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THE PTOLEMAEAN INEQUALITY IN H–TYPE GROUPS

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Abstract. We prove the Ptolemaean inequality and the Theorem of Ptolemaeus in the setting of H–type Iwasawa groups.

1. Introduction

The purpose of this short note is to give an elementary proof of a generalization of the Ptolemaean inequality in the context of H–type Iwasawa groups. These groups are precisely the Iwasawa $\mathfrak{n}$–components of simple Lie algebras of real rank one [1, p.704], thus we recover the results in [7] with a considerably simplified proof.

The Ptolemaean inequality in planar Euclidean geometry states that given a quadrilateral, then the product of the lengths of the diagonals is less or equal to the sum of the products of the lengths of its opposite sides. Moreover, equality holds if and only if the quadrilateral is inscribed in a circle. Many authors have proved generalization of the Ptolemaean inequality in various settings (e.g. normed spaces [8], CAT(0) spaces [2], Möbius spaces [3]).

Let (X, d) be a metric space. The metric d is called *Ptolemaean* if any four distinct points p_1, p_2, p_3 and p_4 in X satisfy the Ptolemaean inequality; that is, for any permutation (i, j, k, l) in the permutation group S_4 we have

$$d(p_i, p_k) \cdot d(p_j, p_l) \leq d(p_i, p_j) \cdot d(p_k, p_l) + d(p_j, p_k) \cdot d(p_l, p_i). \qquad (1)$$

In a Ptolemaean space (X, d), we are most interested in the sets where Ptolemaean inequality holds as an equality (Ptolemaeus' Theorem). A subset Σ of X is called a *Ptolemaean circle* if for any four distinct points $p_1, p_2, p_3, p_4 \in \Sigma$ such that p_1 and p_3 separate p_2 and p_4 we have

$$d(p_1, p_3) \cdot d(p_2, p_4) = d(p_1, p_2) \cdot d(p_3, p_4) + d(p_2, p_3) \cdot d(p_4, p_1).$$

Our main theorem is the following.

Theorem 1. *Let G be an $H-$type Iwasawa group. Then the metric d defined in* (9) *is Ptolemaean and its Ptolemaean circles are $\mathbb{R}-$circles.*

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