mu) - \alpha_\mu$. Hence, $P(u_i, v_i)$ is $(\alpha_\mu, -d_N(\mu))$-monotone. Analogously, $P(u_k, v_k)$ is $(\alpha_\mu, d_S(\mu))$-monotone. Therefore, the path $P(u, v)$ composed of all the paths $P(u_i, v_i)$ is an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. By Lemma 8, $P(u, v)$ is monotone. Hence, by Property 2, the subpath of $P(u, v)$ between w' and w'' is monotone, as well, and Γ_μ satisfies Property A.

We prove that Γ_μ satisfies Property B. Observe that, for each node $\mu_1, \dots, \mu_{k-1}$, $\mathit{boom}(\mu_i)$ is such that $p_N(\mu_i)$ and $p_S(\mu_i)$ lie on segment $\overline{p_N(\mu)p}$, which creates angles equal to $\frac{\alpha_\mu}{2}$ with $\overline{p_N(\mu)p_W(\mu)}$ and with $\overline{p_N(\mu)p_E(\mu)}$. Further, $\beta_{\mu_i} + \alpha_{\mu_i} < \beta_{\mu_i} + 2\alpha_{\mu_i} < \frac{\alpha_\mu}{2}$. Hence, $\mathit{boom}(\mu_i)$ is contained inside $\mathit{boom}(\mu)$. Analogously, it is possible to show that $\mathit{boom}(\mu_k)$ is contained inside $\mathit{boom}(\mu)$. As, by induction (Property B), $\mathit{pert}(\mu_i)$, for each $i = 1, \dots, k$, is drawn inside $\mathit{boom}(\mu_i)$ without crossings, Property B holds for Γ_μ.

We prove that Γ_μ satisfies Property C. Observe that, for every vertex $w \in \mathit{pert}(\mu)$, it is possible to choose a vertex $w' \in \mathit{pert}(\mu)$ such that w and w' do not belong to the same node μ_i and to show that there exists an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path $P(u, v)$ from u to v passing through w and w' with the same argument as in the proof that Γ_μ satisfies Property A. $\square$

Claim 5 *If μ is a P-node, Γ_μ satisfies Properties A, B, and C.*

Proof: We prove that Γ_μ satisfies Property A. Let $\mu_1, \dots, \mu_k$ be the child nodes of μ. Consider any two vertices $w_a, w_b \in pert(\mu)$ and the nodes μ_a and μ_b, with $1 \leq a, b \leq k$, such that $w_a \in pert(\mu_a)$ and $w_b \in pert(\mu_b)$. If $a = b$, then a monotone path between w_a and w_b exists by induction. Otherwise, consider the $(\alpha_{\mu_a}, -d_N(\mu_a), d_S(\mu_a))$-path $P_a(u, v)$ from u to v through w_a and the $(\alpha_{\mu_b}, d_N(\mu_b), -d_S(\mu_b))$-path $P_b(v, u)$ from v to u through w_b, which exist by induction (Property C). Suppose that w_b lies on the $(\alpha_{\mu_b}, d_N(\mu_b))$-monotone path $P(w_b, u)$ from w_b to u that is a subpath of $P_b(v, u)$ (see Fig. 17(b)), the other case being analogous. Consider the $(\alpha_{\mu_a}, -d_N(\mu_a), d_S(\mu_a))$-path $P(u, w_a)$ that is a subpath of $P_a(u, v)$. We show that the path $P(w_b, w_a)$ composed of $P(w_b, u)$ and $P(u, w_a)$ is monotone. Refer to Fig. 17(c). Rotate the coordinate axes in such a way that u and v lie on the y-axis. Then, when translated to the origin of the axes, $d_N(\mu_b)$ is in the second quadrant, $-d_N(\mu_a)$ in the fourth quadrant, and $d_S(\mu_a)$ in the third quadrant. By construction, the wedge delimited by $d_N(\mu_b)$ and $-d_N(\mu_a)$ and containing the third quadrant has an angle smaller than or equal to $\pi - 2\frac{\alpha_\mu}{2k-1}$. Since, by definition, every edge of $P(w_b, u)$ creates an angle with $d_N(\mu_b)$ that is smaller than $\alpha_{\mu_b} = \frac{\alpha_\mu}{2k-1}$ and every edge of $P(u, w_a)$ creates an angle with $-d_N(\mu_a)$ that is smaller than $\alpha_{\mu_a} = \frac{\alpha_\mu}{2k-1}$, it follows that the slopes of all the edges of $P(w_b, w_a)$ lie inside a wedge having an angle smaller than π. Hence, $P(w_b, w_a)$ is monotone.

We prove that Γ_μ satisfies Property B. Each boomerang $boom(\mu_i)$ is contained inside $boom(\mu)$, by construction, and $pert(\mu_i)$ is drawn inside $boom(\mu_i)$ without crossings, by induction (Property B).

We prove that Γ_μ satisfies Property C. It is sufficient to observe that, for each vertex $w \in pert(\mu)$, the $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path passing through w coincides with the $(\alpha_{\mu_i}, -d_N(\mu_i), d_S(\mu_i))$-path passing through w, where μ_i is the node such that $w \in pert(\mu_i)$, which exists by induction (Property C). $\square$

Claim 6 *If μ is an R-node, Γ_μ satisfies Properties A, B, and C.*

Proof: We prove that Γ_μ satisfies Property A. Consider any two vertices w_a and w_b of $pert(\mu)$ and the nodes μ_a and μ_b such that $w_a \in pert(\mu_a)$ and $w_b \in pert(\mu_b)$. Let e_a and e_b be the virtual edges of $skel(\mu)$ corresponding to μ_a and μ_b, respectively. If $a = b$, by induction, there exists a monotone path between w_a and w_b. Otherwise, consider the monotone drawing $\Gamma'(skel(\mu))$ of a subdivision of $skel(\mu)$ and the monotone path $P(u_a, u_b)$ between the subdivision vertex u_a of e_a and the subdivision vertex u_b of e_b. By construction, $\pi - range(P(u_a, u_b)) \geq \gamma$. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\gamma}{2}$, for the path $P(w_a, w_b)$ that is obtained by replacing each edge e_i of $P(u_a, u_b)$ with the corresponding path of $pert(\mu_{e_i})$, it holds that $range(P(w_a, w_b)) < \pi$.

We prove that Γ_μ satisfies Property B. First observe that the constructed drawing of $skel(\mu)$ is planar (even convex) by construction. Also, for each node μ_p whose corresponding virtual edge is incident to the outer face of $skel(\mu)$, points $p_W(\mu_p)$ and $p_E(\mu_p)$ are inside $boom(\mu)$, by construction. Further, as for each node μ_i it holds $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\delta}{2}$, there is no intersection between any two boomerangs $boom(\mu_p)$ and $boom(\mu_q)$, with $p \neq q$. As, by induction, $pert(\mu_i)$, for each $i = 1, \ldots, k$, is drawn inside $boom(\mu_i)$ without crossings, Property B holds for Γ_μ.

We prove that Γ_μ satisfies Property C. Let $\Gamma(G')$ be the drawing of the subgraph of $skel(\mu)$ induced by the vertices of $skel(\mu) \setminus \{v\}$. Observe that, as the position of the vertices in $\Gamma(G')$ has been perturbed before applying the directional-scale $\mathcal{DS}(d_N(\mu) - \frac{\pi}{2}, k(\Gamma(G'), \frac{\alpha_\mu}{2}))$, every edge e in $\Gamma(G')$ is such that $d_N(\mu) - \frac{\alpha_\mu}{2} < slope(e) < d_N(\mu) + \frac{\alpha_\mu}{2}$.

Consider any vertex $w \in pert(\mu)$ and let μ_p be the child such that $w \in pert(\mu_p)$. Let (u_p, v_p) be the virtual edge in $\Gamma(skel(\mu))$ corresponding to μ_p. Two cases are possible: either (u_p, v_p) is adjacent to v or not. In the first case, consider the path $P(u,v)$ from u to v that is a composition of the $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path $P(u, u_p)$ between u and u_p and of the $(\alpha_{\mu_p}, -d_N(\mu_p), d_S(\mu_p))$-path $P(u_p, v_p)$ from u_p to $v_p = v$ passing through w, which exists by induction. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\delta}{2}$, $P(u,v)$ is an $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. In the latter case, assume, without loss of generality, that (u_p, v_p) has a negative projection on $d_N(\mu)$. Consider the path $P(u,v)$ from u to v that is a composition of the $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path $P(u, u_p)$ between u and u_p, of the $(\alpha_{\mu_p}, -d_N(\mu_p), d_S(\mu_p))$-path $P(u_p, v_p)$ from u_p to v_p passing through w, which exists by induction, and of the $(\frac{\alpha_\mu}{2}, -d_N(\mu))$-monotone path $P(v_p, w')$ between v_p and a vertex w' adjacent to v in $skel(\mu)$, and of a $(\frac{\alpha_\mu}{2}, d_S(\mu))$-monotone path $P(w', v)$ from w' to v, which exists by induction. As $\beta_{\mu_i} + 2\alpha_{\mu_i} < \min\{\frac{\delta}{2}, \frac{\gamma}{2}\} \leq \frac{\delta}{2}$, $P(u,v)$ is a $(\alpha_\mu, -d_N(\mu), d_S(\mu))$-path. □

We perform the computational complexity analysis of the algorithm. First, the SPQR-tree $\mathcal{T}$ of G can be computed in linear time [9, 10, 12].

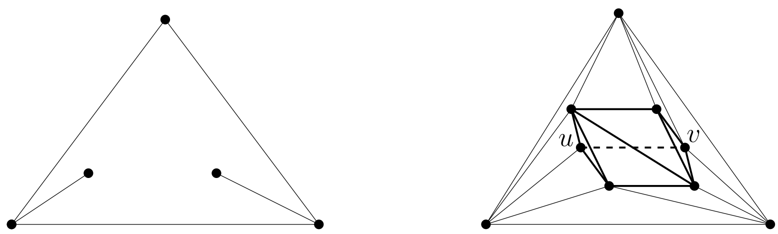
Then, consider a node μ of $\mathcal{T}$. If μ is an S- or a P-node, the algorithm only computes the points and the angles defining the boomerangs inside which a drawing of the children $\mu_1, \ldots, \mu_k$ of μ will be inductively constructed. Such an operation can be performed in $O(k)$ time for each S- or P-node, and hence in total $O(n)$ time over all the S- and the P-nodes of $\mathcal{T}$. If μ is an R-node, a convex drawing of the graph G' obtained by removing vertex v from the skeleton of μ can be computed in $O(k)$ time [7, 6, 5], where k is the number of children of μ. Then, a drawing of the skeleton can be found in $O(k)$ time by simply reinserting v and connecting it to its neighbors. Finally, the computation of the angles α_{μ_i} and β_{μ_i} defining the boomerangs inside which a drawing of the children $\mu_1, \ldots, \mu_k$ of μ will be inductively constructed can be performed in $O(k)$ time. Hence, the computational complexity over all the R-nodes of $\mathcal{T}$ is still $O(n)$, and the statement of Theorem 4 follows. □

6 Conclusions and Open Problems

In this paper we have initiated the study of monotone graph drawings.

Concerning trees, we have shown that every monotone drawing is planar, that every strictly convex drawing is monotone, and that monotone drawings exist on polynomial-size grids. We believe that simple modifications of the algorithms we described allow one to construct strictly convex drawings of trees on polynomial-size grids. Another possible extension of our results is to characterize the monotonicity of a drawing in terms of the angles between adjacent edges. Our definition of slope-disjointness goes in this direction, although it introduces some non-necessary restrictions on the slopes of the edges (like the one that all the edge slopes are between 0 and π).

Further, we have proved that every biconnected planar graph admits a planar monotone drawing. Extending such a result to general connected graphs seems to be non-trivial. Observe that there exist planar graphs that do not have a monotone drawing (see Fig. 18(a)) if the embedding is given. However, we are not aware of any planar graph not admitting a planar monotone drawing for any of its embeddings.



(a) (b)

Figure 18: (a) A planar embedding of a graph with no monotone drawing. (b) A drawing of a planar triangulation that is not strongly monotone.

Several area minimization problems concerning monotone drawings are, in our opinion, worth of study.

- Determining tight bounds for the area requirements of grid drawings of trees appears to be an interesting challenge. An improvement of the bounds achieved by Algorithms DFS-based and BFS-based could be possibly obtained by reusing in the subtree rooted at a vertex v the same slope of the edge connecting v to its parent.

- Modifying our tree drawing algorithms so that they construct grid drawings in general position would lead to algorithms for constructing monotone drawings of non-planar graphs on a grid of polynomial size.

- The drawing algorithm we presented for biconnected planar graphs constructs drawings in which the ratio between the lengths of the longest and of the shortest edge is exponential in n. Is it possible to construct planar monotone drawings of biconnected planar graphs in polynomial area?

Finally, we introduce a new drawing standard which is definitely related to monotone drawings. A path from a vertex u to a vertex v is *strongly monotone* if it is monotone with respect to the half-line from u through v. A drawing of a graph G is *strongly monotone* if for each pair of vertices $u, v \in G$ a strongly monotone path $P(u, v)$ exists. Strong monotonicity appears to be even more desirable than general monotonicity for the readability of a drawing. However, designing algorithms for constructing strongly monotone drawings seems to be harder than for monotone drawings and it appears to be true that only restricted graph classes admit strongly monotone drawings. Note that a subpath of a strongly monotone path is, in general, not strongly monotone; notice also that, while convexity implies monotonicity, it does not imply strong monotonicity, even for planar triangulations (see Fig. 18(b)).

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