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BOOK REVIEW

Geometric Mechanics, Part I: Dynamics and Symmetry, Part II: Rotating, Translating and Rolling, by Darryl D. Holm, Imperial College Press, London, 2008. Part I: xx + 354 pages, ISBN-978-1-84816-195-5, Part II: xvi + 294 pages, ISBN-978-1-84816-155-9. Distributed by World Scientific Publishing Co., Singapore.

1. Presentation of the Book

Geometric Mechanics is currently an important subject of research, in pure and applied mathematics as well as in engineering science. Most existing textbooks on the subject [1–3, 6, 7] are suited for graduate students or professional researchers. In writing the present book, Professor Darryl D. Holm successfully undertook the difficult and important task of presenting the main ideas of Geometric Mechanics so that they can be understood by undergraduate students.

The first volume, *Dynamics and Symmetry*, is based on a course taught by the author for undergraduates in their third year of mathematics at the Imperial College London. In that volume, the ideas of reduction by symmetry and reconstruction are presented and used for several very different problems: geometric optics, the motion of a rigid body, ideal fluid dynamics, resonance of coupled oscillators, elastic spherical pendulum, laser light interaction with matter. The necessary mathematical tools (differential forms, Lie groups and Lie algebras, symplectic structures, Poisson brackets) are introduced when needed. The symmetry group most often used for reduction and reconstruction is the one-dimensional circle S^1.

The second volume, *Rotating, Translating and Rolling*, is based on a course of thirty three lectures taught by the author to fourth year undergraduate students in their last term in applied mathematics at the London Imperial College. The symmetry groups encountered in that volume are of higher dimensions: the rotation group $\mathrm{SO}(3)$ or more generally $\mathrm{SO}(n)$, the group of Euclidean displacements $\mathrm{SE}(3)$, the Galilean group, the unitary group $\mathrm{U}(n)$ are defined and used for several problems, including the motions of a heavy top and of round rolling bodies. The mathematical tools presented in the first volume are, for a large part, discussed again from scratch in the second. Results already obtained in the first volume are not used in the second for the treatment of problems, such as the free motion of a rigid body, dealt with in both volumes. So each volume can be read independently of the other.

In both volumes, many exercises are proposed throughout the text, some of them with hints, others with full answers. Appendix C of the second volume proposes very interesting and pleasing generalizations of the mechanical systems previously considered: motions in an alien worlds in which the time is two-dimensional, or in which the space is modelled on $\mathbb{C}^3$ instead of on $\mathbb{R}^3$, or in which the space is four-dimensional.

2. Contents of the First Volume

The first volume is organized into six chapters and one appendix. The following sentence, taken from the author's preface, perfectly describes its spirit:

"The text surveys a small section of the road to geometric mechanics, by treating several examples in classical mechanics all in the same geometric framework."

Chapter 1 deals with Geometric Optics. Fermat's principle of stationarity of the optical length is presented and used to derive the *eikonal equation*. It is the name in Optics of the Euler-Lagrange equation, for the *optical Lagrangian* which corresponds to Fermat's principle. Huygens wavelets, the Huygens-Fermat complementarity theorem, Snell's law of refraction are discussed. The propagation of light in axisymmetric optical instruments, as well as in a general three-dimensional medium, is considered. Several important mathematical notions are presented in this long chapter: Hamilton-Jacobi equation, Legendre transformation, symplectic structures, Poisson brackets, Hamiltonian vector fields, Lie groups and Lie algebras, momentum maps. Light rays in an axisymmetric medium satisfy Hamilton's equation, the abscissa along the symmetry axis playing the role of time. The invariance of optical properties of the medium under rotations around the axis gives rise to a conserved quantity, the *skewness*, which was discovered by Lagrange and its analogue for a mechanical system with a symmetry axis is the angular momentum's component on that axis. When the optical properties of the medium are invariant under translation in the direction of the axis, another quantity is conserved, the Hamiltonian, exactly as the energy of a time-independent mechanical system is conserved. An example of singular Lagrangian appears with the study of the propagation of light in an anisotropic medium.

Undergraduate students will probably think that this first chapter is rather difficult, because of its richness. However, all the new concepts which appear in it will be seen again in the following chapters.

Chapter 2, “Newton, Lagrange, Hamilton and the Rigid Body”, describes the fundamental laws of motion as they were stated by Newton. These laws are then used for the derivation of the equations of motion of a system of material points. The principle of Galilean relativity, the existence of inertial frames are stated and used to prove that the centre of mass of a closed system moves with a constant velocity. The free rigid rotation of a system around its centre of mass is defined, and its angular momentum is expressed in terms of its moments of inertia and its angular velocity. Several new mathematical concepts are then introduced: smooth manifolds, their tangent and cotangent bundles, vector fields on a manifold and their integral curves. With these concepts, following the method initially used by Lagrange [5], the author expresses (without naming it) the *Lagrange differential* which gives, in terms of local coordinates, the infinitesimal work rate of accelerations of a system of material points for a given infinitesimal virtual displacement. That result proves that the *Lagrange equations* are equivalent to equations obtained by application of Newton’s laws of motion and directly leads to *Hamilton’s principle of stationary action*. The motion of a free particle in a Riemannian space is then discussed, and the motion of a rigid body is shown to be equivalent to the geodesic flow on the Lie group $\mathrm{SO}(3)$ equipped with a left-invariant Riemannian metric. Euler’s equations of motion (written in matrix form on the Lie algebra $\mathfrak{so}(3)$) are derived. Then the author defines the Legendre transformation and describes the transition from Lagrangian to Hamiltonian dynamics. He proves a *phase space action principle* involving the *constrained action*

$$S = \int_{t_a}^{t_b} \left(L(q,\dot{q}) + p\left(\frac{\mathrm{d}q}{\mathrm{d}t} - \dot{q}\right)\right) \mathrm{d}t = \int_{t_a}^{t_b} \left(p\frac{\mathrm{d}q}{\mathrm{d}t} - H(q,p)\right) \mathrm{d}t$$

in which the equality $\dot{q} = \dfrac{\mathrm{d}q}{\mathrm{d}t}$ appears as a constraint and the momentum p as a Lagrange multiplier. This phase space action principle appears several times in the second part of the book under the name of *Hamilton-Pontryagin principle*.

Next, the author defines *canonical transformations* and proves that the flow of a Hamiltonian vector field is a family of such transformations, parametrized by the time. The comparison of Lagrange’s and Hamilton’s approaches for deriving the equations of motion of various mechanical systems (a bead sliding on a rotating hoop, the spherical pendulum) is proposed as exercises. The motion of a free rigid body is fully solved, using the method presented by the author and Jerry Marsden in [4].

Chapter 3, “Lie, Poincaré, Cartan: Differential Forms”, introduces the general concept of a symplectic manifold (already encountered, for special examples, in Chapter 1). The fact that the flow of an Hamiltonian vector field on a cotangent bundle is a family of canonical transformations, already stated in Chapter 2, is presented again for general symplectic manifolds under the name of *Poincaré's theorem*: Hamiltonian flows are symplectic. Then the author develops the general formalism of differential forms, defines the Liouville one-form of a cotangent bundle and the cotangent lift of a diffeomorphism of the base manifold. Hamilton-Jacobi equation and generating functions, already encountered in Chapter 1 for the special example of ray optics, are presented in a more general setting. Differential calculus with differential forms (wedge product, exterior derivative, Lie derivative with respect to a vector field, Stokes theorem) is sketched, with several examples given as exercises.

A long section, devoted to Euler's equations of motion of an ideal, incompressible fluid in a rotating frame, offers an opportunity to use these mathematical tools. Kelvin's circulation theorem, steady flows and their stability, Lamb surfaces and the helicity integral are discussed. The Hodge star operator, the codifferential and the Laplace-Beltrami operator are then defined and at its end the Poincaré lemma (local exactness of closed differential forms) is stated and illustrated by examples.

Chapter 4, "Resonances and S^1 Reduction", studies the resonance of two coupled nonlinear oscillators. This Hamiltonian system has $\mathbb{C}^2$ as phase space, on which the Lie group $\mathrm{U}(2)$ acts by an Hamiltonian action. The author calls $1:1$ *resonant dynamics* the Hamiltonian vector field on $\mathbb{C}^2$ associated to the diagonal action of S^1, and fully describes the $\mathrm{U}(2)$-action on $\mathbb{C}^2$. He explains how, after S^1-reduction, one has to reconstruct the phases of oscillating solutions, and how the phase can be split into a *geometric* part and a *dynamic* part. The Poincaré sphere and the Hopf map play an important part in the geometric analysis of the problem.

A long section reviews the work of M. Kummer on $n:m$-resonances of any type. Various physical applications (quantum computing, MASER dynamics, travelling wave pulses in optical fibres, polarization optics) are evocated. In the reviewer opinion, this chapter is the richest and the most difficult of the book.

Chapter 5, "Elastic Spherical Pendulum", is of easier access because it begins with a clear description of the mechanical system under study. Newton, Lagrange and Hamilton's methods are successively used to derive the equations of motion. The system invariance under rotation around the vertical axis through the fixed point is discussed, along with the corresponding conserved quantity (the vertical component of the angular momentum). Small motions around the stable equilibrium position are then considered. The author describes the Lagrangian averaging method: he assumes that the variations of the coordinates of the moving mass can be described as approximately periodic, with slowly varying amplitudes and a rapidly varying phase factor. He considers the $1:1:2$ resonant motions, in which the period of oscillations of the z coordinate is half that of the x and y coordinates. The resulting equations of motion, called the *three-wave interaction equations*, appear in several physical applications: waves in plasmas, laser-matter interaction. A detailed geometric picture of the motions by *three-wave surfaces* is obtained by using the reduction technique, followed by reconstruction. Equations which govern the precession of the oscillation plane are discussed in the last section.

Chapter 6, "Maxwell-Bloch Laser-matter Equations", begins with a short description of a physical phenomenon, *self-induced transparency*: a travelling electromagnetic wave excites the atoms of a resonant dielectric medium. After a delay, the energy temporarily stored by the atoms is returned to the trailing edge of the travelling electromagnetic wave. Maxwell-Schrödinger equations, which govern the phenomenon, are written for a plane wave, and are shown to obey a variational principle. Approximations similar to those made in the previous chapter lead to a Hamiltonian system, the *Maxwell-Schrödinger envelope equations*. The S^1-symmetry allows the reduction of that system (the Hopf map appears again in the reduction procedure), and yields the *complex Maxwell-Bloch equations*. In the remainder of the chapter, the author considers the subsystem obtained when the variables involved are assumed to be real-valued. It is a first order differential system on $\mathbb{R}^3$ which is Hamiltonian with respect to a continuous family of Poisson structures parametrized by the Lie group $\mathrm{SL}(2,\mathbb{R})$. The author then presents a very nice geometric description of the level sets of the conserved functions, of the phase portrait of the system and of the phase of periodic solutions, similar to that he made with Jerry Marsden for the motion of a rigid body in [4].

Appendix A, "Enhanced Coursework", offers as exercises, in a first section, several examples of dynamical systems: among others, a particle in a potential well, a bead sliding on a rotating hoop, the spherical pendulum, the heavy top. Hints, or for the most difficult full solutions, are given. The following sections contain more detailed discussions of questions dealt with in the previous chapters: canonical transformations, complex phase spaces, resonant oscillators, . . .

3. Contents of the Second Volume

The second volume is organized into twelve chapters and three appendices. As said by the author in its Preface, its aim is to explain the following statement so that it may be understood by undergraduate students in mathematics, physics and engineering:

"Lie symmetry reduction on the Lagrangian side produces the Euler-Poincaré equation, whose formulation on the Hamiltonian side as a Lie-Poisson equation

governs the dynamics of the momentum map associated with the cotangent lift of the Lie-algebra action of that Lie symmetry on the configuration manifold."

Chapter 1, "Galileo", briefly introduces the concepts of time, space and motion in the classical (non-Einsteinian) Mechanics. The principle of the Galilean relativity is stated. The groups of the three-dimensional rotations $\mathrm{SO}(3)$, of Euclidean displacements $\mathrm{SE}(3)$ and their actions on space are defined. The general notion of a semi-direct product of groups is given. The Galilean group, denoted by $\mathrm{G}(3)$ (why not $\mathrm{G}(3,1)$, since it involves time as well as space?) and its action on space-time is defined. Of course, a good part of these notions were already given in the first volume. They are described again without reference to their first appearance.

The much longer Chapter 2, "Newton, Lagrange and Hamilton's Treatments of the Rigid Body", introduces the mathematical tools used for dealing with rigid body dynamics. The author calls *hat map* the correspondence which associates to each vector in the three-dimensional oriented Euclidean vector space an element of the Lie algebra of the infinitesimal rotations of that space (a 3×3 skew-symmetric matrix, once an orthonormal basis of that space is chosen). Several important notions already seen in Chapter 2 of the first volume are treated again with more details: the equations of motion of a free rigid body, Lagrange's equations, Hamilton's principle of stationary action, the phase-space action principle (called in this second volume the Hamilton-Pontryagin principle), the Legendre transformation, Hamilton's equations, Poisson brackets. Lie symmetries are defined, Emmy Noether's theorem which associates conservation laws to such symmetries is stated and proven. Manakov's formulation of the $\mathrm{SO(n)}$ rigid body motion is presented as a first example of the Euler-Poincaré equations, a special form of the Euler-Lagrange equations discovered by Poincaré [8] in 1901 for an action integral involving paths in a Lie algebra.

Chapters 3 and 4, "Quaternions" and "Quaternionic Conjugacy and Adjoint Actions", use Pauli matrices to describe the algebra of quaternions and to derive its main properties. The author explains how pure quaternions can be used to represent vectors in $\mathbb{R}^3$ and to write Newton's equations of motion. The Kepler problem (motion of a planet around the Sun) is treated as an example. The Cayley-Klein coordinates of a quaternion, their use to describe the three-dimensional rotations and rigid body dynamics, the links between quaternionic conjugacy and the Lie groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are explained. The Hopf map $\mathrm{S}^3 \to \mathrm{S}^2$ is discussed again in this new context. Action of unit quaternions by conjugation is taken as a model to introduce the adjoint action of a Lie group on its Lie algebra. Formulae for the derivative of a smooth path in a Lie group and of its pull-back to its Lie algebra by right or left translations are discussed, and the coadjoint action of a Lie group on the dual space of its Lie algebra is defined.

Adjoint and coadjoint actions of two Lie groups specially important in Geometric Mechanics are fully discussed in the two following chapters: “The Special Orthogonal Group $\mathrm{SO}(3)$” and “The Special Euclidean Group $\mathrm{SE}(3)$, Adjoint and Coadjoint Actions”. Orbits of motion of a free rigid body are shown to be intersections of energy surfaces of the Hamiltonian with $\mathrm{SO}(3)$-coadjoint orbits. Formulae for the adjoint and coadjoint actions of a semi-direct product of Lie groups are derived and used for the group $\mathrm{SE}(3)$ of Euclidean displacements.

Chapter 7, “Euler-Poincaré and Lie-Poisson Equations on $\mathrm{SE}(3)$”, and Chapter 8, “Heavy Top Equations”, discuss systems whose configuration space is a Lie group. The Lagrangian of such a system is defined on the tangent space to that Lie group so that Euler-Lagrange equations can be written under a remarkable form, involving the Lie algebra of the group, discovered by Poincaré [8] and called by the author *Euler-Poincaré equations*. After a Legendre transformation, these equations are written, in the Hamiltonian formalism, on the dual of the Lie algebra and involve its canonical Lie-Poisson structure. The heavy top’s configuration space is $\mathrm{SO}(3)$, and the symmetry group of its Lagrangian is the one-dimensional subgroup S^1 of rotations around the vertical axis. In Chapter 8 the author writes that this *symmetry breaking* explains why the Lie-Poisson equations of the heavy top are written on the dual space of the semi-direct product Lie algebra $\mathfrak{s}e(3)$. The reviewer prefers the explanation, which rests on the notion of *completely orthogonal Hamiltonian actions*, given in his book with Paulette Libermann [6] (Chapter IV, theorem 7.12 page 233 and section 10.3 page 254). The author describes the *Clebsch action principle* for the heavy top which is similar to the Hamilton-Pontryagin action principle of Chapter 2, written with different variables. The last section “Kaluza-Klein Construction and the Heavy Top”, suspends the heavy top Hamiltonian system in a higher dimensional phase space.

Chapter 9, “The Euler-Poincaré Theorem”, presents Euler-Poincaré equations, already used in the preceding chapters, in a more general setting, and shows that the Hamilton-Pontryagin principle can be used to derive these equations. The author also discuss an *implicit variational principle* called the “Clebsch Euler-Poincaré principle”, which yields additional informations about the momentum map.

Chapter 10, “Lie-Poisson Hamiltonian Form of a Continuum Spin Chain”, deals with an infinite-dimensional Hamiltonian system: its configuration space is the set of smooth maps defined on an interval with values in a Lie group G. The Euler-Poincaré procedure can be extended and yields Hamiltonian partial-differential equations. The Lie group G is finally taken to be $\mathrm{SO}(3)$ and several possible Lagrangians are discussed.

In Chapter 11, "Momentum Maps", the notions of an Hamiltonian action of a Lie group on a symplectic manifold and of its momentum map are defined, and some of their properties are discussed. Of course, these notions appeared in almost all the preceding chapters. This is a deliberate choice made by the author: mathematical concepts are used in examples before the statements of their definitions and general properties. This choice, probably adequate for many readers, may be disturbing for the most mathematically minded. Tangent and cotangent lifts of a Lie group action on a configuration manifold are defined, and properties of the corresponding momentum map are discussed. The Hopf map again appears when the momentum map of the $\mathrm{SU}(2)$-action on $\mathbb{C}^2$ is determined, and the calculation of several Hamiltonian Lie groups actions are proposed as exercises.

The last chapter deals with round rigid bodies rolling on a flat surface. Two non-holonomic mechanical systems are considered: the Chaplygin's top (a non-homogeneous heavy sphere) and the Euler's disk (a heavy disk), both rolling without sliding on a horizontal plane. The equations of motion are deduced from a *non-holonomic Hamilton-Pontryagin variational principle*. These equations are the same as those deduced from the Euler-Lagrange equations with, in the right hand side, Lagrange multipliers to account for the constraint forces. When the Chaplygin's top mass distribution has a cylindrical symmetry, the system has an additional constant of motion called *Jellet's integral*. The last section of the chapter, *Non-holonomic Euler-Poincaré Reduction*, develops a reduction procedure adapted to non-holonomic mechanical systems.

Mathematical tools (smooth manifolds, tangent and cotangent bundles, vector fields and their flows, Lie groups and Lie algebras, action of a Lie group on a smooth manifold and its tangent and cotangent lifts) are presented, with many examples, in Appendices A and B. As we already spoke, in the general presentation of the book, of Appendix C which offers, as exercises, very interesting and pleasing generalizations of the mechanical systems considered in the book.

4. Conclusion

The two volumes of this book are in fact two separate books. Although the same subjects are often dealt with in both volumes, each one can be read independently of the other. Both are very rich in examples, concepts and physical applications. They will certainly be very useful for many students, and will help to make better known the problems and methods of Geometric Mechanics.

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