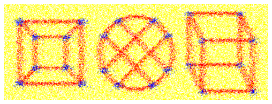
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Kernelization for Maximum Leaf Spanning Tree with Positive Vertex Weights

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Abstract

In this paper we consider a natural generalization of the well-known Max Leaf Spanning Tree problem. In the generalized Weighted Max Leaf problem we get as input an undirected connected graph G, a rational number k not smaller than 1 and a weight function $w : V \mapsto \mathbb{Q}_{\geq 1}$ on the vertices, and are asked whether a spanning tree T for G exists such that the combined weight of the leaves of T is at least k. We show that it is possible to transform an instance $\langle G, w, k\rangle$ of Weighted Max Leaf in polynomial time into an equivalent instance $\langle G', w', k'\rangle$ such that $|V(G')| \leq 5.5k$ and $k' \leq k$. In the context of parameterized complexity this means that Weighted Max Leaf admits a kernel with $5.5k$ vertices. The analysis of the kernel size is based on a new extremal result which shows that every graph $G = (V, E)$ that excludes some simple substructures always contains a spanning tree with at least $|V|/5.5$ leaves. We also prove that Weighted Max Leaf does not admit a polynomial-time factor $\mathcal{O}(n^{\frac{1}{2}-\varepsilon})$ or $\mathcal{O}(\text{OPT}^{\frac{1}{3}-\varepsilon})$ approximation algorithm for any $\varepsilon > 0$ unless P = NP.

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1 Introduction

The area of parameterized complexity theory was pioneered by Downey and Fellows [10] to cope with the "rock of intractability" of NP-complete problems. Much of the complexity theoretic work of the past decades has been spent proving that there is an abundance of natural and important problems that are NP-complete, and for which the existence of a polynomial-time algorithm therefore seems unlikely. Parameterized complexity is an approach to deal with the intractability of such problems; rather than trying to find a polynomial-time algorithm for some decision problem $L \subseteq \Sigma^*$, we look more carefully at problem instances and associate every instance with a *parameter* value k that describes the structure of the instance. We then try to confine the seemingly unavoidable exponential factor in the running time of an algorithm to some function that depends only on k. This leads to the natural definition of a *parameterized problem* as a set $Q \subseteq \Sigma^* \times \mathbb{N}$. For an instance $(x, k) \in Q$ of a parameterized problem we can think of x as the "classical" problem, whereas k is the new parameter that expresses some (structural) property of x. A natural choice of the parameter value is the desired *solution size*. Taking the Vertex Cover problem as an example, where we are asked whether a given graph has a vertex cover of a certain size, we could take the desired size k of the vertex cover as the parameter. Through a long series of successive improvements it was shown that the vertex cover problem can be decided in $\mathcal{O}(1.2738^k + kn)$ time [6]. This shows that if the vertex cover we are looking for is small, then the problem can be solved efficiently — even on very large graphs! One possible formalization of this concept is the notion of the complexity class (strongly uniform) FPT (for *Fixed Parameter Tractable*), that consists of all parameterized problems Q for which there is an algorithm that decides whether $(x, k) \in Q$ running in time $f(k)p(|x|)$, where f is a computable function and p a polynomial. Parameterized complexity theory thus serves as a tool to deal with the intractability of NP-complete problems by exploiting the structure that lots of real-world problems have. This brings about a shift in perspective from negative results (NP-completeness) to positive results (FPT algorithms). We argue that this also calls for a shift in the type of problems that should be considered.

When a problem is *intractable* it is interesting to study the restrictions under which it remains intractable; this yields fundamental information about the structure of the problem, and it might also lead to the conclusion that there are restricted yet practically relevant versions of the problem which are tractable. When dealing with problems that are *tractable* we can ask ourselves a similar question: which generalizations of the problem are still tractable? Since practical problems are highly complex, being able solve more general problems will often allow real-world problems to be modeled (and hence solved) more accurately. This style of research is popular in the community of polynomial-time approximation algorithms; many studies [8, 14, 17, 27] have been undertaken to see which generalizations of well-known problems are still "tractable" to approximate, i.e. can be approximated efficiently with good bounds on the error ratio. One important type of generalization that is often relevant for combina-

torial graph problems is to introduce weights for each vertex: instead of finding a subset of vertices of minimum (or maximum) cardinality that satisfies some criteria, we instead look for a subset of minimum (maximum) weight that satisfies the criteria. These weights can then be used to model costs or benefits in real-world applications. We suggest applying the practical techniques from parameterized complexity theory to such generalized problems.

A powerful technique in parameterized complexity theory is that of polynomial-time kernelization. The goal is to preprocess an instance (x, k) in time $p(|x| + k)$ for some polynomial p to obtain a *reduced* instance (x', k') that preserves the answer to the decision problem, such that $|x'|, k' \leq f(k)$ for a computable function f. A procedure that performs such preprocessing is called a *kernelization algorithm* (or *kernel*), and the function f is the *size* of the kernel. Thus kernelization can be seen as a form of preprocessing with a performance guarantee on the compression that is obtained with respect to the parameter value k. Kernelization algorithms are often valuable in practice because they can be combined with any other type of algorithm (either heuristic or exact in nature); since the kernelization step does not change the answer to the problem, it "never hurts" to start by first kernelizing the instance, and then using a heuristic approach or exact exponential-time algorithm on the reduced instance.

Given the practical importance of weighted problems and the practical relevance of kernelization algorithms, it is surprising to note that only few kernelization algorithms exist for weighted problems. The classic Vertex Cover kernelization by Buss can also be applied for Weighted Vertex Cover if all weights are at least 1, and generalizes to Weighted d-Hitting Set for fixed d. Chlebík and Chlebíková showed how the concept of crown reductions for Vertex Cover can be lifted to a weighted setting [7]. The Weighted Cluster Editing problem where each edge is given an integral weight has a $\mathcal{O}(k^2)$-vertex kernel as shown by Böcker et al. [2], which was recently improved to a kernel with $2k$ vertices by Cao and Chen [5]. Aside from these examples, no kernelization algorithms for weighted problems are known to us.

In this work we will study the fixed-parameter tractability of a generalization of the well-known Maximum Leaf Spanning Tree problem (abbreviated as Max Leaf from now on). In the Max Leaf problem we are given an undirected, connected graph G and an integer k, and are asked whether G has a spanning tree with at least k leaves. The problem was originally proven NP-complete by Garey and Johnson, even when restricted to planar graphs of maximum degree 4. Peter Lemke [23] showed several years later that the problem remains NP-complete when restricted to d-regular graphs for any $d \geq 3$. Max Leaf is APX-complete which means it has a polynomial-time constant-factor approximation algorithm [25], but no PTAS unless P = NP [15] — not even on cubic graphs [3]. The problem of finding a spanning tree with k leaves is equivalent to finding a connected dominating set with $|V| - k$ vertices, and these problems have many applications in circuit layout [26] and network design. The Max Leaf problem has been a popular topic of research from the parameterized complexity standpoint. When parameterized by the requested number of leaves k, it has been shown that the problem has a kernel with $3.75k$ ver-

tices [11] and that there is an algorithm to solve the problem in $\mathcal{O}(4^k k^2 + n^{\mathcal{O}(1)})$ time [22], which was later improved to $\mathcal{O}(3.4575^k \cdot n^{\mathcal{O}(1)})$ [24]. Using a different type of analysis this line of research also lead to a $\mathcal{O}^*(1.8966^n)$ algorithm for the classical (non-parameterized) problem [13]. Max Leaf has also been studied from the perspective of extremal graph theory [26, 16, 21, 4]. The related problem on directed graphs is called Maximum Leaf Out-Branching [22], and is also very interesting from a kernelization point of view because it has been shown that the rooted variant admits a kernel with $\mathcal{O}(k^2)$ edges and vertices [9], but the unrooted variant does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly [12]. This paper focuses on the following natural generalization of the classical problem:

> Weighted Max Leaf
> **Instance:** An undirected connected graph $G = (V, E)$; a weight function $w : V \mapsto \mathbb{Q}_{\geq 1}$ on the vertices; a rational number $k \geq 1$.
> **Parameter:** The value k.
> **Question:** Does G have a spanning tree with leaf set L such that $\sum_{v \in L} w(v) \geq k$?

Observe that this definition requires vertex weights to be rational numbers not smaller than 1. There is a good motivation for this restriction; when vertex weights are allowed to be arbitrarily small fractions then the problem is NP-complete for $k = 1$, since an unweighted graph G has a spanning tree with k leaves if and only if that same graph has a spanning tree with leaf weight 1 if we set all vertex weights to $1/k$. If the weight 0 is allowed then it was shown in the author's Master's thesis [19] that the resulting problem is hard for $W[1]$, through a reduction from Independent Set. Therefore we focus on weights that are at least 1. This may still lead to small (and hence practical) values for the parameter value k since the relative weight differences between vertices may be small, thus yielding a small parameter value for the overall target weight. For technical reasons we assume that each value of the weight function can be encoded as an integer plus a fractional part consisting of a constant number of decimal places. The reason for this assumption will be made clear in Section 4.4.

Our Contribution

The main result of this paper is that the Weighted Max Leaf problem has a kernel with $5.5k$ vertices when every weight is a rational number not smaller than 1. The kernelization is achieved by a small set of simple reduction rules.The reduction rules make non-trivial use of the vertex weights, thus giving an example of how kernelization can be applied to weighted problems. The existing $3.75k$ kernelization by Estivill-Castro et al. [11] does not work in the weighted case, because it relies on the fact that two adjacent degree-2 vertices can always be leaves in an optimal spanning tree if the edge between them is not a bridge. Since this no longer holds in the weighted variant of the problem, we have to devise new reduction rules.

We also show that the multiplicative constant of 5.5 in the kernel size is best-possible with respect to the given set of reduction rules, which means that our analysis of the size of the reduced instances is tight. This analysis relies on a new result in the style of extremal graph theory: we give a constructive proof that every connected undirected graph $G = (V, E)$ that avoids some simple subgraphs (see Definition 1) has a spanning tree with at least $|V|/5.5$ leaves. To prove this result we extend the technique of "amortized analysis by keeping track of dead leaves", which was originally used by Griggs et al. [16] to show that every connected cubic graph G has a spanning tree with at least $\lceil |V|/4+2 \rceil$ leaves.

In a brief excursion to approximation theory we exploit the relationship between the optimization version of Weighted Max Leaf and Independent Set to prove that even when the weights are bounded polynomially in the size of the input graph, Weighted Max Leaf does not have a polynomial-time multiplicative factor $\mathcal{O}(n^{\frac{1}{2}-\varepsilon})$-approximation algorithm or $\mathcal{O}(\text{OPT}^{\frac{1}{3}-\varepsilon})$-approximation algorithm for any $\varepsilon > 0$ unless P = NP.

Organization

We give some preliminaries in Section 2. In Section 3 we obtain a structural result on the existence of spanning trees with many leaves in graphs that avoid some simple subgraphs. Section 4 uses this structural result to present the kernelization algorithm. We consider the optimization version of Weighted Max Leaf in Section 5, and prove that the problem is very hard to approximate.

2 Preliminaries

An undirected graph G is a pair (V, E) where V is the set of vertices and the edge set E is a collection of 2-element subsets of V. We also use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. We only consider simple, undirected, connected graphs. An edge between vertices u and v is denoted as uv. As all graphs are undirected, this is the same object as the edge vu. For $v \in V$ we denote the *open* neighborhood of v by $N_G(v)$ and the *closed* neighborhood by $N_G[v] := N_G(v) \cup \{v\}$. Throughout this work we omit subscripts if this does not lead to confusion. The neighborhood of a set $S \subseteq V$ is $N_G(S) := \cup_{v \in S} N_G(v) \setminus S$. The degree of a vertex v in graph G is denoted by $\deg_G(v)$. We write $G' \subseteq G$ if G' is a subgraph of G. For $X \subseteq V$ we denote by $G[X]$ the subgraph of G that is induced by the vertices in X. Noting that V is the vertex set of G we abbreviate the construction $G[V \setminus X]$ by $G - X$. A *cutset* for a connected graph G is a set $S \subseteq V$ such that $G - S$ is not connected. Vertex v is a *cut vertex* if $\{v\}$ is a cutset.

If T is a tree subgraph of G and e is an edge with $e \in E(G)$ and $e \notin E(T)$, then we say that T *avoids* edge e. If $e \in E(G)$ and $e \in E(T)$ then tree T *uses* edge e. If $T \subseteq G$ is a tree with $V(T) = V(G)$ then T is a *spanning tree* for G. A vertex of degree at most 1 is called a *leaf*. If v is a vertex in a tree and v is not

a leaf, then it is an *internal node* of the tree. The *leaf set* of a graph $G = (V, E)$ is the set of vertices with degree at most 1, denoted as $\textsc{Leaves}(G) := \{v \in V \mid \deg_G(v) \leq 1\}$. If we have a weight function $w : V \mapsto \mathbb{Q}_{\geq 1}$ for graph G then we can define its *leaf weight* as $\textsc{lw}_w(G) := \sum_{v \in \textsc{Leaves}(G)} w(v)$.

A *path component* in a graph G is a path P on vertices $\langle u, s_1, s_2, \ldots, s_q, v\rangle$ such that successive vertices are connected by an edge, all the vertices s_i are distinct and have degree 2, and such that $\deg(u), \deg(v) \neq 2$. Note that we explicitly allow u and v to be the same vertex. The vertices u, v are called the *endpoints* of the path component. The vertices s_i are the *inner vertices* of the path component. We define the *size of a path component* to be equal to the number q of inner vertices.

To simplify the exposition we use K_n to refer to the complete graph on n vertices. The class of d-degenerate graphs consists of all graphs for which every vertex-induced subgraph has a vertex of degree at most d. The set of rational numbers not smaller than 1 is denoted by $\mathbb{Q}_{\geq 1}$. We use a simple folklore lemma regarding spanning trees that will simplify the proofs that follow later. We omit the straight-forward proof, which can be found in the technical report [20].

Lemma 1 *If $S \subseteq V$ forms a cutset for graph $G = (V, E)$ then there is no spanning tree $T \subseteq G$ in which all vertices in S are leaves.*

3 Spanning Trees with Many Leaves in Graphs Without Long Path Components

In this section we prove a lower bound on the number of leaves that can be obtained in spanning trees for graphs that do not contain long path components and which avoid some simple subgraphs.

Definition 1 *Let $\mathcal{C}$ be the class of graphs $G = (V, E)$ that satisfy the following properties:*

- *(i) Graph G is simple and connected.*
- *(ii) The graph is not isomorphic to a simple cycle.*
- *(iii) The maximum size of a path component in G is at most 3.*
- *(iv) Every vertex $v \in V$ with $\deg_G(v) = 1$ is adjacent to a vertex of degree at least 3.*
- *(v) If G contains a triangle on three vertices x, y, z as a subgraph ($\{xy, xz, yz\} \subseteq E$), then at least one of the vertices x, y, z has a degree in G of at least 4.*
- *(vi) If x, y are two distinct degree-2 vertices then $N_G(x) \neq N_G(y)$.*

Theorem 1 *Every graph $G = (V, E) \in \mathcal{C}$ has a spanning tree with at least $|V|/5.5$ leaves.*

The remainder of Section 3 is devoted to the proof of Theorem 1. Consider $G = (V, E) \in \mathcal{C}$. As the theorem obviously holds for K_1, and since connected graphs different from K_1 with fewer than 3 vertices do not satisfy Property (iv),

we may assume in the remainder that $|V| \geq 3$. Our proof uses the method of "amortized analysis by keeping track of dead leaves", as introduced by Griggs et al. [16]. The proof is constructive and consists of a series of operations that can be used to initialize a tree subgraph $T \subseteq G$, and to grow T into a spanning tree step by step. We will prove that the resulting tree T has sufficiently many leaves by showing for every augmentation step that the increase in the total size of the tree is balanced against the increase in the number of leaves of the tree. To analyze the number of leaves in the resulting spanning tree we use the notion of *dead leaves*. A leaf $v \in \textsc{Leaves}(T)$ is called *dead* if all its neighbors in G are also in T. More formally, leaf v is dead if $N_G(v) \subseteq V(T)$. Every leaf vertex that is not dead, is alive. We define the following abbreviation for the set of live leaves:

$$\underset{G}{\textsc{LiveLeaves}}(T) := \{v \in \textsc{Leaves}(T) \mid N_G(v) \setminus V(T) \neq \emptyset\}. \tag{1}$$

If vertex $v \in V$ has a neighbor $u \in N_G(v)$ but the vertex u is not in T, then we say that v has a neighbor u outside the tree. If $u \in N_G(v)$ and $u \in V(T)$ then vertex v has a neighbor u inside the tree. A vertex $x \in N_G(V(T))$ is said to be *adjacent* to the tree T. When referring to the degree of a vertex v in the tree $T \subseteq G$ under construction — as will be done in the next definition — we always mean its degree $\deg_G(v)$ in the graph G, *not* its degree in the subgraph T. The following concept will be used in our amortized analysis.

Definition 2 *For a tree $T \subseteq G$, we say that a vertex v is a* split vertex *if v is a live leaf of degree 2 (i.e. $\deg_G(v) = 2$), and $N_G(v) \setminus V(T) = \{u\}$ for some vertex u with $\deg_G(u) = 2$. We denote the set of all split vertices of T with respect to G by $\textsc{Split}_G(T)$.*

Our analysis uses the following properties of the tree T that is under construction:

- L, the number of *leaves* of T,
- D, number of *dead* leaves of T,
- S, the number of *split vertices* of T,
- N, the total number of vertices of T.

We will give a series of augmentation operations that satisfy the following *incremental inequality*:

$$4\Delta L + 1.5\Delta D + \Delta S \geq \Delta N. \tag{2}$$

The Δ values in this inequality represent the changes in the respective quantities in the new tree compared to the old tree. For example, if the tree had 5 leaves before the augmentation operation and it has 7 leaves after the operation then $\Delta L = 7 - 5 = 2$. The following proposition shows how this inequality will be useful in proving the theorem.

Proposition 1 *If T is a spanning tree of $G = (V, E)$ that is built from an empty tree by successive augmentation operations that respect the incremental inequality, then $|\textsc{Leaves}(T)| \geq |V|/5.5$.*

Proof: Observe that in a spanning tree there can be no live leaves: all neighbors to all vertices must be in the tree. This means that all leaves must be dead and hence $L = D$. Since there are no live leaves in a spanning tree we also have $S = 0$ for such trees. It follows that if we grow a spanning tree such that every augmentation step satisfies the incremental inequality, then by summing the inequalities over the individual augmentation steps we find that the final tree satisfies $4L + 1.5D + S \geq N$; using the fact that $L = D$ and $S = 0$ on spanning trees we can then conclude that $|\textsc{Leaves}(T)| \geq |V|/5.5$ for a tree T that is grown by operations that respect the incremental inequality. □

So to prove Theorem 1 all that remains is to show that there exists a set of augmentation operations that can grow a spanning tree while respecting the incremental inequality.

Tree Initialization The initialization of the tree T is simple: we just pick a vertex v of maximum degree in G as the root of the tree, and we add edges to all neighbors of v to the tree. So after initialization we have a tree with one internal vertex v and leaf set $\textsc{Leaves}(T) = N_G(v)$. For this operation we have $\Delta N = 1 + |N_G(v)|$ since the tree is a star rooted at v, and $\Delta L = |N_G(v)|$ because all neighbors of v have become leaves. The values ΔD and ΔS cannot be negative because there were no leaves before the tree was initialized, so no dead leaves or split vertices can be lost. With this information it can easily be seen that the initialization of the tree satisfies the incremental inequality.

Extending the Tree We need the following definition to describe the operations that augment the tree.

Definition 3 (Vertex Expansion) *Let $T \subseteq G$ be a tree. The* expansion *of a vertex $v \in V(T)$ yields an augmented tree $T' \subseteq G$, where T' is obtained by adding all edges $vu \in E(G)$ with $u \in N_G(v) \setminus V(T)$ to the tree T.*

Figure 1 shows an example of vertex expansions. It follows from the definition that the expansion of a vertex can never decrease the number of leaves in the tree. If v has a neighbor $u \in N_G(v) \setminus V(T)$, then expansion of v will cause v to become internal (decreasing the number of leaves), but it will also cause u to become a new leaf. If v has no neighbors in G that are outside T, then the expansion has no effect. The operations that we will present to augment the tree consist of series of vertex expansions. This means that $\Delta L \geq 0$ for every augmentation step. Since the expansion of a vertex cannot change the fact that a leaf is dead, we also have $\Delta D \geq 0$ for all our augmentation operations. By growing the tree through expansion operations we will also maintain the invariant that for every internal vertex of T, all its neighbors in G are also in T. In other words, we will grow an *inner-maximal* spanning tree. This implies that whenever a vertex $v \in V(G) \setminus V(T)$ is adjacent to some $u \in V(T)$, then u must be a leaf of T. We will use this invariant of the tree construction process in the correctness proofs of the augmentation operations. To simplify the bookkeeping we introduce the following concept.

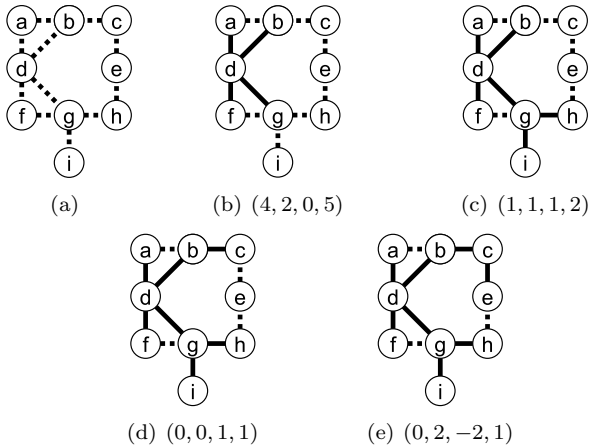


Figure 1: This sequence shows the effect of successive vertex expansions. Dotted lines indicate edges in $E(G) \setminus E(T)$, and solid lines represent edges in $E(G) \cap E(T)$. Each expansion is labeled with the corresponding vector $(\Delta L, \Delta D, \Delta S, \Delta N)$ of changes in the measured quantities. 1(a): the graph G without a tree subgraph. 1(b): initialized T as the star around vertex d, causing a, f to become *dead* leaves and b, g to become *live* leaves. 1(c): expanded g, adding i, h to T. Vertex h is now a *split* vertex. 1(d): expanded b, causing c to become a split vertex as well. 1(e): expanded c, which causes vertex h to transform from a split vertex into a dead leaf. $C_{T \to T'} = \{h\}$ for this step.

Definition 4 *Let $T \subseteq G$ be a tree subgraph of G, and let $T' \subseteq G$ be the tree that is obtained from T by a sequence of vertex expansions. We define the set $C_{T \to T'}$ of* converted split vertices *as the set of vertices that are split vertices in T, but dead leaves in T':*

$$C_{T \to T'} := \left\{ v \in V(G) \mid v \in \underset{G}{\textsc{Split}}(T) \wedge v \in \underset{G}{\textsc{Leaves}}(T') \wedge N_G(v) \setminus V(T') = \emptyset \right\}.$$

Lemma 2 *Let $T \subseteq G$ be a tree subgraph of G, and let T' be the tree that results after the successive expansion of vertices $x_1, \ldots, x_k$. Let $c := |\textsc{Split}_G(T) \cap \{x_1, \ldots, x_k\}|$. For this operation it holds that $\Delta S \geq -|C_{T \to T'}| - c$. All vertices in $C_{T \to T'}$ are live leaves in T and dead leaves in T', implying $\Delta D \geq |C_{T \to T'}|$.*

Proof: Assume the conditions stated in the lemma hold. By definition, all vertices in $C_{T \to T'}$ are split vertices of T and therefore live leaves. The expansion of a vertex cannot change the status of a dead leaf, since all its neighbors were already in the tree before the expansion. Therefore all the vertices that are dead leaves in T, must still be dead leaves in T'. Additionally we have by the definition of $C_{T \to T'}$ that all vertices in $C_{T \to T'}$ were not dead leaves in T, but have become dead leaves in T' which proves $\Delta D \geq |C_{T \to T'}|$.

Now we will prove the lower bound on ΔS. Consider some split vertex $v \in \textsc{Split}_G(T)$ that is not expanded, i.e. $v \notin \{x_1, \ldots, x_k\}$. We will argue that

if $v \notin \textsc{Split}_G(T')$ then v is a dead leaf in T'. So assume that $v \notin \textsc{Split}_G(T')$. By the definition of a split vertex we have $N_G(v) \setminus V(T) = \{x\}$ for some vertex x with $\deg_G(x) = 2$. Every split vertex is a live leaf, and a leaf of the tree can only become internal by expanding that leaf. Since $v \notin \{x_1, \ldots, x_k\}$ we did not expand v when building T'. Therefore the vertex v must still be a leaf in T'. From the definition of a split vertex we can now deduce that the only way for v to cease being a split vertex is if it has in fact become a dead leaf, because its neighbor x was added to T' by the expansion of $x_1, \ldots, x_k$. This shows that all split vertices of T that are not expanded and which are no longer split vertices in T', must have become dead leaves and are therefore in the set $C_{T \to T'}$ of converted split vertices. Since we expand c vertices that are split vertices in T, we lose at most $|C_{T \to T'}| + c$ split vertices by the expansions from which the bound $\Delta S \geq -|C_{T \to T'}| - c$ follows. □

Order of Augmentation Operations We are now ready to present the augmentation operations. When describing an augmentation, we will always assume that no earlier augmentation is applicable. This is important because for some rules their correctness depends on the fact that certain situations are reduced by earlier augmentations. In our description of the operations we use T and T' to denote the tree before and after the augmentation, respectively.

Augmentation Operation 1 *If there is a live leaf $v \in \textsc{LiveLeaves}_G(T)$ with at least 2 neighbors outside of T, then expand v.*

Lemma 3 *Augmentation operation 1 satisfies the incremental inequality.*

Proof: Assume the preconditions for the augmentation hold for a tree $T \subseteq G$, and let T' be the tree after the augmentation. Any tree T is initialized to contain at least two vertices, hence every vertex in T has at least one neighbor inside T. This implies that if v has at least 2 neighbors outside T, then the degree of v is at least three and therefore it is not a split vertex. All neighbors of v that are not in T will have become leaves in T'. The vertex v itself is internal in T'. We now find $\Delta L = |N_G(v) \setminus V(T)| - 1 \geq 1$ and $\Delta N = |N_G(v) \setminus V(T)|$. Since we do not expand any split vertices in this operation, we find by Lemma 2 that $\Delta D \geq |C_{T \to T'}|$ and $\Delta S \geq -|C_{T \to T'}|$, which implies that this operation satisfies the incremental inequality. □

Observation 1 *If augmentation operation 1 is not applicable, every live leaf $v \in \textsc{LiveLeaves}_G(T)$ has exactly one neighbor outside T.*

Augmentation Operation 2 *If there is a simple path $P = \langle x, y \rangle$ in G such that $\{x, y\} \cap V(T) = \{x\}$ and $|N_G(y) \setminus V(T)| \geq 2$ then expand x and afterwards expand y.*

Lemma 4 *Augmentation operation 2 satisfies the incremental inequality.*

Proof: Assume the preconditions for the augmentation hold for a tree $T \subseteq G$, and let T' be the tree after the augmentation. By Observation 1 we may assume that x has y as its unique neighbor outside T. We find that $\Delta N = |N_G(y) \setminus V(T)| + 1$ and $\Delta L = |N_G(y) \setminus V(T)| - 1$. Since y must have a degree of at least 3 we find that x and y cannot be split vertices in T, and therefore $\Delta D \geq |C_{T \to T'}|$ and $\Delta S \geq -|C_{T \to T'}|$ by Lemma 2. This combination satisfies the incremental inequality since $|N_G(y) \setminus V(T)| \geq 2$. □

Augmentation Operation 3 *If there is a vertex $v \in N_G(V(T))$ with at least 2 neighbors x, y inside T, then expand the neighbor x.*

Lemma 5 *Augmentation operation 3 satisfies the incremental inequality.*

Proof: Assume the preconditions for the augmentation hold for a tree $T \subseteq G$, and let T' be the tree after the augmentation. As the constructed tree T is inner-maximal, the neighbors x and y of $v \notin V(T)$ are leaves of T, and must therefore be live leaves. By Observation 1 both x and y have v as their unique neighbor outside T. Expansion of x causes x to become internal in T' (decreasing the number of leaves by one), but this is compensated by v becoming a leaf in T' which implies that $\Delta L = 0$ and $\Delta N = 1$. As y's unique neighbor outside T is added to T by the expansion of x, the expansion turns y into a dead leaf. To determine the values of ΔD and ΔS, we consider the local situation.

1. If $\deg_G(v) \geq 3$ then none of v's neighbors can be split vertices by Definition 2, and hence no split vertices can be involved in the expansion; we have $\Delta S = 0$ and $\Delta D \geq 1$, which implies that this operation satisfies the incremental inequality.
2. If $\deg_G(v) = 2$ then x and y can be split vertices, but since no other split vertices are lost we have $\Delta S \geq -2$. Observe that v only has the vertices x, y as its neighbors, which are both in T and hence in T'. Therefore v is a dead leaf in T'. Using the fact that y is also a dead leaf in T' we find $\Delta D \geq 2$; this also satisfies the incremental inequality.

Since we assumed that v has at least 2 neighbors inside T we know that $\deg_G(v) \geq 2$ and hence the case analysis above is exhaustive. □

Observation 2 *If Operations 2 and 3 are not applicable, then no vertex $v \notin V(T)$ with $\deg_G(v) \geq 3$ is adjacent to T (since such a vertex would have two neighbors inside T, two neighbors outside T, or both), and all vertices outside T have at most one neighbor in T.*

Augmentation Operation 4 *If there is a vertex $v \in \textsc{LiveLeaves}_G(T)$ that has a degree-1 neighbor u outside of T, then expand v.*

Lemma 6 *Augmentation operation 4 satisfies the incremental inequality.*

Proof: Assume the preconditions for the augmentation hold for a tree $T \subseteq G$, and let T' be the tree after the augmentation. Since v has a degree-1 neighbor

outside of T, it is not a split vertex. By Observation 1 we may assume that $N_G(v) \setminus V(T) = \{u\}$. Since u has a degree of 1 it will be a dead leaf in T'. Therefore we find that $\Delta L = \Delta S = 0$ and $\Delta D = \Delta N = 1$ which satisfies the incremental inequality. □

Augmentation Operation 5 *If there is a simple path $P = \langle x, y, z\rangle$ in G such that $\{x, y, z\} \cap V(T) = \{x\}$, $\deg_G(x) \geq 3$ and $\deg_G(y) = \deg_G(z) = 2$, then expand vertex x.*

Lemma 7 *Augmentation operation 5 satisfies the incremental inequality.*

Proof: Assume the preconditions for the augmentation hold for a tree $T \subseteq G$, and let T' be the tree after the augmentation. By Observation 1 the vertex x has y as its unique neighbor outside T. Vertex y is a leaf in T', and in fact must be a split vertex in T' since it has a single neighbor z outside of T' and $\deg_G(y) = \deg_G(z) = 2$. Since vertex x has a degree of at least 3, it is not a split vertex by definition. We find that $\Delta L = \Delta D = 0$ and $\Delta S = \Delta N = 1$. □

Augmentation Operation 6 *If there is a simple path $P = \langle x, y, z\rangle$ in G such that $\{x, y, z\} \cap V(T) = \{x\}$, vertex x is not a split vertex and $\deg_G(z) \geq 3$, then expand vertices x, y and then z.*

Lemma 8 *Augmentation operation 6 satisfies the incremental inequality.*

Proof: Assume the preconditions for the augmentation hold for a tree $T \subseteq G$, and let T' be the tree after the augmentation. By Observation 2 the vertex z is not adjacent to T, and the degree of vertex y must be 2. Therefore all neighbors of z except y will be leaves in T', resulting in $|N_G(z) \setminus \{y\}| = |N_G[z]| - 2$ new leaves. As vertex x is lost as a leaf, the net change is $\Delta L = |N_G[z]| - 3$. Since x is no split vertex by assumption, we find by Lemma 2 that $\Delta D \geq |C_{T \to T'}|$ and $\Delta S \geq -|C_{T \to T'}|$. For the remaining variable we have $\Delta N = |N_G[z]|$ which satisfies the incremental inequality since $|N_G[z]| \geq 4$. □

Lemma 9 *If augmentation operations 1-6 cannot be applied, then every live leaf $v \in \textsc{LiveLeaves}_G(T)$ is a split vertex.*

Proof: Proof by contradiction. Assume there is a live leaf $v \in \textsc{LiveLeaves}_G(T)$ that is not a split vertex, and that none of the presented augmentation operations can be applied. By Observation 1 we know that v has a unique neighbor u outside T, and by Observation 2 we know that u has exactly one neighbor in T and $\deg_G(u) \leq 2$. If $\deg_G(u) = 1$ then Operation 4 is applicable; since this contradicts our assumptions, we find that $\deg_G(u) = 2$. Let $\{w\} = N_G(u) \setminus \{v\}$, and observe that $w \notin V(T)$ by Observation 2. Using the assumption that v is not a split vertex we can conclude (by the definition of a split vertex) that $\deg_G(v) \neq 2$. Since v must have at least one neighbor inside T and one neighbor outside of T (because trees are always initialized to contain at least two vertices, and v is a live leaf) we know that the degree of v must be at least 3. We now consider the situation of w.

1. If $\deg_G(w) = 2$ then Operation 5 is applicable to the path $\langle v, u, w\rangle$.
2. If $\deg_G(w) \geq 3$ then by Observation 2 the vertex w is not adjacent to T. Since v is not a split vertex by assumption, we now find that Operation 6 is applicable to the path $\langle v, u, w\rangle$.

All possibilities lead to the conclusion that an augmentation operation is applicable, which contradicts our assumptions and finishes the proof. □

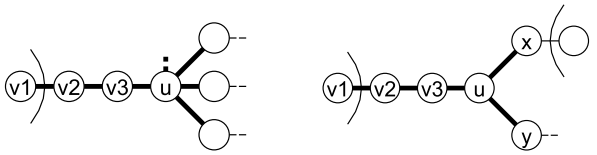
When none of the earlier operations are applicable, the boundary of the tree has a very specific structure. We will use the following lemma to capture this structure before introducing the remaining augmentation operations.

Lemma 10 *Assume none of the earlier operations are applicable and consider a live leaf $v_1 \in \text{LIVELEAVES}_G(T)$, which must be a split vertex by Lemma 9. Let $N_G(v_1) \setminus V(T) = \{v_2\}$. By the definition of a split vertex we know that $\deg_G(v_2) = 2$. Consider the maximal path in G formed by $P = \langle v_1, v_2, \ldots, v_q\rangle$ where $\deg_G(v_i) = 2$ for all $1 \leq i \leq q$. Let $\{u\} = N_G(v_q) \setminus \{v_{q-1}\}$. The following must hold:*

1. $q \leq 3$,
2. $v_q \notin V(T)$,
3. $\deg_G(u) \geq 3$,
4. $u \notin V(T)$.

Proof: Assume the conditions in the lemma hold. The size of a path component in G is bounded by 3 by Property (iii) of Definition 1; this establishes (1) since the v_i form a path component. If $q = 2$ then $v_q \notin V(T)$ follows from the definition of v_2. If $q = 3$ and $v_q = v_3 \in V(T)$ then the vertex v_2 has two neighbors $v_1, v_3 \in V(T)$ which implies that Operation 3 must be applicable — a contradiction. Together these statements prove (2). Since the degree of u must be unequal to 2 (by definition of the vertices v_i as a maximal path of degree-2 vertices) and since no degree-2 vertex is adjacent to a degree-1 vertex (by Property (iv) of Definition 1) we obtain (3). Because u has degree at least 3 we have by Definition 1 that u cannot be a split vertex and therefore by Lemma 9 it is not a leaf of T. Since $v_q \notin V(T)$ the vertex u cannot be an internal vertex of T since a vertex can only become internal by expanding it, and the expansion of u would have added v_q to T. Because u is not a leaf of T and not internal to T, it cannot be in T at all; this establishes (4). □

Augmentation Operations 7-26 We resume the description of the augmentation operations using the structural information of Lemma 10. The augmentation that we perform depends on the local situation of the vertex u. Since the various structural possibilities give rise to a multitude of different augmentation operations, we will not describe each of these operations individually in text; rather we will present these operations graphically to keep the proof as intuitive as possible. The remaining 20 augmentations are described by the illustrations in Figures2–5: each subfigure corresponds to an augmentation operation. For these augmentations we do not give explicit proofs that they satisfy the incremental inequality, but instead we give bounds on the Δ values in the tables



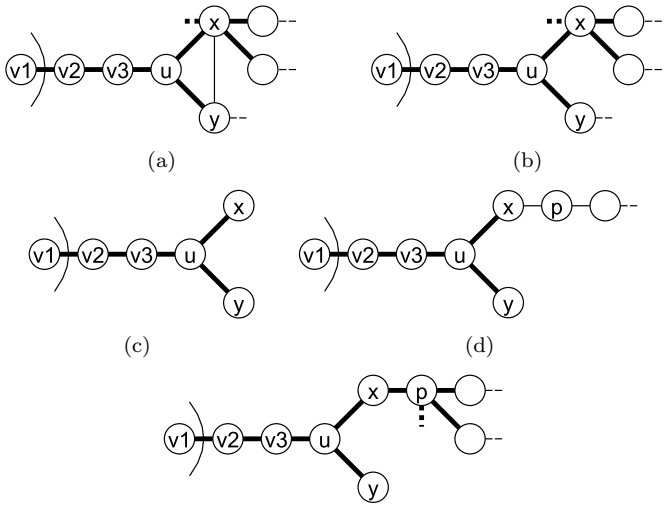
(a) (b)

Case	$\Delta N =$	$\Delta L =$	$\Delta S \geq$	$\Delta D \geq$
2(a)	$\deg_G(u) + 2$	$\deg_G(u) - 2$	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert$
2(b)	5	1	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert + 1 \geq 2$

Figure 2: This figure gives graphical representations of the augmentation operations after number 6. Each subfigure represents the structure of the boundary of the tree to which its augmentation is applicable, i.e. each subfigure shows a substructure of G and the tree T near that substructure. All augmentations start from the structure described in Lemma 10. Vertices of G that are also in T are drawn to the inside of an arc; the remaining vertices are in G, but not yet in T. Vertices incident to a dotted edge may have an arbitrary number of neighbors besides the vertices that are shown in the structure. The degrees of vertices without incident dotted edges should match their degree in the image exactly. The augmentation operation is visualized by drawing all edges that are added to T by the augmentation as thick lines. Edges of the graph that are not added to the tree by the augmentation are drawn as thin lines. This implies that all vertices that are incident to at least two thick edges were expanded during the tree augmentation. The table gives bounds on the Δ values for the relevant variables. The ΔN and ΔL values follow directly from the structure represented in the image. The bounds on ΔS and ΔD often rely on Lemma 2 for the split vertices in T that are transformed into dead leaves by the expansions, noting that each augmentation expands exactly one split vertex (v_1).

below the figures; these bounds imply that the operations satisfy the inequality. In the presentation of these operations we will assume that $q = 3$. The case $q = 2$ can be handled in the same structural way, except that the increase in N is not as high as for $q = 3$. Therefore the situations of $q = 2$ satisfy the incremental inequality whenever the $q = 3$ case does. In the case analysis of the remaining situations we start from the structure of the boundary that is described in Lemma 10 and refine this structure step by step. We first deal with some easy cases.

- By the previous Lemma we have $u \notin V(T)$ and $\deg_G(u) \geq 3$, which implies $u \notin N_G(V(T))$ by Observation 2. If $\deg_G(u) \geq 4$ then expand as in Figure 2(a). Otherwise we can assume that $\deg_G(u) = 3$ in the remainder of the situations.
- Assuming $\deg_G(u) = 3$, let $\{x, y\} = N_G(u) \setminus \{v_q\}$. If one of x, y is adjacent to T then assume w.l.o.g. that this is x, and expand as in Figure 2(b); in the remainder we assume that x, y are not adjacent to T, and we may also assume that $\deg_G(x) \geq \deg_G(y)$.



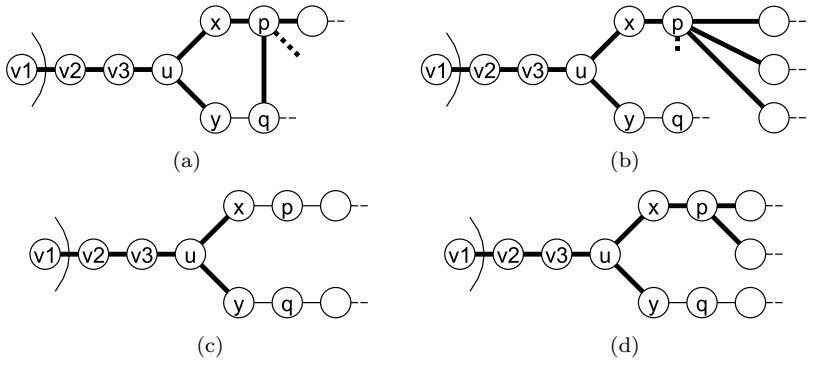
Case	$\Delta N =$	$\Delta L =$	$\Delta S \geq$	$\Delta D \geq$
3(a)	$\deg_G(x) + 3$	$\deg_G(x) - 2$	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert$
3(b)	$\deg_G(x) + 4$	$\deg_G(x) - 1$	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert$
3(c)	5	1	-1	2
3(d)	5	1	0	1
3(e)	$\deg_G(p) + 5$	$\deg_G(p) - 1$	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert + 1$

Figure 3: Tree augmentations. Refer to Figure 2 for the semantics of the images.

We now proceed with a more careful case analysis.

1. If $\deg_G(x) \geq 3$:
   (a) If edge $xy \in E(G)$ then the vertices u, x, y form a triangle in G. By Property (v) of Definition 1, one of the vertices u, x, y must have a degree of at least 4. Since $\deg_G(u) = 3$ by the case distinctions made so far, and because of our assumption that $\deg_G(x) \geq \deg_G(y)$ we find that $\deg_G(x) \geq 4$. We proceed as in Figure 3(a).
   (b) If edge $xy \notin E(G)$ then proceed as in Figure 3(b).
2. If $\deg_G(y) = 1$, then we look at the degree of x. By the existence of the above case we may assume $\deg_G(x) \leq 2$.
   (a) If $\deg_G(x) = 1$ then proceed as in Figure 3(c).
   (b) If $\deg_G(x) = 2$ then let $N_G(x) \setminus \{u\} = \{p\}$. We consider the status of p. Observe that $\deg_G(p) > 1$ by Property (iv) of Definition 1.
      i. If $\deg_G(p) = 2$ then proceed as in Figure 3(d); note that vertex x is a split vertex after the augmentation.
      ii. If $\deg_G(p) \geq 3$ then vertex p is not adjacent to T by Observation 2. Proceed as in Figure 3(e).

Since we assumed $\deg_G(x) \geq \deg_G(y)$ and we handled all cases with $\deg_G(y) = 1$ and $\deg_G(x) \geq 3$, we may assume in the remainder that $\deg_G(x) = \deg_G(y) = 2$.



Case	$\Delta N =$	$\Delta L =$	$\Delta S \geq$	$\Delta D \geq$
4(a)	$\deg_G(p) + 5$	$\deg_G(p) - 1$	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert + 1$
4(b)	$\deg_G(p) + 5$	$\deg_G(p) - 1$	$-\lvert C_{T \to T'} \rvert - 1$	$\lvert C_{T \to T'} \rvert$
4(c)	5	1	1	0
4(d)	8	2	$-\lvert C_{T \to T'} \rvert$	$\lvert C_{T \to T'} \rvert$

Figure 4: Tree augmentations. Refer to Figure 2 for the semantics of the images.

By Property (vi) we know that $N_G(x) \neq N_G(y)$ and hence that x must have some neighbor $p \neq u$, and y must have some neighbor $q \neq u$, such that $p \neq q$. Assume without loss of generality that $\deg_G(p) \geq \deg_G(q)$. We will find that the case $\deg_G(p) = \deg_G(q) = 3$ is the most complex; therefore we will first deal with the remaining cases, finishing with $\deg_G(p) = \deg_G(q) = 3$ at the end of the proof. Note that $\deg_G(p) \geq \deg_G(q) \geq 2$, since no degree-1 vertex can be adjacent to a degree-2 vertex by Property (iv).

1. If $pq \in E(G)$ then the degree of p must be at least 3. To see this, assume that p and q both have a degree of 2. Since they are connected to x and y respectively, which also have a degree of 2, it follows that x, p, q, y is then a path component of length at least 4. Since the length of path components in G is bounded by 3 by Property (iii) of Definition 1, this is not possible. Therefore at least one of p, q must have a degree unequal to 2. Note that p and q cannot have a degree of 1 by Property (iv), since they are adjacent to degree-2 vertices. Therefore at least one of p, q must have degree at least 3. Since we assumed $\deg_G(p) \geq \deg_G(q)$, we may conclude that $\deg_G(p) \geq 3$. We expand the tree as illustrated in Figure 4(a).
2. If $pq \notin E(G)$, we consider the local situation.
   (a) If $\deg_G(p) \geq 4$ then we proceed as in Figure 4(b).
   (b) If $\deg_G(q) = 2$ then we consider the degree of p. By the existence of the previous case, we may assume that $\deg_G(p) \leq 3$. Since p is adjacent to a degree-2 vertex, we know by Property (iv) that $\deg_G(p) \geq 2$.
      i. If $\deg_G(p) = 2$ then proceed as in Figure 4(c). (Note that the un-marked vertices adjacent to p and q might actually represent the same vertex, but this does not affect the augmentation.)
      ii. If $\deg_G(p) = 3$ then proceed as in Figure 4(d). By Observation 2

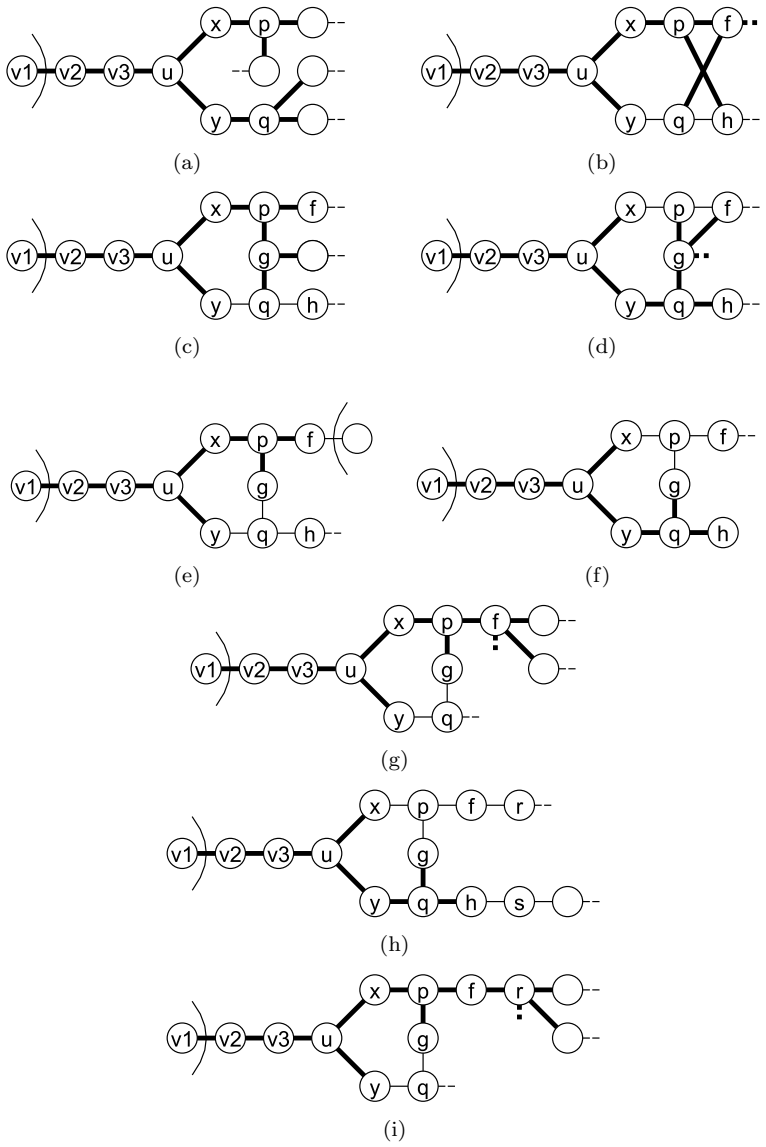
the vertex p is not adjacent to T in this case, and hence all but one of its neighbors will become leaves. The vertex y will be a split vertex after the augmentation.

Since we assumed $\deg_G(p) \geq \deg_G(q)$, the only remaining situation is when $pq \notin E(G)$ and $\deg_G(p) = \deg_G(q) = 3$, which implies by Observation 2 that p and q are not adjacent to T. Note that $\deg_G(q) = 1$ is not possible by Property (iv). We now conclude with an analysis of this final case. For the analysis we consider the set $N_G(p) \cap N_G(q)$, which can be seen to have cardinality at most two by the structure of the local situation.

1. If $N_G(p) \cap N_G(q) = \emptyset$ then expand as in Figure 5(a).
2. If $N_G(p) \cap N_G(q) = \{f, h\}$ for some vertices f, h then expand as in Figure 5(b). Vertices f, h cannot be adjacent to T: if (for example) f had some neighbor in T then $\deg_G(f) \geq 3$ since f also has p, q as neighbors (which are not in T by the earlier observations), which would imply by Observation 2 that f is not adjacent to T.
3. If $N_G(p) \cap N_G(q) = \{g\}$ for a vertex g then we make a further distinction on the situation of g. Let $\{f\} = N_G(p) \setminus \{x, g\}$ and let $\{h\} = N_G(q) \setminus \{y, g\}$.
   (a) If $\deg_G(g) \geq 3$:
      i. If $N_G(g) \cap \{f, h\} = \emptyset$ then expand as in Figure 5(c).
      ii. Otherwise assume by symmetry that $f \in N_G(g)$ and expand as in Figure 5(d).
   (b) Otherwise we must have $\deg_G(g) = 2$, since $pg, qg \in E(G)$.
      i. If one of f, h is adjacent to T then assume w.l.o.g. that this is f and expand as in Figure 5(e).
      ii. If f, h are not adjacent to T, then assume without loss of generality (by symmetry) that $\deg_G(f) \geq \deg_G(h)$.
         - If $\deg_G(h) = 1$ then expand as in Figure 5(f).
         - If $\deg_G(f) \geq 3$ then expand as in Figure 5(g).
         - Otherwise we must have $\deg_G(h) = \deg_G(f) = 2$, since we assumed that $\deg_G(f) \geq \deg_G(h)$. Let r be the neighbor of f that is not p, and let s be the neighbor of h that is not q. Assume w.l.o.g. that $\deg_G(r) \geq \deg_G(s)$.
           – If $\deg_G(s) = 2$ then expand as in Figure 5(h).
           – Otherwise $\deg_G(r) \geq \deg_G(s) \geq 3$; expand as in Figure 5(i). Once again we know that r is not adjacent to T by Observation 2.

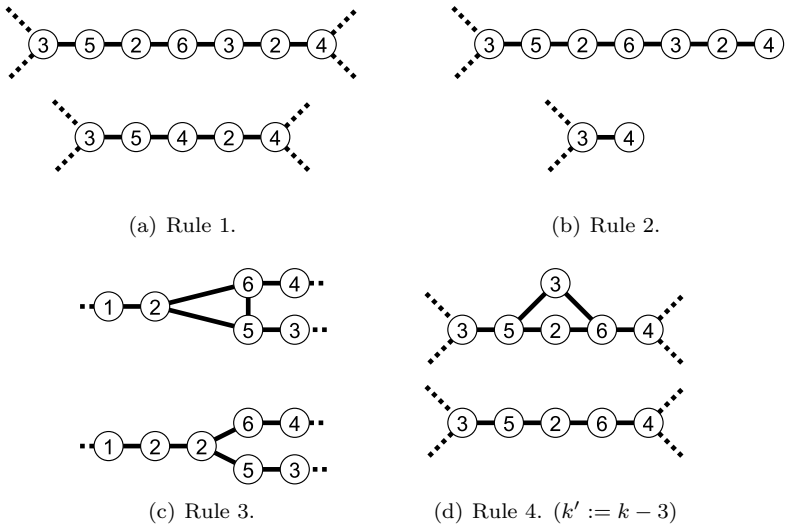
Since this case analysis is exhaustive it shows that in every possible situation where T is not a spanning tree for G, there is some operation to augment T that satisfies the incremental inequality. By Proposition 1 this concludes the proof of Theorem 1.

Strengthening of Theorem 1 We briefly comment on the possibility of strengthening Theorem 1. It is not hard to see that the maximum degree of



Case	$\Delta N =$	$\Delta L =$	$\Delta S \geq$	$\Delta D \geq$
5(a)	11	3	$-\lvert C_{T\to T'}\rvert - 1$	$\lvert C_{T\to T'}\rvert$
5(b)	$\lvert N_G(f) \setminus \{p,h\}\rvert + 8$	$\lvert N_G(f) \setminus \{p,h\}\rvert + 1$	$-\lvert C_{T\to T'}\rvert - 1$	$\lvert C_{T\to T'}\rvert + 2$
5(c)	$\deg_G(g) + 7$	$\deg_G(g)$	$-\lvert C_{T\to T'}\rvert - 1$	$\lvert C_{T\to T'}\rvert + 1$
5(d)	$\lvert N_G(g) \setminus \{h\}\rvert + 7$	$\lvert N_G(g) \setminus \{h\}\rvert$	$-\lvert C_{T\to T'}\rvert - 1$	$\lvert C_{T\to T'}\rvert + 2$
5(e)	8	2	-2	2
5(f)	8	2	-1	1
5(g)	$\deg_G(f) + 7$	$\deg_G(f)$	$-\lvert C_{T\to T'}\rvert - 1$	$\lvert C_{T\to T'}\rvert$
5(h)	8	2	0	0
5(i)	$\deg_G(r) + 8$	$\deg_G(r)$	$-\lvert C_{T\to T'}\rvert - 1$	$\lvert C_{T\to T'}\rvert$

Figure 5: Tree augmentations. Refer to Figure 2 for the semantics of the images.



(a) Rule 1. (b) Rule 2.

(c) Rule 3. (d) Rule 4. ($k' := k - 3$)

Figure 6: Examples of the reduction rules. The reduced structure is shown below the original structure. The numbers represent vertex weights.

any nontrivial graph in $\mathcal{C}$ is at least 3. This results in an excess of at least 8 on the right hand side of the incremental inequality during the initialization of the tree. By keeping track of this excess when summing over the incremental steps, the argumentation given in Proposition 1 shows that the number of leaves that can be obtained on a graph $G = (V, E)$ contained in $\mathcal{C}$ with $|V| \geq 3$ is at least $\lceil \frac{|V|+8}{5.5} \rceil$. The additive constant gained does not improve the kernelization bound, however.

4 Kernelization

In this section we describe the kernelization for Weighted Max Leaf. We first specify a set of reduction rules, then show that these can be applied efficiently and finally we analyze the resulting reduced instances.

4.1 Reduction Rules

We present reduction rules in a structured, 3-stage format. The reduction rules presented here transform an instance $\langle G, w, k \rangle$ into an instance $\langle G', w', k' \rangle$ such that G has a spanning tree with leaf weight at least k if and only if G' has a spanning tree with leaf weight at least k'. Any reduction rule that satisfies these properties is called *safe*. The reduction rules are illustrated by examples in Figure 6.

Reduction Rule 1 *Shrink large path components*

> **Structure:** A path component $P = \langle u, s_1, s_2, \ldots, s_q, v\rangle$ of size $q \geq 4$ with endpoints u, v of degree at least 3, such that $w(s_1) \geq w(s_q)$.
> **Operation:** Remove $s_2, \ldots, s_{q-1}$ and their incident edges from G. Add a new vertex s' with $N_{G'}(s') := \{s_1, s_q\}$, and set $w'(s') := \max_{i=1}^{q-1}(w(s_i) + w(s_{i+1}) - w(s_1))$.
> **Justification:** If a spanning tree avoids an edge with both endpoints inside the path component, it is always optimal to avoid an edge that maximizes the combined weight of its endpoints. The reduced graph offers the same connection and weighting possibilities as the original.

Lemma 11 *Rule 1 is safe:* $\langle G, w, k\rangle$ YES*-instance* $\Leftrightarrow$ $\langle G', w', k'\rangle$ YES*-instance.*

Proof: The reduction rule replaces the path component P by a shorter one. Observe that a spanning tree avoids at most one edge inside a path component, otherwise it is disconnected. Spanning trees in G avoiding an edge incident on an endpoint of the path component can trivially be transformed to spanning trees in G' of the same leaf weight, and vice versa. The only interesting case is if a spanning tree in G avoids an interior edge of the path component. In this case it is optimal to avoid an edge whose endpoints have maximum weight, and avoiding the edge between s' and a maximum-weight neighbor yields the same net value. The other direction is similar. □

Let us remark on why Rule 1 shrinks path components to size 3, and not less. Intuitively, there are four distinct possibilities that we have to encode: an optimal spanning tree can avoid the leftmost or rightmost edge on the path component, it can avoid an interior edge whose endpoints have maximum combined weight, or it can use all edges on the path component. These four choices are potentially all interesting if the endpoints of the path component have higher weight than what can be obtained in the interior. Since making a high-weight endpoint a leaf contributes much weight, but limits which vertices in the remainder of the graph can be made leaves, it is a priori unclear what the optimal decision is. To allow the resulting weight contributions to be stored for all options, we need three interior vertices. For example, it seems impossible to encode the path component with successive weights $(10, 2, 4, 3, 8)$ with fewer interior vertices.

Reduction Rule 2 *Shrink paths leading to a degree-1 vertex*

> **Structure:** Path component $P = \langle u, s_1, s_2, \ldots, s_q, v\rangle$ of length $q \geq 1$ where $\deg_G(v) = 1$.
> **Operation:** Replace $s_1, \ldots, s_q$ and their incident edges by a direct edge uv.
> **Justification:** Every vertex s_i is a cut vertex and will never be a leaf in a spanning tree.

Correctness of the previous rule is easy to see, so we do not give a formal proof.

Reduction Rule 3 *Reduce triangles with simple neighborhoods*

> **Structure:** Triangle on three vertices x, y, z such that every vertex of the triangle has at most one neighbor not in the triangle, and the graph is not isomorphic to K_3. Let x have minimum weight among $\{x, y, z\}$.
> **Operation:** Remove all the edges between vertices x, y, z. Add a new vertex m with $N_{G'}(m) := \{x, y, z\}$ and set $w'(m) = w(x)$.
> **Justification:** Any spanning tree must avoid at least one edge on the triangle; the decision can be encoded using fewer high-degree vertices by adding the vertex m to represent the connectivity.

Although Rule 3 increases the number of vertices, it still effectively simplifies the instance. If the vertex v is in a triangle that is reduced by this rule then the neighborhood of v is simplified by the reduction: if v had degree 3 then its degree is reduced to 2, and if it had degree 2 then its degree is reduced to 1.

Lemma 12 *Rule 3 is safe:* $\langle G, w, k\rangle$ YES*-instance* $\Leftrightarrow$ $\langle G', w', k'\rangle$ YES*-instance.*

Proof: ($\Rightarrow$) Suppose G has a spanning tree T with $\textsc{lw}(T) \geq k$. We make a distinction based on the status of the vertices in the triangle.

- If all vertices x, y, z are leaves in T, we build a spanning tree for G' by adding the isolated vertex m to T. We connect to m from the vertex x of minimum weight. The resulting tree T' is a spanning tree for G' with the same leaf weight as T, since the new leaf m has the same weight as x that became internal to connect to m.
- If there is at least one vertex on the triangle that is internal, then denote this vertex by v. We build a spanning tree $T' \subseteq G'$ by removing the edges from v to the other vertices on the triangle, adding the vertex m and edge vm and finally adding edges um for every $u \in N_T(v) \cap \{x, y, z\}$. Note that this construction does not change the degrees of vertices that are leaves in T; hence $\textsc{Leaves}(T') \supseteq \textsc{Leaves}(T)$ and the claim follows.

($\Leftarrow$) Suppose G' has a spanning tree T' with $\textsc{lw}(T') \geq k'$. By our assumption that G is not isomorphic to K_3 we find that the set $\{x, y, z\}$ is a cutset for G' since it separates m from the vertices outside the triangle. Hence by Lemma 1 there must be at least one vertex among $\{x, y, z\}$ that is internal in T; let v be such a vertex. To build the spanning tree $T \subseteq G$ we make a distinction based on the status of m in T'.

- If m is internal in T', we obtain T by removing m and its incident edges from T' and adding edges vu for every $u \in N_{T'}(m) \setminus \{v\}$. The resulting tree is a spanning tree for G with the same set of leaves.
- If m is a leaf in T' then we obtain T from T' by simply deleting m and its single incident edge. We now claim that the vertex denoted by v (which was internal in T') has become a leaf in T. To see this, observe that v must have had a degree of 2 in T'. By the precondition to the reduction

rule vertex v has at most one neighbor in G that is not contained in the triangle $\{x, y, z\}$, so in graph G' vertex v has at most one neighbor besides m. Therefore v has degree at most 2 in G'. Since v is internal in T' (by assumption) it must have a degree of at least 2 in T'. Hence the degree of v in T' is indeed 2. By deleting m and its incident edge the degree of v becomes 1 in T, and hence it is a leaf. Since $w(v) = w'(v) \geq w'(m)$ by definition of $w'(m)$ we find that $\text{LW}(T) \geq \text{LW}(T')$.

Hence we conclude that the reduction rule is safe. □

Reduction Rule 4 *Reduce degree-2 vertices with identical neighborhoods*

Structure: Two vertices u, v with $w(u) \geq w(v)$ and $N_G(u) = N_G(v) = \{x, y\}$ such that $V \setminus \{u, v, x, y\} \neq \emptyset$.
Operation: Remove u and its incident edges, set $k' := k - w(u)$.
Justification: There is always an optimal spanning tree in which u is a leaf.

Lemma 13 *Rule 4 is safe:* $\langle G, w, k\rangle$ YES*-instance* $\Leftrightarrow$ $\langle G', w', k'\rangle$ YES*-instance.*

Proof: ($\Rightarrow$) Suppose G has a spanning tree T with $\text{LW}(T) \geq k$. We first show that there is always an optimal spanning tree for G in which u is a leaf. Observe that u and v cannot both be internal in T, as this would imply T contains a cycle. If u is internal and v is a leaf, then we can also make v internal and u a leaf; since the weight of u is at least as much as that of v, this does not decrease the leaf weight of the spanning tree. This shows that there is always an optimal spanning tree in which u is a leaf. From such a spanning tree, we obtain a spanning tree for G' with leaf weight at least $k' = k - w(u)$ by removing u and its incident edges.

($\Leftarrow$) Suppose G' has a spanning tree T' with $\text{LW}(T') \geq k'$. The vertices x, y from a cutset for G' since they separate v from the remainder of the graph. By Lemma 1 we know that one of the vertices x, y is internal in T'. We create a spanning tree $T \subseteq G$ by copying T', adding the vertex u and connecting to it from a vertex of x, y that is internal. Since u becomes a leaf in T we know that $\text{LW}(T) = \text{LW}(T') + w(u)$, which proves the claim. □

Since a kernelization is a self-reduction of a problem, it is important that the output instance of a kernelization algorithm satisfies the same restriction on the allowed weight range as the input problem. The following lemma will turn out to be useful to prove this.

Lemma 14 *Let* $\langle G, w, k\rangle$ *be an instance of* Weighted Max Leaf *such that Rule 1 can be applied, and let* $\langle G', w', k'\rangle$ *be the resulting instance after application of Rule 1. Then* $\min\{w'(v) \mid v \in V(G')\} \geq \min\{w(v) \mid v \in V(G)\}$.

Proof: All vertices in G' that also exist in G have the same weight under w as under w'. The only vertex in G' that does not exist in G is the new vertex s' that is created by application of the reduction rule; we will show that its

weight under w' is not less than the minimum weight of a vertex in G under w. The weight of s' is defined as $w'(s') := \max_{i=1}^{q-1}(w(s_i) + w(s_{i+1}) - w(s_1))$. Since the edge s_1s_2 is considered in the maximum, it follows that $w'(s') \geq w(s_1) + w(s_2) - w(s_1) \geq w(s_2) \geq \min\{w(v) \mid v \in V(G)\}$, which proves the claim. □

4.2 Structure of a Reduced Instance

If no reduction rules are applicable to an instance, then such an instance is called *reduced*. The structure of reduced instances is captured by the class $\mathcal{C}$ (see Definition 1), which is proven in the following theorem.

Theorem 2 *If $\langle G', w', k' \rangle$ is a reduced instance of* Weighted Max Leaf *that is not isomorphic to a simple cycle and $|V(G')| \geq 3$, then $G' \in \mathcal{C}$.*

Proof: Suppose $\langle G', w', k' \rangle$ is a reduced instance with $|V(G')| \geq 3$ that is not isomorphic to a simple cycle. We will prove that G' satisfies all properties of Definition 1, in the order in which the properties are listed in the definition.

(i) The input graph is simple and connected by definition. It is easy to verify that the reduction rules preserve these properties.
(ii) By assumption the reduced graph G' is not isomorphic to a simple cycle.
(iii) By Rule 1 the reduced graph G' does not have path components of length larger than 3.
(iv) By Rule 2 the reduced graph G' does not have degree-1 vertices that are adjacent to degree-2 vertices. If the neighbor of a degree-1 vertex also has degree-1 then the graph is isomorphic to K_2 and hence $|V(G')| < 3$; otherwise every degree-1 vertex must be adjacent to a vertex of degree at least 3.
(v) Suppose the reduced graph has a triangle x, y, z such that each vertex on the triangle has degree at most 3. If G' is isomorphic to K_3 then it is isomorphic to a simple cycle, which contradicts the assumption in the theorem. But if G' is not isomorphic to K_3 then Rule 3 is applicable, which is also a contradiction. So the reduced graph cannot have a triangle on vertices of degree at most 3.
(vi) Suppose there are two degree-2 vertices u, v such that $N_G(u) = N_G(v) = \{x, y\}$. If $V \setminus \{u, v, x, y\} \neq \emptyset$ then Rule 4 is applicable; on the other hand if $V = \{u, v, x, y\}$ then either G' is a simple cycle on 4 vertices (which contradicts our assumption) or G' is isomorphic to K_4 minus one edge, which must contain a triangle to which Rule 3 is applicable.

Since G' satisfies all the required properties we may conclude that $G' \in \mathcal{C}$. □

4.3 Reduction Procedure

In this section we consider how the reduction rules can be realized by an algorithm. It is easy to verify that we can test in polynomial time whether a

reduction rule is applicable to the graph, and to apply the reduction if necessary. All reduction rules except Rule 3 reduce the number of vertices. Rule 3 strictly decreases the number of triangles in the input graph. Since the other reduction rules do not create new triangles, Rule 3 can be applied at most $\binom{|V(G)|}{3}$ times. The other reduction rules can be applied at most a polynomial number of times, since they decrease the number of vertices. Hence after applying a polynomial number of reduction rules, we must arrive at a reduced instance. These easy observations show that we can obtain a reduced instance in polynomial time. By an elementary but tedious argument, it can be shown that it is possible to compute a reduced instance in $\mathcal{O}(|V| + |E|)$ time, assuming that arithmetic on the weights takes constant-time. We have chosen to omit the argument here due to its length; interested readers are referred to the technical report [20, Theorem 3].

4.4 Kernelization Algorithm

We now combine the ingredients obtained earlier to prove the existence of a kernelization algorithm for Weighted Max Leaf. We start by presenting a small lemma which shows that the reduction rules do not change a valid instance of the problem into an invalid one.

Lemma 15 *Let $\langle G', w', k' \rangle$ be obtained by applying reduction rules to an instance $\langle G, w, k \rangle$ of* Weighted Max Leaf*. If $k' \geq 1$ then $\langle G', w', k' \rangle$ is also a valid instance of* Weighted Max Leaf*: the graph G' is simple and connected, the weight of a vertex is not smaller than 1, and the precision needed to store the weights under w' is not greater than the precision needed for w.*

Proof: By the specification of Weighted Max Leaf the input graph G must be simple and connected in a valid instance. It is easy to verify that the reduction rules preserve these properties. Let us argue that the weights assigned to vertices by the new weight function w' are all at least 1. The only reduction rule that changes vertex weights is Rule 1. Lemma 14 proves that the minimum vertex weight is not decreased by an application of that rule, which shows that the vertex weights under w' are not smaller than the vertex weights under w. Since $\langle G, w, k \rangle$ is a valid instance with weights not smaller than 1, this proves the second part of the claim. For the final part, observe that a vertex weight under w' is either equal to a weight under w, or the weight under w' is obtained from three weights under w by addition and subtraction through Rule 1. Since subtraction and addition do not change the number of digits after the decimal point which are needed to represent the number, the lemma follows. □

Finally we can state the kernelization theorem and obtain the main result of this paper.

Theorem 3 Weighted Max Leaf *has a kernel with at most $5.5k$ vertices: there is an algorithm that takes an instance $\langle G, w, k \rangle$ of* Weighted Max Leaf *as input, and computes an equivalent instance $\langle G', w', k' \rangle$ of bitsize polynomial*

in k such that $|V(G')| \leq 5.5k$, $k' \leq k$ and $\max_{v \in V(G')} w'(v) \leq k'$ in polynomial time.

Proof: We sketch the kernelization procedure. When supplied with an input $\langle G, w, k \rangle$ the algorithm first computes a reduced instance $\langle G', w', k' \rangle$ in polynomial time using the approach sketched in Section 4.3. The safety of the reduction rules ensures that $\langle G, w, k \rangle$ is a YES-instance if and only if $\langle G', w', k' \rangle$ is a YES-instance. By the definitions of the reduction rules we know that $k' \leq k$. If the graph G' is isomorphic to a simple cycle then the problem can be decided in linear time: the maximum leaf weight that can be obtained by a spanning tree in a simple cycle is equal to $\max_{uv \in E(G')} w'(u) + w'(v)$. So when G' is isomorphic to a simple cycle we can decide the problem and output a trivial 1-vertex instance that yields the same answer.

Assume from now on that G' is not a simple cycle. Observe that for every vertex with weight larger than k', we may decrease its weight to k' without changing whether there is a spanning tree of leaf weight at least k'. Therefore we may assume that $\max_{v \in V(G')} w'(v) \leq k'$. If $k' \leq 1$ then the algorithm outputs a trivial YES instance. If $|V(G')| \geq 5.5k' \geq 5.5$ then the answer to the decision problem must be YES: since G' is not isomorphic to a cycle Theorem 2 shows that $G' \in \mathcal{C}$, which implies by Theorem 1 that there is a spanning tree $T' \subseteq G'$ with at least k' leaves. Noting that every vertex has a weight not smaller than 1, the leaf weight of T' must be at least k' which implies that $\langle G', w', k' \rangle$ is indeed a YES instance. We now output a trivial 1-vertex instance that yields this answer. If $|V(G')| < 5.5k'$ then the size of the output graph is bounded as required since $k' \leq k$, and we output $\langle G', w', k' \rangle$. Lemma 15 shows that a reduced instance is a valid input for Weighted Max Leaf. To see that the reduced instance has a bitsize polynomial in k, observe that we can easily encode the graph on at most $5.5k$ vertices in $O(k^2)$ bits using its adjacency matrix. Recall that the problem definition assumes that the fractional part of a weight is representable by a constant number of decimals. By Lemma 15 this holds for w' if it holds for w, and therefore the number of bits required to encode the weight function w' is polynomial in k: the fractional part of each value uses a constant number of decimal places (and therefore a constant number of bits), and the integer part of a value is at most k'. This concludes the proof. □

5 Hardness of Approximation

Although the focus of this work is on parameterized complexity, we briefly change the perspective in this section and consider Weighted Max Leaf as an optimization problem. In this case an instance consists only of a weighted graph, and the goal is to find a spanning tree with maximum leaf weight. This point of view allows us to study the approximability of the problem, and contrasts the parameterized view of Weighted Max Leaf which treats it as a decision problem. It is not hard to see that the optimization problem related to Weighted Max Leaf is in fact an NP-optimization problem since the value

of a potential solution can be computed in polynomial time. There is an extensive literature [1] devoted to NP-optimization problems and reductions between them, with various notions of approximation-preserving reductions. To simplify the exposition we will *not* use the rigorous formal framework of NP-optimization problems, but rather present the results in an informal manner. It is easy to verify that the claims made here can be formalized in a straight-forward way.

Consider a NP-maximization problem Π, and let f be a function that maps instances x of Π to non-negative numbers. The function f will often only depend on the optimum value of x and on the size of x. We say that an algorithm A is a polynomial-time f-approximation algorithm for Π if algorithm A always computes a feasible solution in polynomial time, such that value v of the resulting solution satisfies $v \geq \textsc{opt}/f(x)$, which means that the value that is found is at most a factor $f(x)$ smaller than the optimum. When f is a constant function then this yields a *constant-factor* approximation algorithm, but we may also allow f to be a function that depends on the instance size (such as a $\frac{n}{\log n}$-approximation), or a function that depends on the optimum value (e.g. a $\sqrt{\textsc{opt}}$-approximation). If Π has a f-approximation and $f \in \mathcal{O}(g(x))$ then we say that Π has a $\mathcal{O}(g(x))$-approximation.

In this section we will show that Weighted Max Leaf is hard to approximate, using a reduction from Independent Set. The approximability of Independent Set has been studied intensively, leading to the following result.

Theorem 4 ([18, 28]) *The* Independent Set *problem on graphs with n vertices does not have a polynomial-time $\mathcal{O}(n^{1-\varepsilon})$-approximation algorithm for any $\varepsilon > 0$ unless $P = NP$.*

Recall that an NP-optimization problem is *polynomially bounded* if the optimum solution value is bounded by a polynomial in the instance size. In some settings it is harder to approximate a problem if it is not polynomially bounded. We will prove that Weighted Max Leaf is hard to approximate, even when its optimum is polynomially bounded and when the input graphs are required to be 2-degenerate (see Section 2). For ease of notation we define the problem *Pb*-Weighted Max Leaf as the restriction of Weighted Max Leaf where the graph G is 2-degenerate and the weights are positive integers not exceeding $|V(G)|^2$. It is easy to verify that under this definition, *Pb*-Weighted Max Leaf is a polynomially bounded optimization problem.

Theorem 5 *We can use an approximation algorithm for* Pb-Weighted Max Leaf *to approximate* Independent Set.

1. *If* Pb-Weighted Max Leaf *has a polynomial-time $\mathcal{O}(n^c)$-approximation algorithm then* Independent Set *has a $\mathcal{O}(n^{2c})$ approximation algorithm.*
2. *If* Pb-Weighted Max Leaf *has a polynomial-time $\mathcal{O}(\textsc{opt}^c)$-approximation algorithm then* Independent Set *has a $\mathcal{O}(n^{3c})$ approximation algorithm.*

The remainder of this section is devoted to the proof of Theorem 5 and a corollary. The proof is based on a reduction from INDEPENDENT SET to *Pb*-WEIGHTED MAX LEAF that preserves approximation properties. We will show that using this reduction, we can use the existence of approximation algorithms for *Pb*-WEIGHTED MAX LEAF to construct approximation algorithms for INDEPENDENT SET.

Consider an instance G of INDEPENDENT SET. We will give an approximation preserving reduction that transforms G into an instance $\langle G', w_{G'} \rangle$ of *Pb*-WEIGHTED MAX LEAF. For the sake of analysis we will partition the vertices of the resulting graph G' into two types: *heavy* vertices that correspond to vertices in graph G, and *light* vertices that are added by the reduction. The reduction proceeds as follows.

1. Initialize G as a copy of G'. All vertices of G' at this stage are *heavy*.
2. Replace every edge uv between heavy vertices by a new light vertex x with edges $\{ux, vx\}$.
3. Add a new light root vertex r and give r edges to all heavy vertices.

The graph that results from the above three steps is used as G'. For ease of notation define $n := |V(G)|$ and $n' := |V(G')|$. The weight function is simple, and corresponds to the intuition. We set the weight of all heavy vertices to n^2, and we set the weight of all light vertices to 1. Figure 7 shows an example of this reduction.

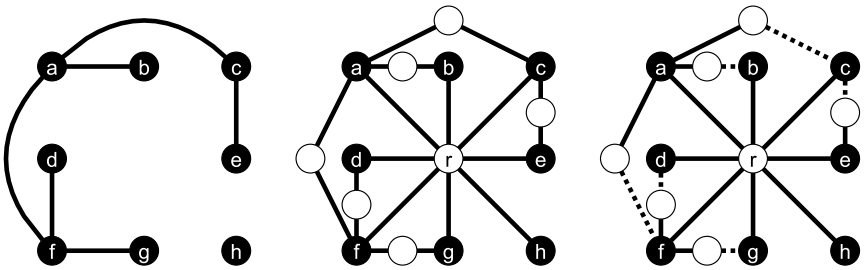
From the definition of the graph G' it follows that $n' = |V(G)| + |E(G)| + 1 \leq n + \binom{n}{2} + 1$ which implies that $n' \leq n^2$ for $n \geq 2$. Since we may assume without loss of generality that $n \geq 2$ we have $n' \leq n^2$. Using the fact that the weight of a vertex in G' is at most $n^2 < (n')^2$ it easily follows that the vertex weights satisfy the restrictions placed on the problem *Pb*-WEIGHTED MAX LEAF. It is not hard to verify that G' is a 2-degenerate graph. Any vertex-induced subgraph that has minimum degree 3 or higher cannot contain any of the light vertices that were used to subdivide edges. But if all such light subdivider vertices are excluded from a vertex-induced subgraph, then such a subgraph must in fact be a star and hence it contains a vertex of degree 1. Therefore all vertex-induced subgraphs of G' have minimum degree at most 2 and hence G' is 2-degenerate.

The following lemma shows the relationship between solutions of the INDEPENDENT SET instance G and the *Pb*-WEIGHTED MAX LEAF instance $\langle G', w_{G'} \rangle$.

Lemma 16 *Instance G has an independent set S of size at least $k \Leftrightarrow$ instance $\langle G', w_{G'} \rangle$ has a spanning tree $T' \subseteq G'$ with at least k heavy leaves.*

Proof: We shall use the fact that the vertex set of G corresponds to the heavy vertices in G'.

($\Rightarrow$) Suppose S is an independent set for G of size $|S| \geq k$. We will show that G' has a spanning tree with at least $|S|$ heavy leaves. We initialize a tree $T' \subseteq G'$ by taking the light vertex r as the root, and adding edges to every heavy vertex in G'. Every heavy leaf in T' now corresponds to a vertex of G. The only vertices that this tree does not reach are the light vertices that were



(a) Graph G. (b) Resulting $\langle G', w_{G'} \rangle$. (c) Tree T'.

Figure 7: Example of the reduction from Independent Set to *Pb*-Weighted Max Leaf. (a) An instance G of Independent Set. (b) The corresponding instance $\langle G', w_{G'} \rangle$ of *Pb*-Weighted Max Leaf that results from the reduction. Heavy vertices with weight n^2 are drawn in black, and light vertices with weight 1 are drawn in white. (c) The spanning tree $T' \subseteq G'$ that corresponds to the independent set $\{b, c, d, g, h\}$ in G. Thick lines represent edges in $E(G') \cap E(T')$ and broken lines represent edges in $E(G') \setminus E(T')$.

added when subdividing edges of G. Since S is an independent set, we know $V(G) \setminus S$ forms a vertex cover for G. By the equivalence between the light subdivider vertices in G' and edges in G, we can augment tree T to a spanning tree by connecting to the light subdivider vertices from the vertices in $V(G) \setminus S$. If we augment T' to a spanning tree in this way, every heavy vertex in G' that is a member of the independent set S in G remains a leaf. So the augmented T' is a spanning tree for G' with at least $|S| \geq k$ heavy leaves.

($\Leftarrow$) Suppose $T' \subseteq G'$ is a spanning tree for G' and S are its heavy leaves with $|S| \geq k$. By the correspondence between heavy vertices in G' and vertices of G, we can also interpret S as a subset of the vertices of G. We will prove that S forms an independent set in G, by showing that if there is an edge uv in G, then u and v are not both heavy leaves in any spanning tree for G'.

So assume G has an edge uv. By the definition of the reduction, this edge was subdivided by some light vertex x when forming G'. Since $N_{G'}(x) = \{u, v\}$ it follows that $\{u, v\}$ is a cutset for G' since it separates w from the light root vertex r. By Lemma 1 we may conclude that at least one of u and v is not a leaf in a spanning tree for G'.

Using this fact we obtain by contraposition that the set S of heavy leaves is an independent set in G, and hence this establishes that G has an independent set of size $|S| \geq k$. □

The correspondence between the two instances allows us to use an approximation algorithm for *Pb*-Weighted Max Leaf to build an approximation algorithm for Independent Set. Suppose we have a polynomial-time $\mathcal{O}(n^c)$-approximation (or $\mathcal{O}(\text{OPT}^c)$ approximation) algorithm A for *Pb*-Weighted Max Leaf. We construct a polynomial-time $\mathcal{O}(n^{2c})$-approximation (resp. $\mathcal{O}(n^{3c})$-approximation) algorithm B for Independent Set by applying

the reduction from Independent Set to Pb-Weighted Max Leaf and running algorithm A on the instance $\langle G', w_{G'} \rangle$. Let $T' \subseteq G'$ denote the spanning tree that is found. By Lemma 16 the heavy leaves of spanning tree T' form an independent set in G; we use this independent set as the output of algorithm B. Since the reduction can be applied in polynomial time, it follows that B runs in polynomial time if the algorithm A exists. The vertex weights that are used in the reduction allow us to easily establish a relationship between the leaf weight of a spanning tree, and the number of heavy leaves that it contains. The total weight of the light vertices is less than n^2, since each light vertex has weight 1 and there at most $\binom{n}{2} + 1$ of such vertices. We can consider the leaf weight of a spanning tree to consist of two parts: the weight of heavy leaves plus the weight of light leaves. Consider the spanning tree $T' \subseteq G'$ that is found by running algorithm A on the instance resulting from the reduction, and let k' denote the number of *heavy* leaves in T'. Since light leaves contribute at most n^2 to the total leaf weight, and because the weight of a heavy vertex is exactly n^2, we find the following:

$$k' \geq \frac{1}{n^2}(\underset{w_{G'}}{\mathrm{LW}}(T') - n^2) \geq \frac{1}{n^2} \underset{w_{G'}}{\mathrm{LW}}(T') - 1. \tag{3}$$

It remains to prove the approximation guarantee for algorithm B. The following observation will be important. Let k be the size of a maximum independent set in G, and let $S \subseteq V(G)$ be an independent set in G of size k. By the correspondence between G and the instance $\langle G', w_{G'} \rangle$ that was derived in Lemma 16 we know that the vertices in S correspond to heavy vertices in G', and that there is a spanning tree $T'' \subseteq G'$ where all the vertices in S are heavy leaves; hence such a spanning tree has leaf weight at least $|S|n^2 = kn^2$. Let $T^* \subseteq G'$ be a spanning tree for G' with maximum leaf weight. Since the leaf weight of T^* must be at least as large as the leaf weight of T'', we find:

$$\underset{w_{G'}}{\mathrm{LW}}(T^*) \geq kn^2. \tag{4}$$

Using these ingredients we are now ready to prove the approximation guarantees for the resulting algorithm B.

Size-based Approximation

We will first prove that B is a $\mathcal{O}(n^{2c})$-approximation algorithm if A is a $\mathcal{O}(n^c)$-approximation algorithm. By the definition of a $\mathcal{O}(n^c)$-approximation algorithm we know that there are constants n_1 and c_1 such that for all instances of Pb-Weighted Max Leaf with size $n' \geq n_1$ the value found by the approximation algorithm is at least $\textsc{opt}\,/c_1(n')^c$. We may assume without loss of generality that $n' \geq n_1$ for the instance of Pb-Weighted Max Leaf that results from the reduction, because for all instances smaller than n_1 we can solve the problem optimally in polynomial time since n_1 is a constant. Recall that T' is the spanning tree found by running algorithm A on the instance resulting from the

reduction. By the assumption on the approximation quality of A we find that:

$$\underset{w_{G'}}{\text{LW}}(T') \geq \underset{w_{G'}}{\text{LW}}(T^*)/(c_1(n')^c) \geq \underset{w_{G'}}{\text{LW}}(T^*)/(c_1 n^{2c}), \tag{5}$$

where the last step follows from the fact that $n' \leq n^2$ for $n \geq 2$ by definition of the reduction. Combining (5) with (4) we find:

$$\underset{w_{G'}}{\text{LW}}(T') \geq kn^2/(c_1 n^{2c}). \tag{6}$$

Combining (6) with (3) we find that:

$$k' \geq \frac{1}{n^2} kn^2/(c_1 n^{2c}) - 1 = k/(c_1 n^{2c}) - 1. \tag{7}$$

So the number k' of heavy leaves in the approximate solution T' is at least $k/(c_1 n^{2c}) - 1$, where k is the optimum of the Independent Set instance. Since the heavy leaves in T' correspond to the independent set that is output by algorithm B, we may conclude from (7) that algorithm B is indeed a $\mathcal{O}(n^{2c})$-approximation for Independent Set since $k/(c_1 n^{2c}) - 1 \geq k/\mathcal{O}(n^{2c})$.

Value-based Approximation

We now prove that B is a $\mathcal{O}(n^{3c})$-approximation algorithm if A is a $\mathcal{O}(\text{OPT}^c)$-approximation algorithm. By definition of this type of approximation guarantee we find that there are constants c_2, n_2 such that for all sufficiently large instances with $n' \geq n_2$ it holds that:

$$\underset{w_{G'}}{\text{LW}}(T') \geq \underset{w_{G'}}{\text{LW}}(T^*)/(c_2(\underset{w_{G'}}{\text{LW}}(T^*))^c) = \frac{1}{c_2}(\underset{w_{G'}}{\text{LW}}(T^*))^{1-c}. \tag{8}$$

Combining (8) with (4) yields:

$$\underset{w_{G'}}{\text{LW}}(T') \geq \frac{1}{c_2}(kn^2)^{1-c}. \tag{9}$$

Combining (9) with (3) we find that:

$$k' \geq \frac{1}{n^2}\frac{1}{c_2}(kn^2)^{1-c} - 1 = \frac{k}{c_2 k^c n^{2c}} - 1 \geq \frac{k}{c_2 n^c n^{2c}} - 1 = \frac{k}{c_2 n^{3c}} - 1, \tag{10}$$

where the last transformation step follows from the fact that $k \leq n$ since the optimum k of the Independent Set instance is at most the size n of the graph G. Hence B is a polynomial-time $\mathcal{O}(n^{3c})$-approximation algorithm for Independent Set. This concludes the proof of Theorem 5. Using Theorem 4 we obtain the following corollary.

Corollary 1 Pb-Weighted Max Leaf*, the polynomially-bounded optimization version of* Weighted Max Leaf *restricted to 2-degenerate graphs on n vertices, does not admit a polynomial-time $\mathcal{O}(n^{\frac{1}{2}-\varepsilon})$-approximation algorithm or $\mathcal{O}(\text{OPT}^{\frac{1}{3}-\varepsilon})$-approximation algorithm for any $\varepsilon > 0$ unless $P = NP$.*

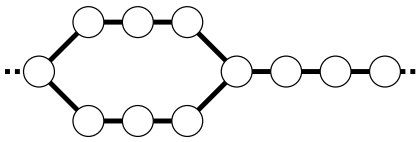


Figure 8: Component which shows that the factor 5.5 in Theorem 1 is best-possible: the graph obtained by replacing every vertex of a simple cycle of length n by this subgraph is a member of $\mathcal{C}$. It has $11n$ vertices and at most $2n + 2$ leaves in any spanning tree.

The corollary shows that the introduction of vertex weights makes Max Leaf much harder to approximate: the unweighted Max Leaf problem has a constant factor 2-approximation [25] and is in fact APX-complete [15], but even the restriction *Pb*-Weighted Max Leaf of the weighted version is not in APX unless P = NP. It is uncommon for optimization problems to exhibit structural differences in hardness of approximation when comparing unweighted versions to polynomially-bounded weighted versions. Crescenzi et al. [8] investigated weighted and unweighted variants of optimization problems with respect to their approximation properties, and found that weights do not structurally alter the hardness of approximation for Minimum Vertex Cover, Minimum Satisfiability, Maximum Cut, Maximum DiCut, Maximum 2-Satisfiability and Maximum Exact k-Satisfiability.

6 Conclusion

We have presented a simple problem kernel with $5.5k$ vertices for Weighted Max Leaf, using new weight-based reduction rules. This serves as an example of how to obtain effective and efficient data reduction on problems with vertex weights. These weights can be used to model real-world problems more accurately. A large part of this work was devoted to the proof of a combinatorial result that graphs excluding some simple substructures always have spanning trees with many leaves; using this result the kernelization effort reduces to eliminating those substructures in the input graph without blowing up the parameter value.

The use of this kernelization algorithm is not restricted to solving the decision variant of Weighted Max Leaf, but it can also be used to construct a spanning tree with the desired leaf weight if one exists. This stems from the fact that all the reduction rules can be reversed to lift a spanning tree for the reduced graph back to the original graph, and from the fact that the combinatorial proof of the extremal result is constructive. When Theorem 1 assures there is a spanning tree with at least k leaves (and hence leaf weight at least k), then such a spanning tree can be found in polynomial time by executing the augmentation operations.

The size of the resulting problem kernel directly corresponds to the extremal

bound from Theorem 1. The factor 5.5 in this theorem is best-possible as shown by the construction in Figure 8, which implies that the analysis of the kernel size is tight. It can be shown that Rule 1 and Rule 2 are sufficient to obtain a kernel of $7.5k$ vertices, and that successively adding Rule 3 and Rule 4 leads to kernels with $7k$ and $5.5k$ vertices, respectively. We found a reduction rule to eliminate the structure shown in Figure 8, but we chose not to incorporate this rule in the presentation since it does not lead to a kernel with less than $5.25k$ vertices while it significantly complicates the proof of the required strengthening of Theorem 1.

The proof of the extremal result of Theorem 1 uses an extension of the method of amortized analysis by keeping track of dead leaves; we extended the method by incorporating a new term ΔS in the incremental inequality, which allows us to exploit the fact that the considered graphs do not have long path components. We believe that this technique may be of independent interest since it can be used to prove similar results about leafy spanning trees in graph classes that avoid other subgraphs.

The kernelization procedure for Weighted Max Leaf raises the question whether the existing FPT algorithms for Max Leaf can be converted to the weighted setting. We have verified that this is indeed the case for the $\mathcal{O}(6.75^k \cdot n^{\mathcal{O}(1)})$ algorithm by Bonsma and Zickfeld [4]; their algorithm uses reduction rules to eliminate "diamonds" and "blossoms" in the input graph, and then guesses the leaf status of the vertices of degree at least 3 in the remaining graph. Given the status of the high-degree vertices the optimal number of leaves can be computed by finding a minimum *edge weighted* spanning tree. This approach carries over to the weighted setting by building weight-aware reduction rules that eliminate the diamonds and blossoms. It is an open question whether the $\mathcal{O}(4^k n^{\mathcal{O}(1)})$ FPT algorithm of Kneis et al. [22] can be adapted to solve the weighted problem. Two obstacles have to be overcome in order to generalize their algorithm. First of all, the bounded-depth search tree algorithm crucially relies on the fact that it can stop once a subtree with k leaves has been found, as such a tree can always be extended to a spanning tree without decreasing the number of leaves. As this does not generalize to the weighted setting — the extension to a spanning tree might increase the number of leaves while decreasing the total leaf weight — one needs to be careful with the stopping criterion. And even more importantly, the algorithm of Kneis et al. crucially exploits the fact that the branching algorithm may "follow paths" outside a partial solution without loss of optimality. When attempting in the unweighted setting to extend a given subtree $T \subseteq G$ to a k-leaf tree, in the presence of a vertex $v \in \textsc{Leaves}(T)$ with exactly one neighbor outside the tree, the following holds: either there is a k-leaf tree in which v remains a leaf, or one may assume without loss of optimality that v, along with all vertices on the path that is traced when starting at v, visiting its unique neighbor outside T, and repeatedly moving to unvisited neighbors outside T as long as these are unique, are internal to the extending tree. This observation makes it possible to limit the number of branching steps needed to exhaustively analyze such paths. In the weighted setting, however, it may be needed to build a tree with fewer leaves to allow

leaves of higher weight to be obtained. Their structural observation therefore fails in the weighted setting, and new insights are needed to reduce the branching factor.

Finally, we have shown that Weighted Max Leaf is hard to approximate: there is no polynomial-time $\mathcal{O}(n^{\frac{1}{2}-\varepsilon})$-approximation algorithm or $\mathcal{O}(\text{OPT}^{\frac{1}{3}-\varepsilon})$-approximation algorithm for any $\varepsilon > 0$ unless P = NP. Hence Weighted Max Leaf is an example of a problem for which the natural parameterization admits a linear-vertex kernel, but where the associated optimization problem does not have a constant-factor approximation algorithm unless P = NP. To our knowledge this is the first problem that shows this kind of behavior; we often find that problems that have a linear-vertex kernel also admit constant-factor approximation algorithms, with Vertex Cover being a notable example. We expect that further study into the parameterized complexity of weighted graph problems will shed more light on the connection between approximation algorithms and kernels. The effects of different parameterizations and weights of 0 on the parameterized complexity of Weighted Max Leaf have been studied in the author's Master's thesis [19].

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