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Box-Rectangular Drawings of Planar Graphs

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Abstract

A plane graph is a planar graph with a fixed planar embedding in the plane. In a box- rectangular drawing of a plane graph, every vertex is drawn as a rectangle, called a box, each edge is drawn as either a horizontal line segment or a vertical line segment, and the contour of each face is drawn as a rectangle. A planar graph is said to have a box-rectangular drawing if at least one of its plane embeddings has a box-rectangular drawing. Rahman et al. [11] gave a necessary and sufficient condition for a plane graph to have a box-rectangular drawing and developed a linear-time algorithm to draw a box-rectangular drawing of a plane graph if it exists. Since a planar graph G may have an exponential number of planar embeddings, determining whether G has a box-rectangular drawing or not using the algorithm of Rahman et al. [11] for each planar embedding of G takes exponential time. Thus to develop an efficient algorithm to examine whether a planar graph has a box-rectangular drawing or not is a non-trivial problem. In this paper we give a linear-time algorithm to determine whether a planar graph G has a box-rectangular drawing or not, and to find a box-rectangular drawing of G if it exists.

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1 Introduction

For the last two decades automatic drawings of graphs have created intense interest due to their broad applications, and as a consequence, a number of drawing styles and corresponding drawing algorithms have emerged [1, 3, 6, 13]. A *plane graph* is a planar graph with a fixed embedding in the plane. A *rectangular drawing* of a plane graph G is a drawing of G, where each vertex is drawn as a point, each edge is drawn as a horizontal or vertical line segment, and each face is drawn as a rectangle. On the other hand a *box-rectangular drawing* of a plane graph G is a drawing of G in which each vertex is drawn as a (possibly degenerated) rectangle, called a *box*, each edge is drawn as a horizontal line segment or a vertical line segment, and the contour of each face is drawn as a rectangle. Figure 1(c) illustrates a box-rectangular drawing of the plane

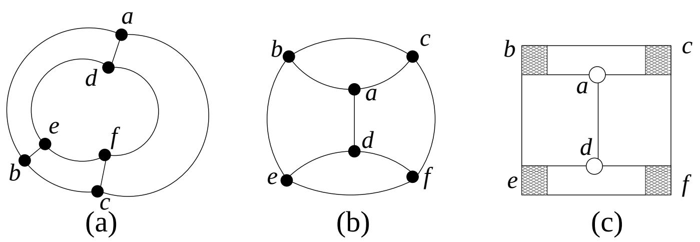

Figure 1: (a) A planar graph G, (b) a plane embedding Γ of G for which a box-rectangular drawing exists, and (c) a box-rectangular drawing of the planar graph G.

graph as in Fig. 1(b). Box-rectangular drawings have practical applications in VLSI floorplanning [7, 8, 11, 12, 14] and architectural floorplanning [2, 9, 10].

All plane graphs do not have box-rectangular drawings. Rahman et al. [11] gave a necessary and sufficient condition for a plane graph to have a box-rectangular drawing and developed a linear-time algorithm to draw a box-rectangular drawing of a plane graph if it exists. Xin He [5] did the same task for proper box- rectangular drawings of plane graphs. A planar graph is said to have a box-rectangular drawing if at least one of its plane embeddings has a box-rectangular drawing. For the plane embedding as illustrated in Fig. 1(a) of the planar graph G there is no box-rectangular drawing. But the plane embedding as in Fig. 1(b) of G has a box-rectangular drawing as illustrated in Figure 1(c). Thus G has a box-rectangular drawing. Since a planar graph G may have an exponential number of planar embeddings, determining whether G has a box-rectangular drawing or not using the algorithm of Rahman et al. [11] for each planar embedding of G takes exponential time. Thus to develop an efficient algorithm to examine whether a planar graph has a box-rectangular drawing or not is a non-trivial problem. In this paper we give a linear-time algorithm to determine whether a planar graph G has a box-rectangular drawing or not, and to find a box-rectangular drawing of G if it exists.

Our approach for finding a box-rectangular drawing of a planar graph is similar to that of Rahman et al. [12] for finding a rectangular drawing of a planar graph. However, our work is not a mere extension of the work of Rahman et al. [12], and we had to face a lot of challenges. In this paper we show that all the plane embeddings of a subdivision of planar 3-connected cubic graph G which is cyclically 4-edge-connected, have box-rectangular drawings, whereas not every such embedding has a rectangular drawing. We denote the maximum degree of a graph G by $\Delta(G)$ or simply by Δ. Rahman et al. [12] deal with planar graphs having $\Delta \leq 3$. But for box-rectangular drawing we deal with planar graphs of the maximum degree 4 or more. We had to face enormous difficulties in dealing with the graphs of maximum degree 4 or more. In [12] Rahman et al. showed that at most four plane embeddings are needed to be checked to determine whether a planar graph has a rectangular drawing. Whereas in case of box-rectangular drawing we showed that at most 81 embeddings are needed to be checked.

The rest of this paper is organized as follows. In section 2, we give some terminologies, and previous results. In Section 3, we describe a necessary and sufficient condition for a planar graph G with $\Delta \leq 3$ to have a box-rectangular drawing and find a drawing if it exists. Section 4 gives a necessary and sufficient condition for a planar graph G with $\Delta \geq 4$ to have a box-rectangular drawing and describes a linear-time algorithm for finding a drawing if it exists. Finally Section 5 concludes the paper. A preliminary version of this paper has been presented at [4].

2 Preliminaries

In this section we give some definitions and present preliminary results.

Throughout the paper we assume that a *graph* G is so called a multigraph which may have *multiple edges*, i.e., edges sharing both ends. If G has no multiple edges, then G is called a *simple* graph. *Subdividing an edge* (u, v) of a graph G is the operation of deleting the edge (u, v) and adding a path $u(= w_0), w_1, w_2, \ldots, w_k, v(= w_{k+1})$ passing through new vertices $w_1, w_2, \ldots, w_k$, $k \geq 1$, of degree 2. A graph G is called a *subdivision* of a graph G' if G is obtained from G' by subdividing some of the edges of G'.

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$. A graph G is called *cyclically 4-edge-connected* if the removal of any three or fewer edges leaves a graph such that exactly one of the components has a cycle [15]. The graph in Fig. 2(a) is cyclically 4-edge-connected. On the other hand, the graph in Fig. 2(b) is not cyclically 4-edge-connected, since the removal of the three edges drawn by dotted lines leaves a graph with two connected components each of which has a cycle. We often use the following operation on a planar graph G. Let v be a vertex of degree d in a plane graph Γ of the planar graph G, let $e_1 = vw_1, e_2 = vw_2, \ldots, e_d = vw_d$ be the edges incident to v, and assume that these edges $e_1, e_2, \ldots, e_d$ appear clockwise around v in this order. Replace v with a cycle $v_1, v_1v_2, v_2, v_2v_3, \ldots, v_dv_1, v_1$, and

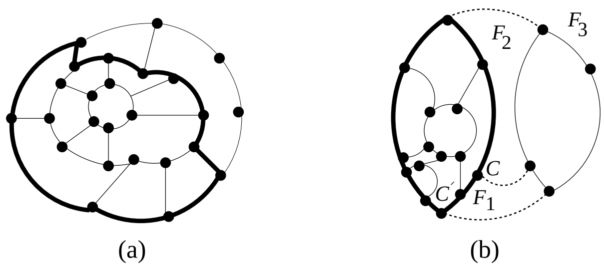

Figure 2: (a) A cyclically 4-edge-connected graph, and (b) a graph which is not cyclically 4-edge-connected.

replace the edges vw_i with v_iw_i for $i = 1, 2, \ldots, d$. We call the operation above *replacement of a vertex by a cycle*. The cycle $v_1, v_1v_2, v_2, v_2v_3, \ldots, v_dv_1, v_1$ in the resulting graph is called the *replaced cycle* corresponding to the vertex v of Γ.

Let G be a planar graph, and Γ be an arbitrary plane embedding of G. The plane graph Γ divides the plane into connected regions called *faces*. The unbounded region is called the *outer face* of Γ, and is denoted by $F_o(\Gamma)$. We sometimes denote by $F_o(\Gamma)$ the contour of the outer face for the sake of simplicity. For a cycle C of Γ, we call the plane subgraph of Γ inside C (including C) the *inner subgraph* $\Gamma_I(C)$ for C, and we call the plane subgraph of Γ outside C (including C) the *outer subgraph* $\Gamma_O(C)$ for C. An edge which is incident to exactly one vertex of a cycle C and located outside C is called a *leg* of C. The vertex of C to which a leg is incident is called a *leg-vertex* of C. A cycle C in Γ is called a *k-legged cycle* of Γ if C has exactly k legs and there is no edge which joins two vertices on C and is located outside C [12]. In each of Figs. 2(a) and 2(b), a 3-legged cycle is drawn by thick lines. We call a face F of Γ a *peripheral face* for a 3-legged cycle C in Γ if F is in $\Gamma_O(C)$ and the contour of F contains an edge on C. Clearly there are exactly three peripheral faces for any 3-legged cycle in Γ. In Fig. 2(b), F_1, F_2, F_3 are the three peripheral faces for the 3-legged cycle C drawn by thick lines. A k-legged cycle C is called a *minimal k-legged cycle* if $\Gamma_I(C)$ does not contain any other k-legged cycle of Γ. The 3-legged cycle C drawn by thick lines in Fig. 2(b) is not minimal, while the 3-legged facial cycle C' in Fig. 2(b) is minimal. We say that cycles C and C' in Γ are *independent* if $\Gamma_I(C)$ and $\Gamma_I(C')$ have no common vertex. A set S of cycles is *independent* if every pair of cycles in S are independent. A cycle C in Γ is called *regular* if the plane graph $\Gamma - \Gamma_I(C)$ has a cycle. Similarly an edge of Γ which is incident to exactly one vertex of a cycle C in Γ and located inside C is called a *hand* of C. The vertex of C to which a hand is incident is called a *hand-vertex* of C. A cycle C is called a *k-handed cycle* if C has exactly k hands in Γ and there is no edge which joins two vertices on C and is located inside C. A k-handed cycle C is called a *maximal k-handed cycle* if $\Gamma_O(C)$ does not contain any other k-handed cycle of Γ. We call a k-handed cycle C a *regular k-handed cycle* if $\Gamma - \Gamma_O(C)$ contains a cycle.

We now give some definitions regarding a box-rectangular drawing of a plane graph Γ [11]. We say that a vertex of graph Γ is drawn as a *degenerated box* in a box-rectangular drawing D if the vertex is drawn as a point in D. We often call a degenerated box in D a *point* and call a nondegenerated box a *real box*. We call the rectangle corresponding to $F_o(\Gamma)$ the *outer rectangle*, and we call a corner of the outer rectangle simply a *corner*. A box in D containing at least one corner is called a *corner box*. A corner box may be degenerated. If $n = 1$, that is, Γ has exactly one vertex, then the box-rectangular drawing is trivial: the drawing is just a degenerated box corresponding to the vertex. Thus in the paper, we can assume that $n \geq 2$.

Rahman at al. [11] gave a necessary and sufficient condition for a plane graph Γ to have a box-rectangular drawing, and developed a linear algorithm for finding a drawing of Γ if it exists, as stated in the following lemma.

Lemma 1 *[11] Let G be a connected planar graph with $\Delta \leq 3$, and let Γ be a plane embedding of G. Assume that Γ has neither a vertex of degree 1 nor a 1-legged cycle. Then Γ has a box-rectangular drawing if and only if Γ satisfies the following two conditions:*

(br1) every 2- or 3- legged cycle in Γ contains an edge on $F_o(\Gamma)$; and

(br2) $2c_2 + c_3 \leq 4$ for any independent set ξ of cycles in Γ, where c_2 and c_3 are the numbers of 2- and 3- legged cycles in ξ respectively.

In the problem box-rectangular drawing of a plane graph Γ for $\Delta \geq 4$ Rahman et al. [11] constructed a new plane graph Φ from Γ by replacing each vertex v of degree four or more in Γ by a cycle. Thus $\Delta(\Phi) \leq 3$. The following lemma is their main result for $\Delta \geq 4$.

Lemma 2 *[11] Let Γ be a plane connected graph with $\Delta \geq 4$, and let Φ be the graph transformed from Γ as above. Then Γ has a box-rectangular drawing if and only if Φ has a box-rectangular drawing.*

It is not difficult to derive a characterization of a connected planar graph to have a box-rectangular drawing if we know a characterization of a biconnected planar graph to have a box-rectangular drawing. We can assume in our paper that, G is connected and has neither a vertex of degree 1 nor a 1-legged cycle; otherwise the planar graph G does not have a box-rectangular drawing as all the faces of the graph can not be drawn as rectangles simultaneously. If a planar graph G has neither a vertex of degree 1 nor a 1-legged cycle, and if the graph G is 1-connected, then the cut vertex v must be of degree 4 or more. If G is not a multi graph and has a box-rectangular drawing D_G, then all the cut vertices must reside on the outer face $F_o(D_G)$ of the drawing D_G, and all the biconnected components with respect to cut vertices have box-rectangular drawings separately. If any of the biconnected components contains exactly one cut vertex, then that component must have a box-rectangular drawing where the cut vertex is drawn as a corner box containing two corners. If any of the biconnected components contains two cut vertices, then that component must have a box-rectangular drawing where each corner vertex is drawn as a corner box and each corner box contains exactly two corners. No component contains 3 or more cut vertices; otherwise no box-rectangular drawing exists for G since the outer face $F_o(G)$ must be a rectangle in the drawing. We can obtain a box-rectangular drawing D_G of G by merging the box-rectangular drawings of the biconnected components.

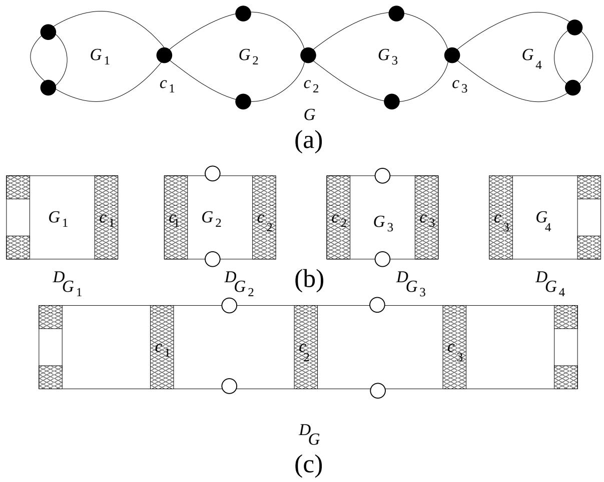

Figure 3: Box-rectangular drawing of a planar graph G with cut vertices.

Figure 3(a) illustrates such a simple planar graph G where c_1, c_2, and c_3 are the cut vertices, and G_1, G_2, G_3, and G_4 are the biconnected components of G with respect to cut vertices. G_1 and G_4 contain one cut vertex each, and G_2 and G_3 contain two cut vertices each. $D_{G_1}, D_{G_2}, D_{G_3}$ and D_{G_4} are the box-rectangular drawings of the biconnected components G_1, G_2, G_3, and G_4, respectively as illustrated in Fig. 3(b). Finally a box-rectangular drawing D_G of the simple planar graph G is illustrated in Fig. 3(c). One can observe that a box-rectangular drawing also exists for G, even if G is a multigraph satisfying the above conditions. Throughout this paper we thus assume that the planar graph G is biconnected.

3 Box-Rectangular Drawings of Planar Graphs with $\Delta \leq 3$

In this section we give a necessary and sufficient condition for a planar graph G with $\Delta \leq 3$ to have a box-rectangular drawing. We first consider the case where G is a subdivision of a planar 3-connected cubic graph. G has an $O(n)$ number of embeddings, one for each face chosen as the outer face. Examining by the linear algorithm in Lemma 1 whether the two conditions (br1) and (br2) hold for each of the $O(n)$ embeddings, one can examine in time $O(n^2)$ whether the planar graph G has a box-rectangular drawing. However, we obtain the following necessary and sufficient condition for G to have a box-rectangular drawing, which leads to a linear-time algorithm to examine whether G has a box-rectangular drawing. We also give a linear-time algorithm to find a drawing if it exists.

Theorem 1 *Let G be a subdivision of a planar 3-connected cubic graph, and let Γ be an arbitrary plane embedding of G.*

(a) Suppose first that G is cyclically 4-edge-connected, that is, Γ has no regular 3-legged cycle. Then the planar graph G has a box-rectangular drawing.

(b) Suppose next that G is not cyclically 4-edge-connected, that is, Γ has a regular 3-legged cycle C. Let F_1, F_2, and F_3 be the three peripheral faces for C, and let Γ_1, Γ_2, and Γ_3 be the plane embeddings of G taking F_1, F_2, and F_3 respectively as the outer face. Then the planar graph G has a box-rectangular drawing if and only if at least one of the three embeddings Γ_1, Γ_2, and Γ_3 has a box-rectangular drawing.

We only prove here Theorem 1(a), the proof of Theorem 1(b) is similar to that of Theorem 3.1(b) in [12]. Before giving a proof of Theorem 1(a), we need the following Lemmas 3 and 4. Lemma 3 is needful to prove the Lemma 4.

Lemma 3 *Let G be a subdivision of planar 3-connected cubic graph, and Γ be an arbitrary plane embedding of G. If G is cyclically 4-edge-connected, then $2c_2 + c_3 \leq 2$ for any independent set ξ of cycles in Γ, where c_2 and c_3 are the numbers of 2- and 3-legged cycles in ξ, respectively.*

Proof: Let G be a subdivision of planar 3-connected cubic graph, and Γ be an arbitrary plane embedding of G. Assume that G is cyclically 4-edge-connected. We first show that Γ does not have two or more independent 2-legged cycles. Assume for a contradiction that Γ has two independent 2-legged cycles, C_1 and C_2. Then removal of the two legs of either C_1 or C_2 leaves a graph with two connected components, each of which has a cycle, contrary to the definition of a cyclically 4-edge-connected graph. Similarly we can prove that Γ can not have two independent 3-legged cycles. Similarly we can also prove that Γ can not have two cycles, one is 2-legged, and another is 3-legged, which are independent. That is, $2c_2 + c_3 \leq 2$ for any independent set ξ of cycles in Γ, where c_2 and c_3 are the numbers of 2- and 3-legged cycles in ξ, respectively. □

Lemma 4 *Let G be a subdivision of planar 3-connected cubic graph. If G is cyclically 4-edge-connected, then all the plane embeddings of the planar graph G, satisfy (br1) and (br2) of Lemma 1.*

Proof: Let Γ be a plane embedding of G. We first show that Γ satisfies (br1) in Lemma 1. Assume for a contradiction that a 2-legged or a 3-legged cycle C has no edge on the outer face of Γ. Then the removal of the legs of C will result in two connected components having cycles, and G would not be a cyclically 4-edge connected graph, a contradiction. By Lemma 3, Γ satisfies (br2) of Lemma 1. $\square$

Proof of Theorem 1(a). By Lemma 4, every plane embedding Γ of G satisfies Conditions (br1) and (br2) of Lemma 1; and hence Γ has a box-rectangular drawing by Lemma 1. Therefore the planar graph G has a box-rectangular drawing. $\square$

We now consider the other case. It can be trivially shown that every biconnected planar graph G having two vertices of degree 3 has a box-rectangular drawing. We may thus assume that G has three or more vertices of degree 3. Then any planar embedding Γ of G has a regular 2-legged cycle; otherwise, G would be a subdivision of a 3-connected cubic graph. In this case we have the following theorem.

Theorem 2 *Let G be a planar biconnected graph with $\Delta \leq 3$ which is not a subdivision of a planar 3-connected cubic graph. Let Γ be a planar embedding of G such that every 2-legged cycle in Γ has leg-vertices on $F_o(\Gamma)$, let Γ have exactly two independent 2-legged cycles, and let C_1 and C_2 be the two minimal 2-legged cycles in Γ. Let $\Gamma_1(=\Gamma)$, Γ_2, Γ_3, and Γ_4 be the four embeddings of G obtained from Γ by flipping $\Gamma_I(C_1)$ or $\Gamma_I(C_2)$ around the the leg vertices of C_1 and C_2. Then G has a box-rectangular drawing if and only if at least one of the four embeddings Γ_1, Γ_2, Γ_3, and Γ_4 has a box-rectangular drawing.*

Using a method similar to that used in the proof of Theorem 3.4 in [12], we can prove Theorem 2. Note that G does not always have such an embedding Γ; if G has no such embedding, then G has no box-rectangular drawing.

4 Box-Rectangular Drawings of Planar Graphs with $\Delta \geq 4$

In this section we give a necessary and sufficient condition for a planar graph G with $\Delta \geq 4$ to have a box-rectangular drawing. We also give a linear-time algorithm to find a drawing if it exists. In Subsection 4.1 we consider the case where G is a subdivision of a planar 3-connected graph with $\Delta \geq 4$ and in Subsection 4.2 we consider the other cases.

4.1 Case for a Subdivision of a Planar 3-Connected Graph with $\Delta \geq 4$

Let G be a subdivision of a planar 3-connected graph with $\Delta \geq 4$, and Γ be an arbitrary plane embedding of G. We construct a new planar graph H from Γ by replacing each vertex v of degree four or more in Γ by a cycle.

Figures 5(a), 5(b), and 5(c) illustrate G, Γ, and H respectively. A replaced cycle corresponds to a real box in a box-rectangular drawing of G. We do not replace a vertex of degree 2 or 3 by a cycle since a vertex of degree 3 may be drawn as a point, and a vertex of degree 2 is always drawn as a point. Thus $\Delta(H) \leq 3$. The following theorem is the main result of this subsection.

Theorem 3 *Let G be a subdivision of a planar 3-connected graph with $\Delta \geq 4$, and Γ be an arbitrary plane embedding of G. Let H be the graph transformed from Γ as above. Then G has a box-rectangular drawing if and only if the planar graph H has a box-rectangular drawing.*

It is rather easy to prove the necessity of Theorem 3.

Proof for Necessity of Theorem 3. Let Γ be an arbitrary plane embed-

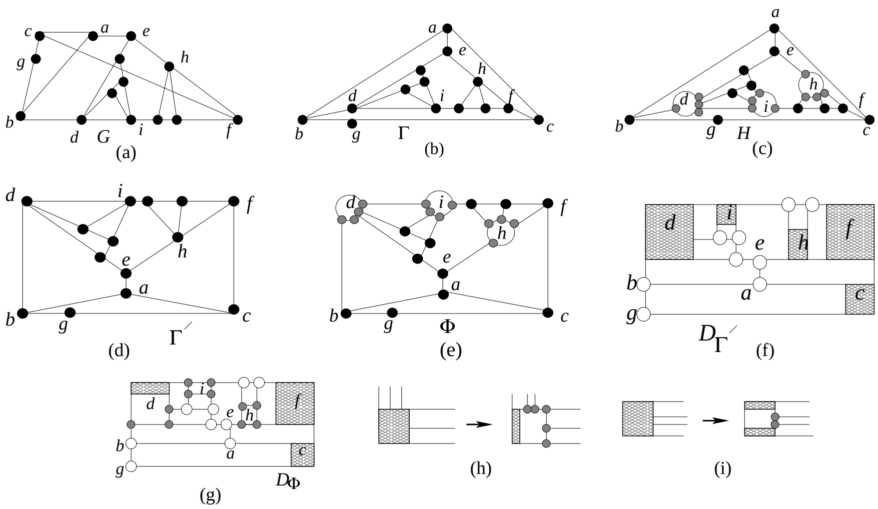

Figure 4: Illustration of G, Γ, H, Γ', Φ, $D_{\Gamma'}$, D_{Φ}, and the two transformations.

ding of the planar graph G. G and Γ are illustrated in Figs. 4(a) and 4(b), respectively. Assume that G has a box-rectangular drawing for any plane embedding Γ' as in Fig. 4(d) of G. Therefore, Γ' has a box-rectangular drawing $D_{\Gamma'}$ in which every vertex of degree 4 or more is drawn as a real box [11], as illustrated in Fig. 4(f). Then, as illustrated in Fig. 4(g), one can obtain a box-rectangular drawing D_{Φ} for the plane graph Φ as in Fig. 4(e) from $D_{\Gamma'}$ by the following transformation, where Φ is actually a plane graph of the planar graph H as illustrated in Fig. 4(c):

(i) regard each noncorner real box in $D_{\Gamma'}$ as a face in D_{Φ};

(ii) if a corner box in $D_{\Gamma'}$ corresponds to a vertex of degree 3 in Γ then regard it as a corner box in D_{Φ}; and

(iii) if a corner box in $D_{\Gamma'}$ corresponds to a vertex of degree four or more in Γ, then transform it to a drawing of a replaced cycle with one or more real boxes as illustrated in Figs. 4(h) and 4(i) where the box in $D_{\Gamma'}$ contains one corner in Fig. 4(h) and contains two corners in Fig. 4(i).

□

Figures 4(f) and 4(g) illustrate $D_{\Gamma'}$ and D_{Φ}, respectively. Box h in $D_{\Gamma'}$ is a noncorner real box, and it is regarded as a face in D_{Φ}. Corner boxes c and f in $D_{\Gamma'}$ correspond to vertices of degree 3, and they remain as boxes in D_{Φ}. Corner box g in $D_{\Gamma'}$ corresponds to a vertex of degree 2, it remains as a degenerated box in D_{Φ}. Corner box d in $D_{\Gamma'}$ correspond to a vertex of degree four or more is transformed to a drawing of a replaced cycle with one real box in D_{Φ} as illustrated in Fig. 4(h).

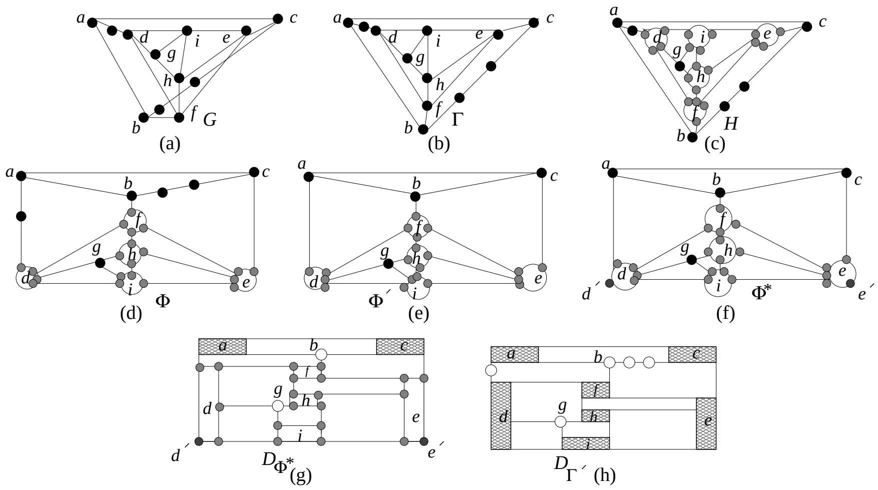

Figure 5: Illustration for a box-rectangular drawing of a subdivision of a planar 3- connected graph G with $\Delta \geq 4$.

We need some definitions before proving the sufficiency. We replace the vertices of degree 4 or more in Γ by cycles. Each vertex of degree 2 or 3 in Γ has a corresponding vertex of the same degree in H, and we call such a vertex in H an *original vertex.* A vertex on a replaced cycle is called a *replaced vertex.* Now each vertex in H is either a replaced vertex or an original vertex.

Assume that, an arbitrary plane embedding Φ of the planar graph H has a box-rectangular drawing D_{Φ}. Therefore, Φ satisfies (br1) and (br2) of Lemma 1. We can easily transform D_{Φ} to a box-rectangular drawing $D_{\Gamma'}$ for any plane embedding Γ' of the planar graph G if only original vertices are drawn as corner boxes in D_{Φ}, because then each replaced vertex is a point in D_{Φ}, and each replaced cycle in Φ is a rectangular face in D_{Φ}, and hence D_{Φ} can be transformed to $D_{\Gamma'}$ by regarding each replaced cycle as a box. The problem is the case where a replaced vertex is drawn as a corner box in D_{Φ}. Because such a drawing D_{Φ} cannot always be transformed to a box-rectangular drawing $D_{\Gamma'}$ of Γ'. However we show that a plane graph Φ^* as illustrated in Fig. 5(f) obtained from Φ as in Fig. 5(d) through an intermediate graph Φ' as in Fig. 5(e) with slight modification has a particular box-rectangular drawing D_{Φ^*} which can be easily transformed to a box-rectangular drawing of Γ' as illustrated in Fig. 5(h). Transformation is also not possible when the outer face of Φ is a replaced cycle. However, we are released from the problem by proving the following Lemmas 5 and 6 which are on a planar graph with $\Delta \geq 4$. Lemma 5 is needful to prove the Lemma 6.

Lemma 5 *Let G be a planar graph with $\Delta \geq 4$, and Γ be an arbitrary plane embedding of G. Let H be the transformed graph of Γ by replacing each vertex v of degree four or more in Γ by a cycle, and Φ be an arbitrary plane embedding of the planar graph H. Denote the total number of 2-legged and 3-legged cycles in Γ by l_Γ, and the total number of 2-handed and 3-handed cycles in Γ by h_Γ. Also denote the total number of 2-legged and 3-legged cycles in Φ by l_ϕ, and the total number of 2-handed and 3-handed cycles in Φ by h_Φ. If $p_\Gamma = l_\Gamma + h_\Gamma$, and $p_\Phi = l_\Phi + h_\Phi$, then $p_\Gamma = p_\Phi$.*

Proof:

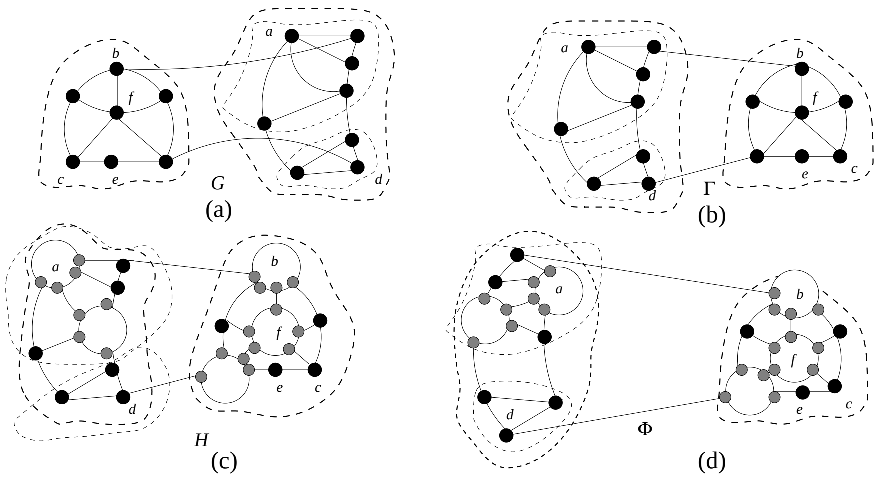

Figure 6: Illustration of Lemma 5.

(See Fig. 6) Since a 2-legged cycle in any plane embedding of G has a corresponding 2-legged or a 2-handed cycle in another plane embedding of G, and a 2-handed cycle in any plane embedding of G has a corresponding 2-handed or a 2-legged cycle in another plane embedding of G, different plane embeddings of a same planar graph do not change the number of 2-legged cycles and 2-handed cycles in total. Similarly, different plane embeddings of a same planar graph do not change the number of 3-legged cycles and 3-handed cycles in total. One can easily observe that, the total number of 2-legged and 2-handed cycles in Φ remains same with the total number of 2-legged and 2-handed cycles in Γ after transformation. Similarly the total number of 3-legged and 3-handed cycles in Φ remains same with the total number 3-legged and 3-handed cycles in Γ, after transformation. Because in Φ every replaced cycle is either a 4- or more handed, or a 4- or more legged cycle. Therefore, if $p_\Gamma = l_\Gamma + h_\Gamma$, and $p_\Phi = l_\Phi + h_\Phi$, then $p_\Gamma = p_\Phi$. □

Lemma 6 *Let G be a planar graph with $\Delta \geq 4$, and Γ be an arbitrary plane embedding of G. Let H be the transformed graph of Γ by replacing each vertex v of degree four or more in Γ by a cycle. Let Φ_R be any arbitrary plane embedding of the planar graph H, such that $F_o(\Phi_R)$ is a replaced cycle in H. Then G is cyclically 4-edge-connected if and only if Φ_R has a box-rectangular drawing.*

Proof: *Necessity.* Assume G is cyclically 4-edge-connected. Then H is cyclically 4-edge connected, since H is obtained from a plane embedding Γ of G by replacing each vertex of degree 4 or more by a cycle. Then by Theorem 1(a), Φ_R has a box-rectangular drawing.

Sufficiency. Assume Φ_R has a box-rectangular drawing. That is, Φ_R satisfies both (br1) and (br2) of Lemma 1. There is no 2- or 3-legged cycle in Φ_R, which does not contain an edge on $C_o(\Phi_R)$. Furthermore, $C_o(\Phi_R)$ is also a 4- or more handed cycle. One can easily observe that removal of any 3 or fewer edges leaves a graph in Φ_R such that exactly one component has a cycle. So Φ_R is cyclically 4-edge-connected. Another plane embedding Φ of H, which does not have a replaced cycle as the outer face is also cyclically 4-edge-connected, as Φ and Φ_R are the two different plane embeddings of the same planar graph H. By Lemma 5, Γ and Φ do not change in total number of 2-legged, 2-handed, 3-legged, and 3-handed cycles. So, Γ is cyclically 4-edge-connected. That is, G is a cyclically 4-edge-connected graph. □

We are now ready to prove the sufficiency of Theorem 3.

Sufficiency of Theorem 3. Assume that H has a box-rectangular drawing. Then H has a plane embedding Φ which has a box-rectangular drawing. We now show that G has a plane embedding Γ' which has a box-rectangular drawing. Before entering into the cases we give a definition. If the outer face of Φ is not a replaced cycle, then the replaced cycle on $F_o(\Phi)$ corresponding to a vertex of degree 4 or more in G contains exactly one edge on $F_o(\Phi)$. We call such an edge in Φ a *green edge*. We have two cases to consider.

Case 1. Φ *does not contain a replaced cycle as the outer face.*

Assume that Φ as in Fig. 5(d) has a box-rectangular drawing. Let Φ' be the minimal graph homeomorphic to Φ as illustrated in Fig. 5(e). Since G is a subdivision of a 3-connected graph, Φ' is a 3-connected cubic graph and there is no 2-legged cycle in Φ'. Since Φ has a box-rectangular drawing, Φ satisfies Conditions (br1) and (br2) in Lemma 1. Hence Φ' also satisfies Conditions (br1) and (br2) in Lemma 1. Using the similar approach used in [11], we can designate four vertices as corner vertices after slight modification in Φ'. Let Φ^* be the resulting graph as illustrated in Fig. 5(f). Note that each of the four designated vertices in Φ^* is either an original vertex or a dummy vertex of degree 2 on a green edge of Φ' [11]. Clearly, every 3-legged cycle in Φ^* contains at least one designated vertex and every 2-legged cycle in Φ^* contains at least three designated vertices. (Although Φ' has no 2-legged cycle, Φ^* may have a 2-legged cycle.) Hence, Φ^* has a box-rectangular drawing $D_\Phi{}^*$ with the four designated vertices as corner boxes, as illustrated in Fig. 5(g). Inserting the removed vertices of degree 2 on some vertical and horizontal line segments in $D_\Phi{}^*$ and regarding the drawing of each replaced cycle as a box, we immediately obtain a box-rectangular drawing $D_{\Gamma'}$ for a plane embedding Γ' of the planar graph G from $D_\Phi{}^*$, as illustrated in Fig. 5(h).

Case 2. Φ *contains a replaced cycle as the outer face.*

In this case $\Phi = \Phi_R$, as in Lemma 6. By Lemma 6, if Φ_R has a box-rectangular drawing, then H is cyclically 4-edge connected. Hence by Theorem 1(a), another plane embedding Γ' of H, whose outer face is not a replaced cycle has also a box-rectangular drawing D. Thus by using the method used in Case 1 we can obtain a box-rectangular drawing of Γ'. □

4.2 The Other Case for a Planar Graph G with $\Delta \geq 4$

It can be trivially shown that every graph G with $\Delta \geq 4$ having two vertices has a box-rectangular drawing. Note that in this case the graph G is a multigraph.

We may thus assume that G is a planar biconnected graph with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph. In this case the following fact holds.

Fact 1 *Let G be a biconnected planar graph with $\Delta \geq 4$. Let Γ_1 and Γ_2 be the two arbitrary plane embeddings of G. A minimal 3-legged cycle in Γ_1 has a corresponding minimal 3-legged or a maximal 3-handed cycle in Γ_2, and a minimal 2-legged cycle in Γ_1 has a corresponding minimal 2-legged or a maximal 2-handed cycle in Γ_2. Similarly a maximal 3-handed cycle in Γ_1 has a corresponding maximal 3-handed or a minimal 3-legged cycle in Γ_2, and a maximal 2-handed cycle in Γ_1 has a corresponding maximal 2-handed or a minimal 2-legged cycle in Γ_2.*

Proof: Let G be a biconnected planar graph with $\Delta \geq 4$. Let Γ_1 and Γ_2 be the two arbitrary plane embeddings of G. Let C_1 be a minimal 3-legged cycle in Γ_1. If any face in $\Gamma_I(C_1)$ of Γ_1 be the outer face of Γ_2, then one can easily observe that C_1 in Γ_1 has a corresponding maximal 3-handed cycle C_2 in Γ_2. If any face in $\Gamma_O(C_1)$ of Γ_1 be the outer face of Γ_2, then one can easily observe that C_1 in Γ_1 has a corresponding minimal 3-legged cycle C_2 in Γ_2. (Note that C_1 in Γ_1 and C_2 in Γ_2 may be the same cycle.) Similar scenario occurs in all other cases also. □

Let G be a planar biconnected graph with $\Delta \geq 4$ but not a subdivision of a planar 3-connected graph, and Γ be an arbitrary plane embedding of G. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pairs of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand-vertices of a regular 2-handed cycle. If there is a plane embedding Γ' of G having a box-rectangular drawing, then the outer face $F_o(\Gamma')$ must contain all vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$; otherwise, Γ' would have a 2-legged cycle containing no vertex on $F_o(\Gamma')$, and Γ' would not have a box-rectangular drawing.

Because after replacing the vertices of degree 4 or more by cycles in Γ', according to Lemma 5, 2-legged cycles will remain same and the total number of 2-legged cycles will also remain same. The graph is denoted by Φ after transformation from Γ'. If Γ' has a 2-legged cycle containing no vertex on $F_o(\Gamma')$, then by Lemma 5, Φ also has a 2-legged cycle containing no vertex on $F_o(\Phi)$. Hence by (br1) of Lemma 1, Φ does not have a box-rectangular drawing, and consequently by Lemma 2, Γ' does not have a box-rectangular drawing. Similarly, if Γ' has a box-rectangular drawing, then by Lemma 2, Φ has a box-rectangular drawing and by [11] exactly two leg vertices of every minimal 3-legged cycle in Φ are on the outer face of the box-rectangular drawing. Thus by Lemma 5, $F_o(\Gamma')$ contains exactly two leg vertices of every minimal 3-legged cycle in Γ'. Hence by Fact 1 exactly two leg vertices of every minimal 3-legged cycle, and exactly two hand vertices of every maximal 3-handed cycle in Γ must be on the outer face $F_o(\Gamma')$.

Let p be the largest integer such that a number p of minimal 2-legged and maximal 2-handed cycles in Γ are independent with each other, and q be the largest integer such that a number q of minimal 3-legged and maximal 3-handed cycles in Γ are independent with each other. If $p > 2$ or $q > 4$, then by [11], there is no plane embedding Γ' of G for which a box-rectangular drawing exists. Assume the worst case, that is, $p = 2$ and $q = 4$ in Γ. Independent minimal 3-legged and maximal 3-handed cycles in Γ are denoted by C_1, C_2, C_3, and C_4. Let $\{a_k, b_k, c_k\}$ be the set of leg vertices or hand vertices in C_k, for $k = 1, 2, 3$, or 4. We can choose two vertices from each C_1, C_2, C_3, or C_4 in 3 ways. The combinations are $\{(a_k, b_k), (b_k, c_k)$, and $(c_k, a_k)\}$, for $k = 1, 2, 3$ or 4. If we want to choose eight vertices from the four cycles, C_1, C_2, C_3, and C_4, two vertices from each C_k, for $k = 1, 2, 3$ and 4, we can choose in 3 x 3 x 3 x 3 = 81 number of ways. The combinations are $S_1 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$, $S_2 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (b_4, c_4)\}$, $S_3 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (c_4, a_4)\}$, ..., and $S_{81} = \{(c_1, a_1), (c_2, a_2), (c_3, a_3), (c_4, a_4)\}$.

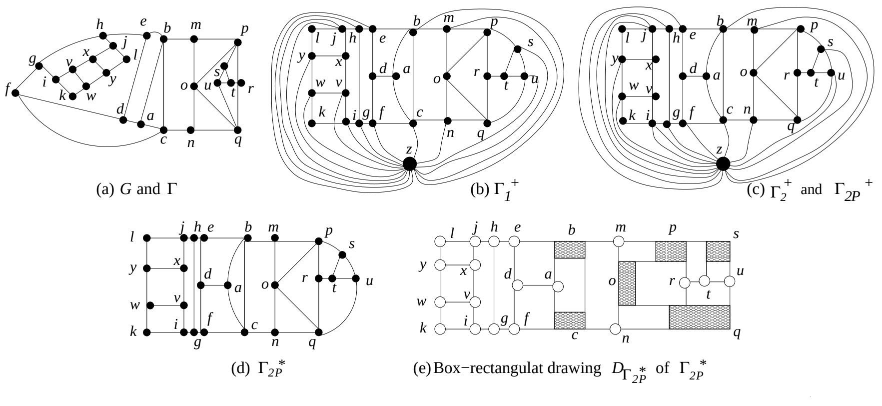

Figure 7: Illustration for a box-rectangular drawing of a biconnected graph G with $\Delta \geq 4$ but not a subdivision of a 3-connected graph .

Let G be a planar biconnected graph with maximum degree 4 or more but not a subdivision of a planar 3-connected graph, and Γ be an arbitrary plane embedding of G as in Fig. 7(a). Let $(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l)$ be all pairs of vertices such that x_i and y_i, $1 \leq i \leq l$, are the leg vertices of a regular 2-legged cycle or the hand-vertices of a regular 2-handed cycle, and $\{a_k, b_k, c_k\}$ be the set of leg vertices or hand vertices in C_k, for $k = 1, 2, 3$ and 4. A dummy vertex z is added in the outer face of Γ. Construct a graph ${\Gamma_j}^+$, for any j = 1, 2, 3, ..., or 81, by adding dummy edges (x_i, z) and (y_i, z) for all indices $i, 1 \leq i \leq l$, and by adding eight dummy edges from z to all vertices in the set S_j. In this way we can get 81 number of graphs ${\Gamma_j}^+$, for $j = 1, 2, 3, \ldots$, and 81. ${\Gamma_1}^+$ and ${\Gamma_2}^+$ are two such graphs as illustrated in Fig. 7(b) and in Fig. 7(c) respectively. G may have a box-rectangular drawing, only if, any one of the graphs ${\Gamma_j}^+$, for $j = 1, 2, 3, \ldots$, and 81, has a planar embedding such that z is embedded in the outer face. ${\Gamma_{2P}}^+$ in Fig. 7(c) is such a planar embedding of the graph ${\Gamma_2}^+$, but ${\Gamma_1}^+$ in Fig. 7(b) has no such a planar embedding. That is why, the planar graph G as illustrated in Fig. 7(a) may have a box-rectangular drawing. Delete the dummy vertex z from ${\Gamma_{2P}}^+$. The graph is then called ${\Gamma_{2P}}^*$ as in Fig. 7(d). Lastly by Lemma 2 and by the approach used in [11] we can test in linear time whether the plane graph ${\Gamma_{2P}}^*$ has a box-rectangular drawing and find a drawing if it exists. $D_{{\Gamma_{2P}}^*}$ is a box rectangular of the plane graph ${\Gamma_{2P}}^*$ as well as of the planar graph G, as illustrated in Fig. 7(e).

We thus have the following theorem.

Theorem 4 *Let G be a planar biconnected graph with $\Delta \geq 4$ which is not a subdivision of a planar* 3*-connected graph. Then one can determine whether G has a box-rectangular drawing or not by checking at most* 81 *graphs constructed from G as mentioned above. Furthermore, each of the* 81 *graphs can be checked in linear time.*

5 Conclusion

In this paper we addressed the problem for finding box-rectangular drawings of planar graphs. We gave a necessary and sufficient condition for a planar graph to have a box-rectangular drawing and developed a linear-time algorithm for finding a drawing if it exists. In this paper we have shown that, at most 81 graphs are required to be checked to take a decision whether a planar biconnected graph with $\Delta \geq 4$ has a box-rectangular drawing or not. In future one may try to minimize the number of graphs required to be checked to take the decision whether the planar biconnected graph with $\Delta \geq 4$ has a box-rectangular drawing or not.

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References

[1] T. C. Biedl. Optimal orthogonal drawings of triconnected plane graphs. In *Proceedings of SWAT (96)*, volume 1097 of *Lecture Notes in Computer Science*, pages 333–344. Springer-Verlag, Berlin/New York, 1996. doi:10.1007/3-540-61422-2_143.

[2] A. L. Buchbaum, E. R. Gansner, C. M. Procopiuc, and S. Venkatasubramanian. Rectangular layouts and contact graphs. *ACM Transactions on Algorithms*, 4(1):8.1–8.28, 2008. doi:10.1145/1328911.1328919.

[3] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing*. Prentice Hall, Englewood Cliffs, New Jersey, 1999.

[4] M. M. Hasan, M. S. Rahman, and M. R. Karim. Box-rectangular drawings of planar graphs. In *Proceedings of the 7th International Workshop on Algorithms and Computation (WALCOM 2013)*, volume 7748 of *Lecture Notes in Computer Science*, pages 334–345. Springer-Verlag, 2013. doi:10.1007/978-3-642-36065-7_31.

[5] X. He. A simple linear time algorithm for proper box rectangular drawings of plane graphs. *Journal of Algorithms*, 40(1):82–101, 2001. doi:10.1006/jagm.2001.1161.

[6] G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16:4–32, 1996. doi:10.1007/BF02086606.

[7] K. Kozminski and E. Kinnen. An algorithm for finding a rectangular dual of a planar graph for use in area planning for VLSI integrated circuits. In *Proceedings of 21st DAC, Albuquerque*, pages 655–656, 1984. doi:10.1109/DAC.1984.1585872.

[8] T. Lengauer. *Combinatorial Algorithms for Integrated Circuit Layout*. Wiley, Chichester, 1990.

[9] S. Munemoto, N. Katoh, and G. Imamura. Finding an optimal floor layout based on an orthogonal graph drawing algorithm. *Journal of Architecture, Planning and Environmental Engineering (Transactions of AIJ)*, 524:279–286, 2000.

[10] T. Nishizeki and M. S. Rahman. *Planar Graph Drawing*. World Scientific, Singapore, 2004.

[11] M. S. Rahman, S. Nakano, and T. Nishizeki. Box-rectangular drawings of plane graphs. *Journal of Algorithms*, 37:363–398, 2000. doi:10.1006/jagm.2000.1105.

[12] M. S. Rahman, T. Nishizeki, and S. Ghosh. Rectangular drawings of planar graphs. *Journal of Algorithms*, 50:62–78, 2004. doi:10.1016/S0196-6774(03)00126-3.

[13] R. Tamassia, I. G. Tollis, and J. S. Vitter. Lower bounds for planar orthogonal drawings of graphs. *Information processing Letters*, 39:35–40, 1991. doi:10.1016/0020-0190(91)90059-Q.

[14] K. Tani, S. Tsukiyama, S. Shinoda, and I. Shirakawa. On area-efficient drawings of rectangular duals for VLSI floor-plan. *Mathematical Programming*, 52:29–43, 1991. doi:10.1007/BF01582878.

[15] C. Thomassen. Plane cubic graphs with prescribed face areas. *Combinatorics, Probability and Computing*, 1:371–381, 1992. doi:10.1017/S0963548300000407.