Pinning balloons with perfect angles and optimal area

Immanuel Halupczok[1] *André Schulz*[1]

[1]Institut für Mathematische Logik und Grundlagenforschung, Universität Münster, Germany.

Abstract

We study the problem of arranging a set of n disks with prescribed radii on n rays emanating from the origin such that two neighboring rays are separated by an angle of $2\pi/n$. The center of the disks have to lie on the rays, and no two disk centers are allowed to lie on the same ray. We require that the disks have disjoint interiors and that for every ray the segment between the origin and the boundary of its associated disk avoids the interior of the disks. Let $\widetilde{r}$ be the sum of the disk radii. We introduce a greedy strategy that constructs such a disk arrangement that can be covered with a disk centered at the origin whose radius is at most $2\widetilde{r}$, which is best possible. The greedy strategy needs $O(n)$ arithmetic operations.

As an application of our result we present an algorithm for embedding unordered trees with straight lines and perfect angular resolution such that it can be covered with a disk of radius $n^{3.0367}$, while having no edge of length smaller than 1. The tree drawing algorithm is an enhancement of a recent result by Duncan et al. [Symp. of Graph Drawing, 2010] that exploits the heavy edge tree-decomposition technique to construct a drawing of the tree that can be covered with a disk of radius $2n^4$.

E-mail addresses: math@karimmi.de (Immanuel Halupczok) andre.schulz@uni-muenster.de (André Schulz)

1   Introduction

When a graph is drawn in the plane, the vertices are usually represented as small dots. From a theoretical point of view a vertex is realized as a point, hence as an object without volume. In many applications, however, it makes sense to draw the vertices as disks with volume. The radii of the vertices can enhance the drawing by visualizing associated vertex weights [2, 4]. This idea also finds applications in so-called *bubble drawings* [8], and *balloon drawings* [10, 11].

In the following we consider only drawings with no edge crossings. Two important quality measures for aesthetically pleasant drawings are the *area* of a drawing and its *angular resolution*. The area of a drawing denotes the area of the smallest disk that covers the drawing with no edge lengths smaller than 1. The angular resolution in a planar drawing denotes the minimum angle between two neighboring edges emanating at a vertex. Unfortunately, drawings of planar graphs with bounded angular resolution require exponential area [12]. On the other hand, by a recent result of Duncan et al. [5], it is possible to draw any unordered tree as plane straight-line graph with *perfect angular resolution* where the edges incident to a vertex v are separated by an angle of at least $2\pi/\mathrm{degree}(v)$, and polynomial area. In the same paper it was observed that an ordered tree drawn with perfect angular resolution requires exponential area. Surprisingly, even ordered trees can be drawn in polynomial area with perfect angular resolution when the edges are drawn as circular arcs [5].

The following sub-problem appears naturally in tree drawing algorithms. Suppose we have drawings of all subtrees of the children of the root. How can we group the subtrees around the root, such that the final drawing is densely packed? Often one assumes that every subtree lies exclusively in some region, say a disk. Hence, at its core, a tree drawing algorithm has to arrange disjoint disks "nicely" around a new "root" vertex. Furthermore this task is also a fundamental base case for bubble drawing algorithms or for algorithms that realize vertices as large disks. In the paper we show how to layout the balloons with perfect angular resolution and optimal area.

More formally, let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be a set of n disks. To distinguish the disks B_i from other disks we call them *balloons*. The balloon B_i has radius r_i, and the balloons are sorted in non-decreasing order of their radii. We are interested in layouts in which the balloons of $\mathcal{B}$ have disjoint interiors and are evenly angularly spaced. In particular, we draw for every balloon a *spoke*, that is, a line segment from the origin to the balloon center. The spokes have to avoid the interior of the other balloons and two neighboring spokes are separated by an angle of $2\pi/n$. Furthermore the drawing should require only small area. We measure the extent of the balloon layout by the radius of the smallest disk that is centered at the origin and covers all balloons. An example of a balloon layout is presented in Figure 1.

Results.   We show how to locate the balloons with perfect angular resolution such that the drawing can be covered with a disk of radius $2\widetilde{r}$, for $\widetilde{r}$ being the sum of the radii. This is clearly the best possible result in the worst case, since
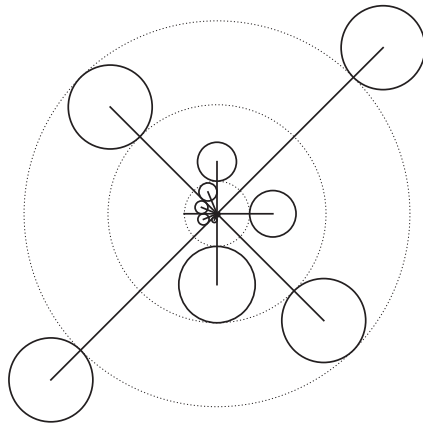
Figure 1: An example of a balloon layout obtained by the strategy presented in this paper.

when $|\mathcal{B}| = 1$, the the covering disk requires a radius of at least $2r_1$. Even for larger sets $\mathcal{B}$ the best possible covering disk might require a radius of $2\widetilde{r} - \varepsilon$ for an arbitrary small $\varepsilon > 0$. The worst case example is obtained by taking one balloon with radius $\widetilde{r} - \varepsilon/2$, and all other balloons with radius $\varepsilon/(2n-2)$. We also study a modified version of the balloon layout problem that finds application in a tree drawing algorithm. Here, one or two spokes may remain without balloon, but the angle between the two unused spokes has to be at least $2\pi/3$. In this setting we obtain a balloon drawing that can be covered with a disk of radius $(1 + \sqrt{2 - 2/\sqrt{5}})\widetilde{r} < 2.0515\widetilde{r}$, which is optimal in the worst case. The induced algorithm draws unordered trees with perfect angular resolution that can be covered with a disk of radius $n^{3.0367}$.

Related work.   Without being explicitly stated, Duncan et al. [5] studied the balloon layout problem (with one or two unused spokes) as part of their drawing algorithm for unordered trees and obtained a bound of $4\widetilde{r}$ for the radius of the covering disk. The induced tree drawing algorithm produces drawings which can be covered by a disk of radius $2n^4$. For the special case of orthogonal straight-line drawings of ternary trees (they automatically guarantee perfect angular resolution) Frati [6] provided an algorithm whose drawings require $O(n^{1.6131})$ area; the drawing of the complete ternary tree requires $O(n^{1.262})$ area. Bach-maier et al. obtained a drawing of the complete 6-regular tree with perfect angular resolution with area $O(n^{1.37})$ [1].

In contrast to our setting the so-called balloon drawings are plane drawings of trees, where the children of every node v are all placed at the same distance

from v. Lin and Yen [10] presented an $O(n^{3/2} \log n)$ algorithm for optimizing the angular resolution or the aspect ratio for balloon drawings of ordered trees. Recently, Lin et al. [11] studied balloon drawings for unordered trees. Some incarnations of their optimization problems for unordered trees became NP-complete, whereas others where solvable in polynomial time. Also related are the *ringed circular layouts* [14]. Here, children are arranged in equidistant layers around the parent with perfect angular resolution for every layer. On the downside, edge crossings appear often in this framework. Without the perfect angular resolution constraint, trees can be drawn with straight-lines and area $\Theta(n)$ [7].

Conventions. We normalize the radii of the balloons such that they sum up to 1. In intermediate stages of the drawing algorithm a spoke may be without a balloon. In this case we consider the spoke as a ray emanating from the origin that fulfills the angular resolution constraint. When we say that "we place balloon B on s at distance x" we mean that the balloon B is placed on a spoke s (that had no associated balloon yet) such that its center lies on s at Euclidean distance x from the origin. In the remainder of the paper all disks covering the balloons are considered as centered at the origin.

2   The greedy strategy

2.1   Wedges and layers

In the following section we introduce the greedy strategy for placing $\mathcal{B}$ with perfect angles. To keep things simple we assume for now that the number of balloons n is a power of two. The general case is discussed later.

   We place the balloons in non-decreasing order of their radii. Thus we start with the smallest balloon and end with the largest balloon. The placement of the balloons is carried out in *rounds*. In every round we place half of the balloons that have not been placed yet. Thus, we "consume" a certain number of spokes in each round. Let S be the list of spokes that are available in the beginning of a round in cyclic order. In every round we select every other spoke as a spoke on which a balloon is placed in the current round. This ensures that consecutive spokes that receive a balloon in round i are separated by an angle of $\alpha_i := 2^{i+1}\pi/n$. For every round we define the *safe disk* SD_i centered at the origin with radius safe$_i$. The safe disk is the smallest disk covering all balloons that were placed in previous rounds. In round i we place all balloons such that they avoid the interior of the safe disk SD_i. Thus, the best we can hope for is to place the balloons such that they touch SD_i. Whenever this is possible we speak of a *contact situation*, depicted in Figure 2(a). The safe disks ensure that balloons placed in the current round will not intersect the interior of the balloons that were placed in previous rounds. However, we have to guarantee that balloons placed in the same round will also not interfere with the remaining spokes. Suppose that B_j is assigned to the spoke s_k. We enforce

B_j to lie inside a wedge with opening angle α_i centered at s_k. This wedge is named W_k. Since the spokes that are used in round i are separated by α_i, the wedges of round i have disjoint interiors. Whenever a balloon touches the boundary of its associated wedge we speak of a *wedge situation*, as shown in Figure 2(b).
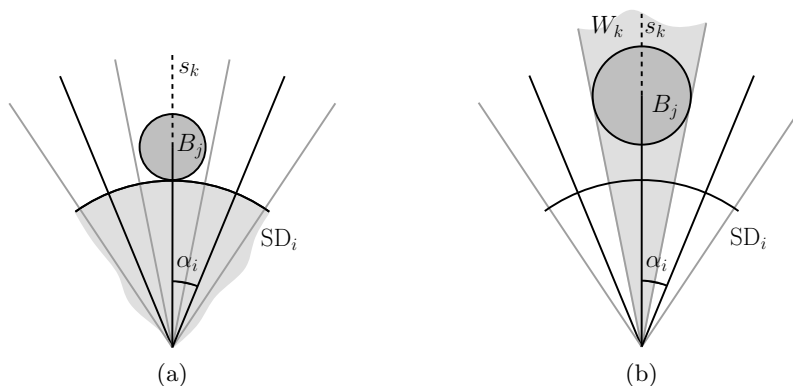

(a)   (b)

Figure 2: In a contact situation (a) we place B_i such that it touches SD_i. In contrast, in a wedge situation (b), we place B_i such that it touches the boundary of W_k (when it is placed on s_k).

The greedy strategy tries first to place B_j at its spoke s_k, such that it touches SD_i. If this would imply that B_j is not contained inside W_k, we move the center of B_j on s_k away from the origin, until B_j touches the boundary of W_k. In case a wedge situation occurs, we can compute the location of the center of B_j with help of the following lemma, which was also proven in a slightly different form by Duncan et al. [5].

Lemma 1 *Let W be a wedge with opening angle φ centered at a spoke s. Further let B be a balloon with radius r that is placed such that (1) its center lies on s, and (2) it touches the boundary of W. Then B is contained inside a disk centered at the origin with radius*

$$\frac{1 + \sin(\varphi/2)}{\sin(\varphi/2)} \cdot r.$$

Proof. Let t be one of the points where B touches the boundary of W and c the center of B as shown in Figure 3. The triangle spanned by the origin, t, and c has a right angle at t and an angle of $\varphi/2$ at the origin. Therefore, c has distance $r/\sin(\varphi/2)$ from the origin. To cover B we add r to the radius of the disk that touches c. The resulting radius equals $r + r/\sin(\varphi/2)$.   □

In the remainder of the paper we use as notation

$$\alpha(\varphi) := \frac{1 + \sin(\varphi/2)}{\sin(\varphi/2)}. \tag{1}$$
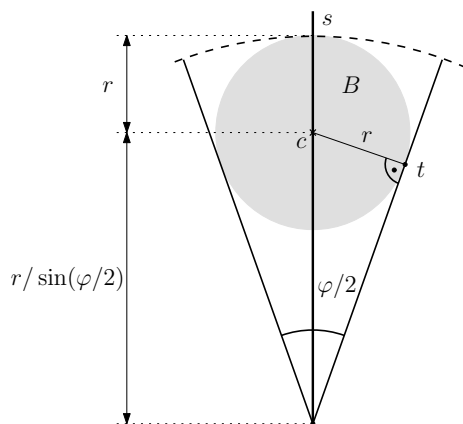
Figure 3: The construction used in the proof of Lemma 1

Notice that when a wedge situation occurs in round i, then in particular a wedge situation has to occur for the last balloon that is added in round i, since the balloons are sorted by increasing radii. All balloons placed in round i are sandwiched between SD_i and SD_{i+1}. We call the region $SD_{i+1} \setminus SD_i$ the i-th layer L_i.[1] The *width* of layer L_i is defined as $safe_{i+1} - safe_i$. When a wedge situation occurs in round i, the layer L_i is called a *wedge layer*, otherwise a *contact layer*. Notice that there is one particular balloon that defines the width of a layer. If all wedges have the same opening angle then the largest balloon inside a layer determines its width. Unless the number of spokes is a power of two, we can not guarantee that all opening angles of wedges are the same, and hence also other balloons might define the width. An example of a wedge layer is shown in Figure 4.
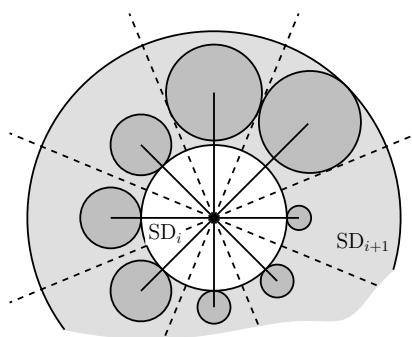

Figure 4: A wedge layer (shaded) that is filled with balloons by the greedy strategy.

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[1] By convention SD_1 is the origin, and for i being the last round, SD_{i+1} = smallest disk covering all balloons.

2.2 Splitting the set of spokes

We now come back to the case where n is not necessarily a power of two. In this setting there might be an odd number of spokes k in some round. In such a round we place only $\lfloor k/2 \rfloor$ balloons, such that no two of them are assigned to consecutive spokes. This however has two drawbacks: First, the angles might not split evenly, and second, the layers will be filled with less balloons.

   We show that we can always pick $\lfloor k/2 \rfloor$ spokes such that in the remaining set of spokes at most two separating angles are smaller than the others, which are all equal. Moreover, the two smaller angles are adjacent, and each of them is at least half as big as the remaining angles. We call every set of spokes for which this property holds *well-separated*. Furthermore we assume that a well-separated set of spokes is ordered such that the two smaller angles are realized between the first and second, and between the second and third spoke. Algorithm 1 describes a strategy that picks $\lfloor k/2 \rfloor$ of the spokes and ensures that the remaining set of spokes is still well-separated if the original set was well-separated.

Algorithm 1: SplitSpokes(S)

    Input   : S sequence of k spokes, indexed in radial order $1 \ldots k$
    Output: (T, T'), such that T' are the spokes that will be used in the current
               round, $T = S \setminus T'$.
1 $T' \leftarrow$ every spoke of S with an even index
2 $T \leftarrow$ every spoke of S with an odd index
3 reorder T by putting the last spoke in front
4 **return** (T, T')

Lemma 2 *Let S be a well-separated set of at least three spokes and let φ denote the size of the large angles in S. Let (T, T') be the pair of spokes returned by Algorithm 1.*

(1) *If $|T| > 2$, then T is well-separated.*

(2) *If $|T| = 2$, then the smaller angle between the two spokes is at least $2\pi/3$.*

(3) *The wedge with angle φ centered at the first spoke in T' contains no spoke of S in its interior.*

(4) *A wedge with angle 2φ centered at a spoke in T' that is not the first spoke contains no spoke of S in its interior.*

Proof. Let the angle between the first and second spoke in S be γ_1, and let the angle between the second and third spoke in S be γ_2. Since S is well-separated, we have $\varphi/2 \le \gamma_1, \gamma_2 \le \varphi$. The first spoke in T' is the second spoke in S and the wedge centered at the second spoke of S with angle φ does not contain any other spoke of S in its interior, which proves (3). The remaining spokes in T' are the spokes in S with even index. Property (4) is due to the fact that every spoke in S with even index larger than 2 is separated from its neighboring spokes by an angle of φ.
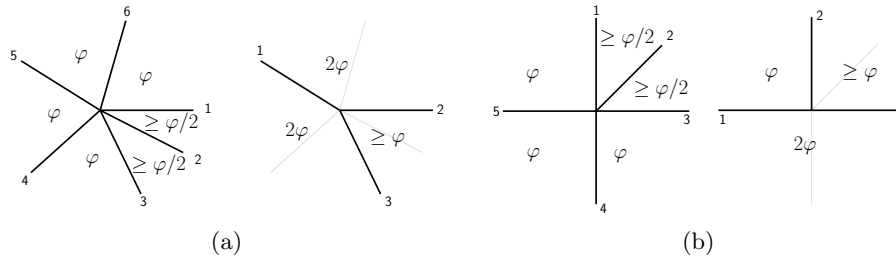
Figure 5: The outcome of Algorithm 1 as example: in case we have an even number of spokes (a), and in case we have an odd number of spokes (b). The spoke indices are shown as small numbers. The left pictures show the set S, the right pictures the set T.

After line 2 of Algorithm 1, the angle between the first and second spoke of T equals $\gamma_1 + \gamma_2 \geq \varphi$. The remaining spokes in T are separated by an angle of 2φ with one exception: In case that S is odd, the last spoke of T forms an angle of φ with the first spoke of T. Thus all separating angles of T have size 2φ, except the two angles around the first spoke in T, which have size at least φ but at most 2φ. After reordering the set T as done in line 3 of Algorithm 1, T is clearly well-separated. Figure 5 illustrates the outcome of the algorithm for the case $|S| = 6$ and $|S| = 5$.

To see that (2) is true, notice the following. T contains two spokes if S contains three or four spokes. In case S contains 4 spokes, the sum of the two small angles is at least $2\pi/3$. In case S contains three spokes, the sum of the two small angles between the spokes is at least π. The large angle between the spokes in S is at least $2\pi/3$. This angle appears also between the spokes in T. □

To ensure that the balloons of each layer cannot interfere with each other and with the remaining spokes, we place them inside the wedges defined by Lemma 2(3–4). All wedges have the same opening angle, say 2φ, except the first wedge, whose opening angle is at least φ. The balloon with the smallest radius in each round is placed inside the wedge with the (possible) smaller opening angle.

2.3   The final layer

It is important to analyze the situation where the greedy strategy has to stop. In every round we reduce the number of spokes from k to $\lceil k/2 \rceil$. If we subdivide the spokes in this fashion we will come to a point where exactly two spokes are left. The final two balloons are placed in the last round as follows: (1) The balloon B_n will be placed such that it touches the safe disk. (2) The balloon B_{n-1} will be placed such that it is contained inside a wedge with opening angle $\pi/3$, centered at its spoke, while avoiding the interior of the current safe disk.

Lemma 3 *When the balloons are placed as discussed in the previous paragraph, then one of the following is true:*

1. *The width of the last layer is $2r_n$.*

2. *All balloons can be covered with a disk of radius $3/2$.*

Proof. Let φ be the smaller of the two angles between the spokes in the final round i. Due to Lemma 2, φ is at least $2\pi/3$. The line orthogonal to the spoke of B_n touching SD_i separates B_n from the spoke of B_{n-1}. Since the angle between this tangent and the spoke of B_{n-1} is at least $\varphi - \pi/2 \geq \pi/6$ it is safe to place B_{n-1} inside a wedge centered at its spoke with opening angle $\pi/3$. Thus, either B_n or B_{n-1} defines the radius of the covering disk. In the former case (see Figure 6(a)) the width of the last layer is $2r_n$, in the latter case (see Figure 6(b)) the radius of the covering disk is at most $\alpha(\pi/3)r_{n-1} \leq 3/2$, since $r_{n-1} \leq 1/2$. □
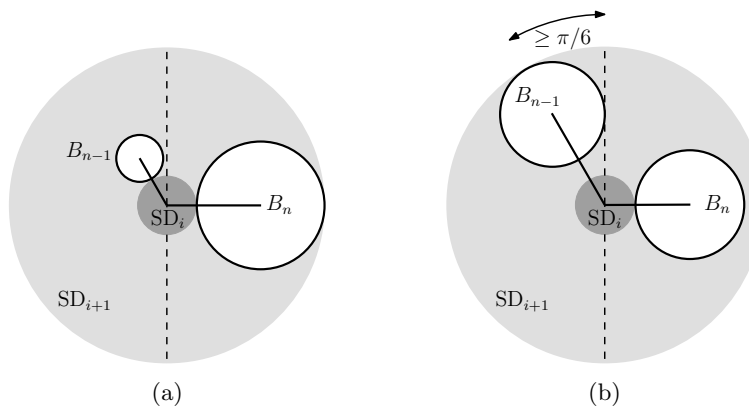

Figure 6: The two possible cases in the final layer as discussed in the proof of Lemma 3. Either B_n determines the width of the layer (a), or B_{n-1} determines the width (b).

Due to Lemma 3 we can assume that the width of the last layer equals $2r_n$. Thus even if B_{n-1} defines a wedge situation, we consider the last layer to be a contact layer. We summarize the discussion in Algorithm 2.

2.4   Quality of the greedy strategy

We denote by R the radius of the smallest disk that covers all balloons. In order to determine R we have to consider only certain balloon radii.

Lemma 4 *The radius of the smallest disk R that covers all balloons drawn with Algorithm 2 can be determined with the knowledge of*

1. *the number of spokes,*

Algorithm 2: GreedyBalloon(S).

Input : S: spokes in cyclic order, balloon radii
Output: Balloon drawing

1 $k \leftarrow 0$ // number of balloons placed so far
2 safe $\leftarrow 0$ // radius of the current safe disk
3 while $|S| > 2$ **do**
4 $(T, T') \leftarrow$ Splitspokes(S)
5   width $\leftarrow 0$ // width of the current layer so far
6   **for** $i \leftarrow k + 1$ **to** $k + |T'|$ **do**
7 $s \leftarrow (i - k)$-th spoke of T'
8 $\varphi \leftarrow 2$(minimal angle between s and its neighboring spokes)
9 $c \leftarrow \max\{\alpha(\varphi)r_i - r_i, \text{safe} + r_i\}$ // center of B_i
10   place B_i on s at distance c
11   width $\leftarrow \max\{\text{width}, c + r_i - \text{safe}\}$
12   **end**
13   safe $\leftarrow$ safe + width
14 $k \leftarrow k + |T'|$
15 $S \leftarrow T$
16 end
17 let s_1, s_2 be the spokes in S
18 place B_n on s_1 at distance safe + r_n
19 place B_{n-1} on s_2 at distance $\max\{2r_{n-1}, \text{safe} + r_{n-1}\}$

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2. *the radius of the largest and smallest balloon in the outermost wedge layer,*

3. *the radii of the largest balloons in each of the contact layers following the outermost wedge layer.*

Proof. Suppose the last wedge situation occurs in round i. Then the radius of SD_{i+1} is determined by a balloon $\widehat{B}$ that touches its wedge. There are two possibilities for $\widehat{B}$. Either $\widehat{B}$ is the last balloon in the layer, since this balloon is the largest, or $\widehat{B}$ is the first balloon in the layer, since its wedge might have a smaller opening angle compared to the other wedges in the layer. The following layers are all contact layers. Their width is determined by the diameter of the largest balloons in each layer. The radius R equals therefore the radius of SD_{i+1} with the addition of the widths of the following contact layers.   □

Since we are interested in a worst case bound for R we make the following assumptions to simplify the analysis of the algorithm.

Lemma 5 *Let w be the index of the balloon, whose wedge situation determined the width of the last wedge layer L_j. The radius R of the smallest covering disk is maximized when*

$$r_w = r_{w+1} = r_{w+2} = \cdots = r_{n-1}, \text{ and}$$
$$r_1 = r_2 = r_3 = \cdots = r_{w-1} = 0.$$

Proof. We are interested in the worst case situation with respect to the initial radii. Since no radius of a balloon with smaller index than w matters for R, we set these radii to zero to save resources. If B_w is the smallest balloon in its layer, all radii of balloons in L_j have radius r_w in the worst case. Otherwise we could shrink some of these balloons without changing the width of L_j and spend the resources to increase r_n and therefore R.

Only the balloon added last in each contact layer determines the width of its layer. We select the radii of the other balloons in contact layers as small as possible, i.e., as large as the radius of the largest balloon in the previous layer. If any of these radii would be larger we could make such a radius smaller and increase r_n instead, which would increase R.

Assume we have at least two contact layers following L_j. The situation is schematically depicted in Figure 7. Let B_c be the largest balloon in the contact layer L_{j+1}, that is, the balloon last added in L_{j+1}. Due to the discussion in the previous paragraph we can assume that the balloon B_{c+1} in the next layer has radius r_c. If $r_c > r_w$, we could lower the radius by $\delta := r_c - r_w$ for B_c and B_{c+1} each. By this we can increase r_n by 2δ. As a consequence the width of L_{j+1} decreases by δ and the width of the last layer increases be 2δ. Thus, the radius R increases by δ and all radii in L_{j+1} equal r_w in the worst case. By an inductive argument the radii in the last contact layers are all r_w. The only exception is the largest balloon B_n.   □

Before proving the main theorem we show a technical lemma, that gives a lower bound for the opening angles of the wedges.
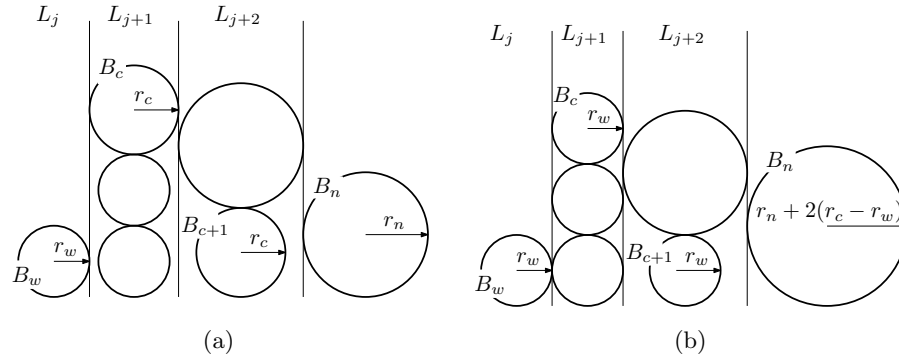
Figure 7: If there is a balloon in between B_w and B_n with radius $> r_w$ (a), then there is another radii assignment that requires a larger covering disk (b).

Lemma 6 *Let L be a wedge layer with k balloons and a well-separated set of spokes. Further, let φ be the opening angle of the wedge that defines the width of L.*

(1) If the largest balloon in L defines its width, then

$$\varphi \geq 4\pi/k.$$

(2) If the smallest balloon in L defines its width, then

$$\varphi \geq 2\pi/(k - 1/2).$$

Proof. (1) By construction the last balloon is placed inside the wedge with largest opening angle (in this round). Therefore its opening angle φ is minimized, when the angles between all pairs of neighboring spokes are equal. Therefore two spokes are separated by $2\pi/k$ and $\varphi \geq 4\pi/k$.

(2) Due to Lemma 2 the angles between two neighboring spokes are all of size ψ except two angles, which are at least $\psi/2$ (the small angles). The angle φ is twice the minimum of the two small angles, and hence minimized when one of the small angles has size ψ and the other has size $\psi/2$. In this case we have $k - 1$ angles of size ψ and one angle of size $\psi/2$. Since all angles sum up to 2π, we have $\psi = 2\pi/(k - 1/2)$, which is a lower bound for φ.   □

Theorem 1 *Algorithm 2 constructs a drawing of balloons with disjoint interiors and spokes that intersect only the interior of their associated balloon that can be covered with a disk of radius two, which is best possible in the worst case.*

Proof. We define as $\bar{L}_i$ the i-th last layer such that $\bar{L}_1$ is the last layer. Suppose there were k spokes left, before the last wedge layer was filled. We denote the

number of contact layers that follow the last wedge layer by ℓ. By Algorithm 1 the number ℓ is given by a function $\ell = f(k)$, which is defined as follows

$$f(k) := \begin{cases} 1 & \text{if } 3 \leq k \leq 4, \\ 1 + f\left(\frac{k}{2}\right) & \text{if } k > 4, \text{ even}, \\ 1 + f\left(\frac{k+1}{2}\right) & \text{if } k > 4, \text{ odd}. \end{cases} \qquad (2)$$

By induction, $f(k) \leq \log(k-1)$, where log denotes the binary logarithm. The radius of the covering disk R equals the radius of $\bar{L}_\ell$'s safe disk plus the width of the last ℓ contact layers. Let B_w be the balloon that determined safe$_\ell$. By Lemma 5 we can assume that all balloons following B_w have radius r_w, except B_n. All other radii are zero.

As previously discussed, the balloon B_w is either the first or the last balloon in the last wedge layer. We discuss the two possibilities by case distinction. Let us first assume that B_w is the last balloon of layer $\bar{L}_{\ell+1}$. By Lemma 6(1) we have that $\varphi \geq 4\pi/k$. Furthermore, we have $\ell - 1$ layers of width $2r_w$, and one layer of width $2r_n$ following $\bar{L}_{\ell+1}$. In layer $\bar{L}_{\ell+1}$ we place no more than $k/2$ balloons and therefore in the last ℓ layers we have at least $k/2$ balloons in total. Since there is one balloon in $\bar{L}_{\ell+1}$ with radius r_w and only one balloon in the last ℓ layers with radius different from r_w, we get $r_n \leq 1 - r_w k/2$. This leads to

$$R \leq \alpha(\varphi)r_w + 2(\ell-1)r_w + 2r_n \leq 2 + [\alpha(4\pi/k) + 2\log(k-1) - k - 2]\, r_w.$$

The last wedge layer must contain at least three spokes. Since $\alpha(4\pi/k) + 2\log(k-1) - k - 2$ is decreasing[2] for $k \geq 4$ and negative for $k = 3, 4$, we get $R \leq 2$.

We assume now that B_w was placed first in $\bar{L}_{\ell+1}$. Again, let φ be the angle of the wedge that contains B_w centered at its spoke. Due to Lemma 6(2) we have that $\varphi \geq 2\pi/(k - 1/2)$. Notice that all balloons in $\bar{L}_{\ell+1}$ have radius r_w, hence we have $k - 1$ balloons of radius r_w, and therefore $r_n \leq 1 - (k-1)r_w$. We conclude with

$$R \leq \alpha(\varphi)r_w + 2(\ell-1)r_w + 2r_n \leq 2 + [\alpha(2\pi/(k-1/2)) + 2\log(k-1) - 2k]\, r_w.$$

For $k \geq 2$ the expression $\alpha(2\pi/(k-1/2)) + 2\log(k-1) - 2k$ is negative and decreasing and the theorem follows. $\square$

2.5   Running time

In this section we discuss the running time of Algorithm 2. Recall that we assumed in the beginning that the balloons are sorted by radii. Our estimation for the running time will also be valid for the later modifications of the algorithm.

Lemma 7 *The running time of Algorithm 2 is $O(n)$.*

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[2]The estimation of this expression and of similar following expressions was obtained by computer algebra software.

Proof. The algorithm consists mainly of a while-loop. The set S denotes the set of spokes, which have not been assigned with a balloon yet. In every iteration of the loop we compute the spokes that will get a balloon in the current round, and assign them with balloons. This takes $O(|S|/2)$ time per round. Since we are halving the set of spokes in every round, the total running time is $O(n)$. □

Even when $\mathcal{B}$ is not ordered by radii we can still guarantee a running time of $O(n)$ for Algorithm 2. This can be achieved by sorting the set of balloons only *partially* in a preprocessing step as follows. Recall that r_i denotes the radius of the balloon B_i. We denote the median of the radii of $\mathcal{B}$ with r_M.

Definition 1 (weakly ordered sequence) *We say that the sequence of balloons $\mathcal{B}$ is* weakly-ordered, *iff*

1. $r_1 = \min\{r_i\}$,

2. *for all $i < \lfloor n/2 \rfloor$ we have $r_i \leq r_M$,*

3. $r_{\lfloor n/2 \rfloor} = r_M$, *and*

4. $(B_{\lfloor n/2 \rfloor+1}, \ldots, B_n)$ *is weakly-ordered.*

The median of n elements can be found in linear time [3]. It follows from the recursive definition that $\mathcal{B}$ can be weakly-ordered in linear time. On the other hand it is indeed sufficient for $\mathcal{B}$ being weakly-ordered, since for the location of the balloons within each round only the smallest and largest balloon matters. A permutation of the balloons in between has no influence on the necessary width of the corresponding layer. We summarize our observations with the following theorem.

Theorem 2 *We can compute the balloon layout as determined by Algorithm 2 in $O(n)$ time, even if the balloons $\mathcal{B}$ are not sorted by radii.*

3   Free spokes

3.1   Modifications

In this section we study a variant of the balloon layout problem that finds application in a tree drawing algorithm, which is presented in Section 4. In contrast to the original setting we require that one or two spokes remain without an assigned balloon. Hence the number of spokes exceeds the number of balloons which we denote with n. A spoke that remains without an assigned balloon is called *free spoke*. As additional constraint we require that if there are two free spokes, the smaller separating angle is at least $2\pi/3$. Allowing free spokes makes the performance of the greedy strategy worse, since the available angular space between the spokes is reduced. In order to achieve good bounds for this modified problem, we change the greedy strategy slightly. In particular, we change the terminal cases for the layout algorithm and we introduce a construction that allows us to move some balloons inside their safe disk. The rest of the greedy strategy remains unaltered.

3.2   New terminal cases

We have two terminal cases for the scenario with one free spoke, and two terminal cases for the scenario with two free spokes. The new terminal cases are covered by the Lemmata 8–10; see also Table 1. Notice that for any number of original spokes the greedy strategy has to come to one of these terminal cases.

	free spokes	remaining spokes	remaining balloons
Lemma 8	1	2	1
	1	3	2
Lemma 9	2	3	1
Lemma 10	2	4	2

Table 1: The terminal cases.

Lemma 8 *Suppose we have either two spokes and one balloon, or three well-separated spokes and two balloons left while executing the greedy strategy. We can place the remaining balloons, such that either all balloons can be covered with a disk of radius two, or the width of the last layer is $2r_n$.*

Proof. The case when there are two spokes and one balloon left is trivial. For the remaining case we assume that the spokes are labeled such that the largest angle is realized between s_3 and s_1, and the second largest angle is realized between s_2 and s_3. Balloon B_n is placed at s_3 such that it touches the safe disk, which is possible, since the angle between s_2 and s_3 is at least $\pi/2$. Let t be the tangent of B_n at the intersection with the safe disk. Since s_3 and s_1 are separated by an angle of at least $2\pi/3$, t and s_1 are separated by an angle of at least $\pi/6$. Similarly to the construction of Lemma 3, we can place B_{n-1} at a wedge with opening angle $\pi/3$ centered at s_1. If this would result in a wedge situation, the disk covering all balloons except possibly B_n would have radius $\alpha(\pi/3)r_{n-1} = 3r_{n-1} \leq 3/2$.   □

Lemma 9 *Suppose we have three well-separated spokes and one balloon left while executing the greedy strategy. We can place B_n, such that the width of the last layer is $(1 + \sqrt{2 - 2/\sqrt{5}})r_n$, and the smaller angle between the two remaining spokes is at least $2\pi/3$.*

Proof. Let $\varphi_1 \leq \varphi_2 \leq \varphi_3$ be the angles that separate the three spokes. Since the spokes are well-separated, we know that $2\varphi_1 \geq \varphi_2, \varphi_3$. Hence, $5\varphi_1 \geq \varphi_1 + \varphi_2 + \varphi_3 = 2\pi$. Thus we can place B_n on the spoke incident to the two smaller angles such that it touches the safe disk inside a wedge with opening angle $4\pi/5$. Hence, B_n can be covered with a disk of radius $\alpha(4\pi/5)r_n = (1 + \sqrt{2 - 2/\sqrt{5}})r_n < 2.0515r_n$. The remaining spokes are separated by the former larger angle. By Lemma 2 this angle is at least $2\pi/3$ and at most π. □

Lemma 10 *Suppose we have four well-separated spokes and two balloons left while executing the greedy strategy. We can place B_n and B_{n-1}, such that either all balloons can be covered with a disk of radius two, or the width of the last layer is $2r_n$. The smaller angle between the two remaining spokes is at least $2\pi/3$.*

Proof. By well-separatedness, we can assume that the two larger angles (which are at least $\pi/2$) are realized between the spokes s_3, s_4, and s_1. We place B_n at s_4 such that it touches the safe disk. The smallest angle is minimized when all other angles are equal. In this case, the smallest angle is $2\pi/7$. Hence, we can place B_{n-1} at s_2 inside a wedge with opening angle $4\pi/7$. If this would result in a wedge situation, the disk covering all balloons would have radius $\alpha(4\pi/7)r_{n-1} < 2$. The angle between the two remaining (free) spokes is at least the sum of the two small angles, which is at least $2\pi/3$.   □

3.3   Compaction

The following construction allows us to place a balloon such that it slightly overlaps the previous safe disk; this is needed in a few special cases.

Lemma 11 *Suppose that s_1 and s_2 are two spokes that are separated by an angle β and that B is a balloon placed on s_1 such that it is disjoint from s_2 and such that it can be covered with a disk of radius s. Then a balloon B' with radius r' placed on s_2 at distance $s \cdot (\sin(\beta) + \cos(\beta))/(\sin(\beta) + 1) + r'$ from the origin will be disjoint from B.*

Proof. In the worst case B is as large as possible, i.e., it touches both, the spoke s_2 and the border of the disk of radius s. Let the radius of B in this case be r. By Lemma 1 we have $s = r + r/\sin(\beta)$ (see Figure 8) and hence $r = s \cdot \sin(\beta)/(\sin(\beta) + 1)$. To ensure that B' is disjoint from B, it suffices to place its center r' units above the line h that is perpendicular to s_2 and touches B'. The distance of this line from the origin is

$$s' = r + \cot(\beta)r = r \cdot \frac{\sin(\beta) + \cos(\beta)}{\sin(\beta)} = s \cdot \frac{\sin(\beta) + \cos(\beta)}{\sin(\beta) + 1}.$$

□

3.4   Analysis of the modified greedy strategy

The analysis follows the presentation in Section 2. As before, the layer of the last round is always considered as contact layer, even when a wedge situation determined its width.

Theorem 3 *Assume that the number of spokes exceeds the number of balloons by one. Algorithm 2 with base case as described in Lemma 8 produces a drawing of balloons with disjoint interiors and one free spoke that can be covered with a disk of radius two.*
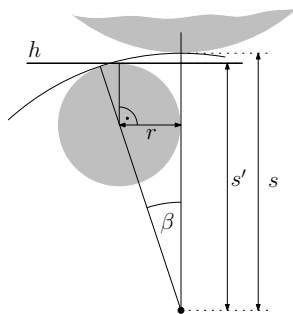
Figure 8: By Lemma 11 it is possible to push the balloons slightly inside the safe disk.

Proof. We denote by k the number of spokes in the last wedge layer. We reuse the estimations for the angles given in Lemma 6. There are ℓ layers following the last wedge layer. The number $\ell = f(k)$ can bounded in terms of k by the following recursion

$$f(k) := \begin{cases} 1 & \text{if } 4 \leq k \leq 6, \\ 1 + f\left(\frac{k}{2}\right) & \text{if } k > 6, \text{ even}, \\ 1 + f\left(\frac{k+1}{2}\right) & \text{if } k > 6, \text{ odd}. \end{cases}$$

The recursion yields $f(k) \leq \log(2(k-1)/3)$, which can by checked by induction.

Let B_w be the balloon that determined the width of the last wedge layer. B_w can be either the first or last balloon of the layer. Assume that B_w was placed last in $\bar{L}_{\ell+1}$. In this case $r_n \leq 1 - (k/2 - 1)r_w$, since we have one balloon less compared to the proof of Theorem 1, but we have the same bounds for the angles, namely $\varphi \geq 4\pi/k$. This gives

$$R \leq \alpha(\varphi)r_w + 2(\ell-1)r_w + 2r_n \leq 2 + [\alpha(4\pi/k) + 2\log(2(k-1)/3) - k]\, r_w.$$

Since $\alpha(4\pi/k) + 2\log(2(k-1)/3) - k$ is non-positive for all $k \geq 4$, we have $R \leq 2$ in this case.

Assume now that B_w was the first balloon of $\bar{L}_{\ell+1}$. By Lemma 6 we have $\varphi \geq 2\pi/(k - 1/2)$. Since we have one balloon less, we get $r_n \leq 1 - (k-2)r_w$. We deduce

$$\begin{aligned} R &\leq \alpha(\varphi)r_w + 2(\ell-1)r_w + 2r_n \\ &\leq 2 + [\alpha(2\pi/(k-1/2)) + 2\log(2/3(k-1)) - 2k + 2]\, r_w. \end{aligned}$$

Since $\alpha(2\pi/(k-1/2)) + 2\log(2/3(k-1)) - 2k + 2 < 0$ for $k \geq 4$ the theorem follows. □

Theorem 4 *Assume that the number of spokes exceeds the number of balloons by two. Algorithm 2 with base cases as described in Lemma 9, and 10, and the construction described in Lemma 11 produces a drawing of balloons with*

disjoint interiors and two free spokes that can be covered with a disk of radius $(1 + \sqrt{2 - 2/\sqrt{5}}) < 2.0515$.

Proof. The proof is similar to the proof of Theorem 3. So again, let k be the number of spokes in the last wedge layer. We stop the greedy strategy when three or four spokes are left.

Let ℓ denote the numbers of layers following the last wedge layer. We have

$$\ell = f(k) := \begin{cases} 1 & \text{if } 5 \le k \le 8, \\ 1 + f\left(\frac{k}{2}\right) & \text{if } k > 8 \text{ even}, \\ 1 + f\left(\frac{k+1}{2}\right) & \text{if } k > 8 \text{ odd}. \end{cases}$$

The solution to this recurrence gives $f(k) \le \log(k-1) - 1$, which can be checked easily by induction.

Assume that B_w is the last balloon in its layer. We place $\lfloor k/2 \rfloor$ balloons in the last wedge layer, and therefore $\lceil k/2 \rceil - 2$ in the final ℓ layers. This gives $r_n \le 1 - (\lceil k/2 \rceil - 2)r_w$. Let $\kappa := \left(1 + \sqrt{2 - 2/\sqrt{5}}\right)$, by Lemma 9 the width of the last layer is at most $\kappa \cdot r_n$. We obtain

$$R \le \alpha(\varphi)r_w + 2(\ell - 1)r_w + \kappa r_n$$
$$\le \kappa + \left[\alpha(4\pi/k) + 2\log(k-1) - 4 + \kappa(2 - \lceil k/2 \rceil)\right]r_w.$$

A numerical analysis shows that $R < \kappa$ when $k \ge 7$. Thus in the two remaining cases ($k = 5, 6$) we apply Lemma 11 to enhance the result by moving B_n slightly inwards. In both cases, the last layer $\bar{L}_1$ contains only B_n, the last wedge layer is $\bar{L}_2$, and we have $r_n \le 1 - r_w$. The angle between two spokes in layer $\bar{L}_2$ is at least $2\pi/(2k-1)$, so we can use $\beta = 2\pi/(2k-1)$ in Lemma 11. In this way, we obtain

$$R \le \alpha(\varphi)r_w \cdot (\sin(\beta) + \cos(\beta))/(\sin(\beta) + 1) + \kappa r_n$$
$$\le \kappa + \left[\alpha(4\pi/k) \cdot (\sin(\beta) + \cos(\beta))/(\sin(\beta) + 1) - \kappa\right]r_w.$$

This is less than κ in both cases, $k = 5$ and $k = 6$.

Finally we have to consider the case when B_w is the smallest balloon in its layer. In this setting we have $r_n \le 1 - (k-3)r_w$ and $\varphi \ge 2\pi/(k - 1/2)$, which yields

$$R \le \alpha(\varphi)r_w + 2(k-1)r_w + \kappa \cdot r_n$$
$$\le \kappa + \left[\alpha(2\pi/(k-1/2)) + 2\log(k-1) - 4 - \kappa(k-3)\right]r_w.$$

We obtain, $R \le \kappa$, since $\alpha(2\pi/(k-1/2)) + 2\log(k-1) - 4 - \kappa(k-3) < 0$ for $k \ge 5$. □

4   Drawing unordered trees with perfect angles

The greedy strategy can be used to construct drawings of unordered trees with perfect angular resolution and small area. In fact, the balloon layout problem studied in Section 2 is a subproblem of the drawing algorithm of Duncan et al. [5], where it is used to draw "depth-1" trees. With the help of the so called heavy edge tree-decomposition (see Tarjan [13]) these trees are combined to the original tree. Since our proposed strategy uses significantly smaller area, it implies an improvement for the area of the tree drawing.

We start with a brief review of the heavy edge tree-decomposition. Let u be a non-leaf of the rooted tree T with root r. We denote by T_u the subtree of T rooted at u. Let v be the child of u such that T_v has the largest number of nodes (compared to the subtrees of the other children of u), breaking ties arbitrarily. We call the edge (u,v) a *heavy edge*, and the edges to the other children of u *light edges*. The heavy edges induce a decomposition of T into (maximal) paths, called *heavy paths*, and light edges; see Figure 9 on the left. We call the node on a heavy path that is closest to r its *top node*. The subtree induced by a heavy path is the subtree rooted at its top node. The light edge that links the top node with its parent in T is called the *light parent edge*. The *height* of a heavy path P is defined as follows: If P is not incident to light parent edges of other heavy paths it has height one. Otherwise we obtain the height of P by adding one to the maximal height of all heavy paths linked to P by some light parent edge (but not the light parent edge of P). By construction, every root-leaf path in T visits at most $\lceil \log n \rceil$ many light edges, and thus no path has height larger $\lceil \log n \rceil$.
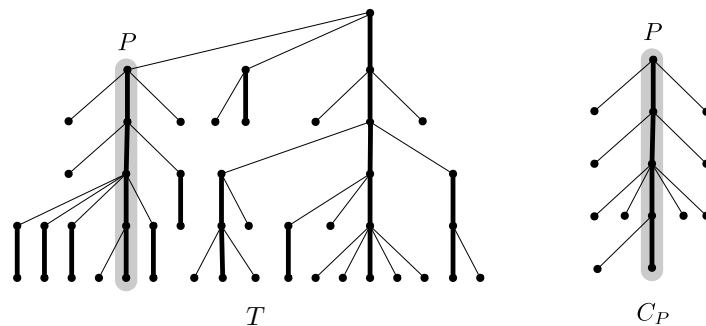

Figure 9: An example of a heavy-edge tree-decomposition. The path P has height two.

The drawing of the tree will be obtained by recursively combining drawings of subtrees. In particular, we construct drawings for all subtrees rooted at a heavy path's top node. For every such subtree S we define an *exclusive disk* X with the following properties: (1) The drawing of S is contained inside X, (2) the light parent edge of S crosses the boundary of X orthogonally, and (3) the center of X coincides with the top node of the corresponding heavy path.
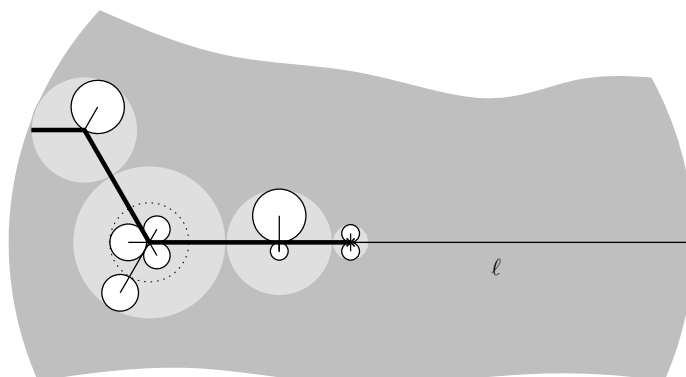
Figure 10: Constructing the drawing of C_P with light parent edge ℓ. Its exclusive disk is drawn darkly shaded. The white circles save the space for the subtrees rooted at the corresponding children. The lighter shaded circles depict the extend of the of the balloon layouts constructed for every non-leaf vertex of the heavy path.

Roughly speaking, we draw S by drawing the corresponding heavy path with its children, and for every children q we save an exclusive region, large enough such that subtree of T rooted at q can be inserted here. The drawing of such a subtree is depicted in Figure 10.

More formally, let P be a heavy path of T and let C_P be the union of P with its incident light edges (but without the light parent edge) as shown in Figure 9. The leaves of C_P represent (possibly degenerate) subtrees of heavy paths with smaller height. Assume we have constructed the drawings for all these subtrees and we constructed an exclusive disk for each of the drawings. Let u be a non-leaf tree node of P. We apply the balloon layout algorithm to draw u and its incident edges. We introduce a spoke for every light edge incident to u. The balloon that is placed on a spoke represents the exclusive disk of the corresponding subtree. The heavy edges incident to u are represented as free spokes. If u is the top node of P, we also add another free spoke (representing the parent edge) unless u is the root of T. In fact, the root of T is the only place for which we have one free spoke, otherwise we always have two free spokes.

More In order to link the balloon layouts for the nodes of P via its heavy edges (free spokes) we apply the technique of Duncan et al. [5, full version, Lemma 2.3]. This Lemma states that the combined drawing fits inside an exclusive disk of radius $2\sum_i x_i$, where x_i is the radius of the disk that covers the balloon layout of the i-th node on P. Figure 10 illustrates this construction. The base case in the recursion draws the leaves of T (degenerate heavy paths) with their incident light parent edges. This is done by drawing these light parent edges with length one and placing the exclusive disk centered at the leaf node with radius one. The following theorem presents a bound on the area of the constructed tree drawing.

Theorem 5 *Let $\kappa = (1 + \sqrt{2 - 2/\sqrt{5}})$ be the constant derived in Lemma 10.*

Using Algorithm 2 in the framework of Duncan et al. produces a drawing of an unordered tree with n nodes that has perfect angular resolution and that can be covered with a disk of radius $n^2 \cdot n^{\log \kappa} < n^{3.0367}$, while having no edge with length smaller than 1.

Proof. Let N_u denote the number of nodes in the subtree rooted at u. We show by induction that any subtree of a heavy path P of height i can be covered with a disk of radius less than $(2\kappa)^i N_u$, for u being the top node of P. This statement is certainly true for $i = 1$. Assume that we have already built the drawings of all height $(i - 1)$ heavy path subtrees. We apply the construction of the previous paragraph to combine the drawings. In order to achieve this we have to apply the greedy strategy for the balloon layout for every node of P. By Theorem 3 and 4, the balloon layout requires a covering disk of radius κ times the sum of the balloon radii. We denote the necessary radius of the covering disk at node z by x_z, and the number of nodes of the subtrees that are linked to z by a light parent edge by M_z. By the recursion hypothesis we have $x_z \le \kappa(2\kappa)^{(i-1)} M_z$. The construction of Duncan et al. [5, full version, Lemma 2.3] combines the balloon layouts of the nodes of P to a tree drawing with perfect angular resolution and the drawing fits inside an exclusive disk of radius at most $2 \sum_{z \in P} x_z = \sum_{z \in P} (2\kappa)^i M_z = (2\kappa)^i N_u$.

Since every root-leaf path in T traverses at most $\log n$ light edges, the height of the root of T is at most $\log n$. This shows that the radius of the covering disk is at most

$$(2\kappa)^{\log n} N_r = n^{\log 2\kappa} \cdot n = n^2 n^{\log \kappa} < n^{3.0367}.$$

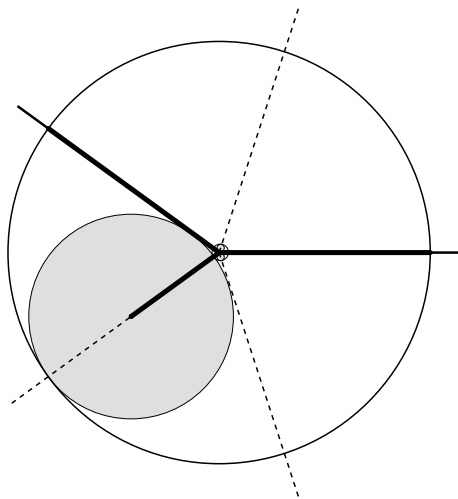Notice that by construction all edges have length at least one.   □

Figure 11: Three balloons with radius $\varepsilon, \varepsilon, 1 - 2\varepsilon$ and 5 spokes. Separating the unused spokes by an angle $\ge 2\pi/3$ yields a covering disk with radius $\alpha(2\pi/5) = \kappa$ when ε approaches zero.

5   Discussion

The only case, where we obtain no strict inequalities in the proof of Theorem 1, is when $|\mathcal{B}| = 1$. By placing all balloons slightly inside the wedges, resp., slightly outside the safe disks we can therefore modify all constructions such that no balloons touch.

Although our balloon layout strategy is the best possible in some scenarios, it would be interesting if one could obtain a denser layout for special cases. In particular, we think that better layouts exist if the fraction of the largest and the smallest radius is bounded. It would be interesting to find an algorithm that produces a layout which gives a guarantee on the area in terms of n and this ratio.

As a final remark we point out that Theorem 1 can be generalized such that it holds for one or two free spokes, while guaranteeing that the whole balloon drawing can be covered with a disk of radius 2. However, as depicted in Figure 11, the slightly worse bound of κ cannot be avoided if one has to guarantee that the smaller angle between the two unused spokes is at least $2\pi/3$. This requirement is however necessary to apply Lemma 2.3 of Duncan et al. [5, full version].

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