Not all planar graphs are in PURE-4-DIR

Daniel Gonçalves [1]

[1]LIRMM, Univ Montpellier, CNRS, Montpellier, France.

Abstract

We prove that some planar graphs are not intersection graphs of segments if only four slopes are allowed for the segments, and if parallel segments do not intersect. This refutes a conjecture of D. West [14].

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1   Introduction

The *intersection graphs of segments* class SEG is a widely studied class of graphs. Any graph $G \in$ SEG admits a representation in the plane, where each vertex $v \in V(G)$ corresponds to a segment $\mathbf{v}$, and where two segments $\mathbf{u}$ and $\mathbf{v}$ intersect if and only if $uv \in E(G)$. The class SEG can be refined by bounding the number of slopes used in the representation. A graph belongs to k-DIR if it has an intersection representation with segments using at most k different slopes. Actually for any $k \geq 1$ [11],

$$k\text{-DIR} \subsetneq (k+1)\text{-DIR} \subsetneq SEG.$$

One should be aware that here the set of allowed slopes is not set, and that for $k \geq 4$ the choice of the slopes would lead to different graph classes [6]. Let us also introduce the class PURE-k-DIR of graphs having a *pure k-DIR* representation, that is where parallel segments do not intersect.

The relation between planar graphs and these intersections graphs goes back to the PhD thesis of E.R. Scheinerman [13]. He proved that outerplanar graphs belong to PURE-3-DIR and he conjectured that every planar graph belongs to SEG. This conjecture was confirmed by J. Chalopin and the author [1], and more recently by the author, L. Isenmann, and C. Pennarun [8] using a simpler method.

D. West conjectured a strengthening of Scheinerman's conjecture [14]. He asked whether every planar graph has a PURE-4-DIR representation. In a pure representation parallel segments form a stable set. Thus this conjecture would also strengthen the 4-color theorem. Actually planar graphs with a smaller chromatic number have such property. Indeed for any $k \leq 3$, k-colored planar graphs belong to PURE-k-DIR [4, 5, 7, 9]. We here refute West's conjecture.

Theorem 1 *There exist planar graphs that do not belong to PURE-4-DIR.*

In the following section we study the PURE-4-DIR representations of some planar gadget. We then prove Theorem 1 in Section 3 using signed planar graphs that are not 4-colorable [10] (in the sense of signed graphs).

2   Planar gadget

Consider any set of four slopes $\{s_0, s_1, s_2, s_3\}$, ordered such that drawing four concurrent lines $\ell_0, \ell_1, \ell_2, \ell_3$ with these slopes and going around the intersection point one successively crosses $\ell_0, \ell_1, \ell_2, \ell_3, \ell_0, \ell_1, \ell_2$, and ℓ_3. As there exists a linear mapping that maps s_0 to the horizontal slope, and s_2 to the vertical slope, we have that a graph has a 4-DIR (resp. a PURE-4-DIR) representation using slopes $\{s_0, s_1, s_2, s_3\}$ if and only if it has a 4-DIR (resp. a PURE-4-DIR) representation using the horizontal s_-, some negative slope $s_\searrow$, the vertical $s_|$, and some positive slope $s_\nearrow$. Thus in the following we only consider the slopes $s_-, s_\searrow, s_|$, and $s_\nearrow$. In the pictures $s_\searrow$, and $s_\nearrow$ form $45°$ angles with s_- and $s_|$,

but all the claims hold for any negative slope $s_\searrow$, and for any positive slope $s_\nearrow$. Let us now study the PURE-4-DIR representations of the graph H depicted in Figure 1.
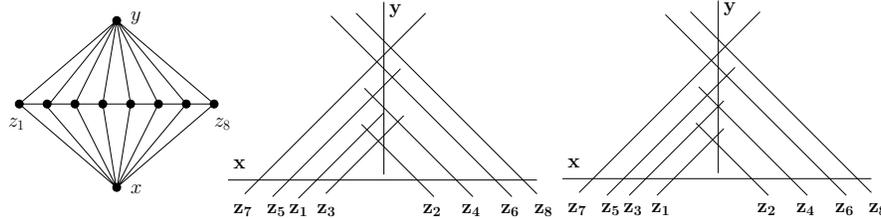

Figure 1: The graph H, and two of its PURE-4-DIR representations.

Consider a PURE-4-DIR representation of H where the vertices x and y are represented with segments $\mathbf{x}$ and $\mathbf{y}$ of slopes s_- and $s_|$. As the considered representation is pure, we have that the segments $\mathbf{z_i}$ with i even have the same slope among $s_\searrow$ and $s_\nearrow$, and that the other slope is the one of all the segments $\mathbf{z_i}$ with i odd. Let $\ell_\mathbf{x}$ and $\ell_\mathbf{y}$ be the lines supporting $\mathbf{x}$ and $\mathbf{y}$, respectively. Note that the intersection point of $\ell_\mathbf{x}$ and $\ell_\mathbf{y}$, $\mathbf{p}$, belongs to exactly one of $\mathbf{x}$ and $\mathbf{y}$. Indeed, it cannot belong to both as $\mathbf{x}$ and $\mathbf{y}$ should not intersect. One the other hand, if it belongs to none of them, say that $\mathbf{x}$ and $\mathbf{y}$ are on the left and above $\mathbf{p}$ respectively, then there cannot be segments of slope $s_\searrow$ intersecting both of them, a contradiction. We can thus define triangles $\triangle_{ij}$ as follows.

Definition 2 *Given a PURE-4-DIR representation of H where the vertices x and y are represented with segments $\mathbf{x}$ and $\mathbf{y}$ of slopes $s_|$ and $s_|$, $\triangle_{ij}$, with $1 \le i = j - 1 < 8$, is the triangle formed by the segments $\mathbf{z_i}$, $\mathbf{z_j}$, and $\mathbf{x}$ (resp. and $\mathbf{y}$) if the intersection point of $\ell_\mathbf{x}$ and $\ell_\mathbf{y}$, $\mathbf{p}$, belongs to $\mathbf{x}$ (resp. $\mathbf{y}$).*

Remark 3 *Consider any PURE-4-DIR representation of H where the vertices x and y are represented with segments $\mathbf{x}$ and $\mathbf{y}$ of slopes s_- and $s_|$, and where $\triangle_{45}$ is bounded by $\mathbf{z_4}$, $\mathbf{z_5}$, and $\mathbf{x}$. Then, for every vertex z that is adjacent to both x and y, but to none of z_4 and z_5, we have either that*

- *$\triangle_{45}$ contains $\mathbf{y} \cap \mathbf{z}$, the intersection point of $\mathbf{y}$ and $\mathbf{z}$, or that*

- *$\triangle_{45}$ does not intersect $\mathbf{z}$.*

The following lemma tells us that both cases occur among $\mathbf{z_1}$ and $\mathbf{z_8}$.

Lemma 4 *Consider any PURE-4-DIR representation of H where the vertices x and y are represented with segments $\mathbf{x}$ and $\mathbf{y}$ of slopes s_- and $s_|$, and where $\triangle_{45}$ is bounded by $\mathbf{z_4}$, $\mathbf{z_5}$, and $\mathbf{x}$. Then either*

- *$\triangle_{45}$ contains $\mathbf{y} \cap \mathbf{z_1}$ and does not intersect $\mathbf{z_8}$, or*

- *$\triangle_{45}$ contains $\mathbf{y} \cap \mathbf{z_8}$ and does not intersect $\mathbf{z_1}$.*

Proof: Let us consider the triangles $\triangle_{12}$, $\triangle_{45}$, and $\triangle_{78}$, that are all bounded by **x**.

Claim 5 *One of $\triangle_{45}$ and $\triangle_{12}$ (resp. one of $\triangle_{45}$ and $\triangle_{78}$, and one of $\triangle_{12}$ and $\triangle_{78}$) contains the other.*

This is obvious as otherwise $\mathbf{z_4}$ or $\mathbf{z_5}$ would intersect $\mathbf{z_1}$ or $\mathbf{z_2}$ (resp. $\mathbf{z_4}$ or $\mathbf{z_5}$ would intersect $\mathbf{z_7}$ or $\mathbf{z_8}$, and $\mathbf{z_1}$ or $\mathbf{z_2}$ would intersect $\mathbf{z_7}$ or $\mathbf{z_8}$).

Claim 6 *Triangles $\triangle_{12}$ and $\triangle_{78}$ cannot both contain (resp. be contained in) $\triangle_{45}$.*

Towards a contradiction assume that the triangles $\triangle_{12}$ and $\triangle_{78}$ both contain (resp. are contained in) $\triangle_{45}$. By symmetry of H, assume that $\triangle_{45} \subsetneq \triangle_{78} \subsetneq \triangle_{12}$ (resp. $\triangle_{12} \subsetneq \triangle_{78} \subsetneq \triangle_{45}$). Note that in that case, the segment $\mathbf{z_3}$ that has to intersect $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z_2}$, and $\mathbf{z_4}$ necessarily intersects $\mathbf{z_8}$, a contradiction.

By the above two claims, either $\triangle_{12} \subsetneq \triangle_{45} \subsetneq \triangle_{78}$ or $\triangle_{78} \subsetneq \triangle_{45} \subsetneq \triangle_{12}$. By Remark 3, this clearly implies the lemma. □

3   Proof of Theorem 1

A *signed graph G* is a graph where the edges are signed by a mapping $\sigma : E(G) \longrightarrow \{-, +\}$. A signed graph is $2k$-colorable if there exists a mapping $c : V(G) \longrightarrow \{-k, 1-k, \ldots, -1, 1, \ldots, k-1, k\}$ such that for each edge $uv \in E(G)$, $c(u) \neq \sigma(uv)c(v)$. This definition of signed coloring appeared in [12] and note that it is distinct from the one in [15].

Our construction is based on the construction G of Kardoš and Narboni [10] depicted in Figure 2. This signed plane graph is not 4-colorable, and this contradicts a conjecture of E. Máčajová, A. Raspaud, and M. Škoviera [12]. Note that every non-signed plane graph is 4-colorable in the classical sense. We hence have the following.

Remark 7 *For every classical 4-coloring c, with colors $-2, -1, +1, +2$, of the plane graph induced by the positive (i.e. black) edges of G, one of the negative (i.e. red) edges uv is such that $c(u) = -c(v)$.*

We define the (non-signed) graph G^H from G by replacing each negative (i.e. red) edge uv by a copy of H, identifying the vertices x and y of H with the vertices u and v respectively, and by then triangulating the obtained plane graph without creating a new edge among the original vertices of G (See Figure 2 right).

Towards a contradiction consider a PURE-4-DIR representation of G^H. Consider the 4-coloring c of G^H, provided by the mapping that associates the slopes $s_|, s_\diagdown, s_\diagup$, and s_-, with $-2, -1, +1$, and $+2$ respectively. By Remark 7, let us consider the copy of H such that $c(x) = -c(y)$. We can assume that the segments **x** and **y** have slopes s_- and $s_|$, respectively (*i.e.* we assume that $c(x) = -c(y) = 2$). If the segments **x** and **y** have a different pair of slopes
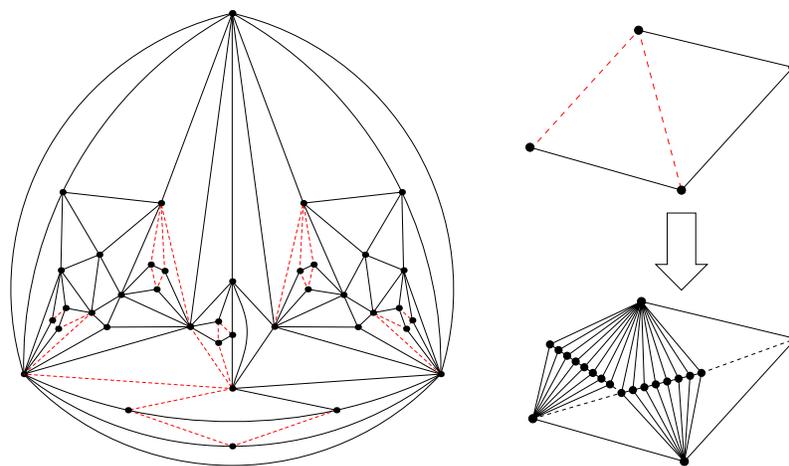
Figure 2: Left: The construction G of Kardoš and Narboni [10], where black and red edges are positive and negative, respectively. Right: Replacing each red edge with a copy of H, and triangulating (with dashed edges).

(*i.e.* if $c(x) = -c(y) = -2$ or if $c(x) = -c(y) = \pm 1$), there exists a linear mapping of the plane that would turn the given PURE-4-DIR representation into a PURE-4-DIR representation with the mentionned slopes for $\mathbf{x}$ and $\mathbf{y}$. Let us also assume, by symmetry of vertices x and y in H, that $\triangle_{45}$ is bounded by $\mathbf{x}$, $\mathbf{z_4}$, and $\mathbf{z_5}$.

By Lemma 4, we have some constraints on the locations of $\mathbf{z_1}$ and $\mathbf{z_8}$, and let us assume by symmetry of vertices z_1 and z_8 in H, that $\triangle_{45}$ contains the point $y \cap z_1$ and does not intersect the segment z_8. As G^H is a triangulation, there is a $z_1 z_8$-path P contained in $N(y) \setminus \{z_2, \ldots, z_7\}$, where $N(y)$ is the set of y's neighbors in G^H (see Figure 3). Moreover by Remark 3, P has a subpath P' whose ends, z' and z'' are common neighbors of x and y, while none of the internal vertices is in $N(x)$, and such that $\triangle_{45}$ contains the point $y \cap z'$, and does not intersect the segment z''.

As $\mathbf{z''}$ lies outside $\triangle_{45}$, and as none of the internal vertices of P' is in $N(x)$, $N(z_4)$ or $N(z_5)$, all the corresponding segments lie outside $\triangle_{45}$ too. In particular, this is the case for the neighbor of z' in P', w. Note that $\mathbf{w}$ should intersect $\mathbf{y}$ and $\mathbf{z'}$, so it cannot use the same slope, but it can use slope s_-. In that case $\mathbf{w}$ cannot intersect $\mathbf{y}$ and $\mathbf{z'}$ (see Figure 3), a contradiction. Thus G^H does not admit a PURE-4-DIR representation.

4   Conclusion

It now remains open whether plane graphs belong to PURE-k-DIR, for some constant $k > 4$. Actually it is even open whether every plane graph with
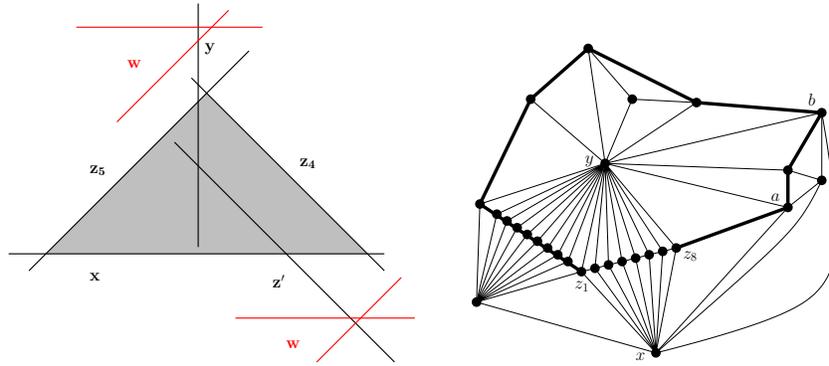
Figure 3: Left: The 4-DIR representation of H with $\triangle_{45}$, $\mathbf{y}$ and $\mathbf{z}'$. The red segments are possible positions for $\mathbf{w}$, not intersecting $\triangle_{45}$, and with slopes s_- or $s_{\diagup}$. In none of the cases $\mathbf{w}$ intersects $\mathbf{y}$ and $\mathbf{z}'$. Right: Neighborhood of y in G^H. The path P corresponds to bold edges and the common neighbors of x and y are z_1, z_8, a and b. So the ends of P' should belong to $\{z_1, z_8, a, b\}$.

n vertices belongs to PURE-$(n - f(n))$-DIR, for some function f tending to infinity. Another question is whether plane graphs belong to k-DIR, for some $k > 0$. In a non-pure k-DIR representation, parallel segments induce an interval representation. Actually, if every planar graph belongs to k-DIR, for some $k > 0$, we can ask parallel segments to induce just a path forest. Indeed, given a plane graph G, let G' be the graph G where for each vertex v, a leaf v' has been attached. Note that G' is planar, and that its k-DIR representations are such that for each $v \in V(G)$ the segment $\mathbf{v}$ has a point (in $\mathbf{v} \cap \mathbf{v}'$) that does not belong to any other segment $\mathbf{u}$ for $u \in V(G)$. The interval representations where each interval has a private point clearly correspond to path forests. As some plane graphs do not admit a vertex partition into two path forests [2,3] we have that some planar graphs do not admit a 2-DIR representation. An anonymous referee provided us an example, described below, that does not admit a 3-DIR representation. Up to our knowledge it is open to know whether every plane graph belongs to 4-DIR.

Consider the planar graph obtained from K_4 by stacking, for each face (including the external face), a new vertex connected to the three incident vertices. Proceed to another step of vertex stacking in every face. As observed earlier, if planar graphs belong to 3-DIR, then our graph has a 3-DIR representation such that each slope induces a path forest. Consider the initial K_4 and denote its vertices v_1, v_2, v_3, v_4.

If the corresponding four segments use the three different slopes, say $\mathbf{v_1}$, $\mathbf{v_2}$ are horizontal, $\mathbf{v_3}$ is vertical, and $\mathbf{v_4}$ is diagonal. In that case, note that $\mathbf{v_3}$ and $\mathbf{v_4}$ intersect $\mathbf{v_1}$ and $\mathbf{v_2}$ on their common part. Consider the vertex w stacked onto v_1, v_3, v_4. Note that $\mathbf{w}$ cannot be horizontal as it should intersect $\mathbf{v_1}$ and $\mathbf{v_3}$ but avoid $\mathbf{v_2}$. Note that $\mathbf{w}$ cannot be vertical neither, as it should intersect

$\mathbf{v_3}$ and $\mathbf{v_1}$ but avoid $\mathbf{v_2}$. Similarly, $\mathbf{w}$ cannot be diagonal.

If the corresponding four segments $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, and $\mathbf{v_4}$ use only two slopes, say $\mathbf{v_1}$, $\mathbf{v_2}$ are horizontal, and $\mathbf{v_3}$, $\mathbf{v_4}$ are vertical. Note that neccessarily $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, and $\mathbf{v_4}$ all meet in a common point. Consider again the vertex w stacked onto v_1, v_3, v_4. the segment $\mathbf{w}$ can neither be horizontal, nor vertical as it avoids $\mathbf{v_2}$. But then v_1, v_3, v_4, w form a K_4 with three slopes and the above argument applies.

Acknowledgments

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