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# SEVERAL INTEGRAL INEQUALITIES

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ABSTRACT. In the article, some integral inequalities are presented by analytic approach and mathematical induction. An open problem is proposed.

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## 1. SEVERAL INTEGRAL INEQUALITIES

In this article, we establish some integral inequalities by analytic method and induction.

**Proposition 1.1.** *Let $f(x)$ be differentiable on $(a,b)$ and $f(a)=0$. If $0 \leqslant f'(x) \leqslant 1$, then*

$$\int_a^b \big[f(x)\big]^3 \,\mathrm{d}x \leqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^2. \tag{1.1}$$

*If $f'(x) \geqslant 1$, then inequality* (1.1) *reverses. The equality in* (1.1) *holds only if $f(x) \equiv 0$ or $f(x) = x-a$.*

*Proof.* For $a \leqslant t \leqslant b$, set

$$F(t) = \left(\int_a^t f(x)\,\mathrm{d}x\right)^2 - \int_a^t \big[f(x)\big]^3\,\mathrm{d}x.$$

Simple computation yields

$$F'(t) = \left\{2\int_a^t f(x)\,\mathrm{d}x - \big[f(t)\big]^2\right\} f(t) \triangleq G(t)f(t),$$
$$G'(t) = 2\big[1-f'(t)\big]f(t).$$

Since $f'(t) \geqslant 0$ and $f(a)=0$, thus $f(t)$ is increasing and $f(t) \geqslant 0$.

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(1) When $0 \leqslant f'(t) \leqslant 1$, we have $G'(t) \geqslant 0$, $G(t)$ increases and $G(t) \geqslant 0$ because of $G(a) = 0$, hence $F'(t) = G(t)f(t) \geqslant 0$, $F(t)$ is increasing. Since $F(a) = 0$, we have $F(t) \geqslant 0$, and $F(b) \geqslant 0$. Therefore, the inequality (1.1) holds.
(2) When $f'(t) \geqslant 1$, we have $G'(t) \leqslant 0$, $G(t)$ decreases, $G(t) \leqslant 0$, $F'(t) \leqslant 0$, and $F(t)$ is decreasing, then $F(t) \leqslant 0$, the inequality (1.1) reverses.
(3) Since the equality in (1.1) holds only if $f'(t) = 1$ or $f(t) = 0$, substitution of $f(t) = t+c$ into (1.1) and standard argument leads to $c = -a$.

The proof is completed. □

**Corollary 1.2** ([3, p. 624]). *Let $f(x)$ be a continuous function on the closed interval $[0,1]$ and $f(0) = 0$, its derivative of the first order is bounded by $0 \leqslant f'(x) \leqslant 1$ for $x \in (0,1)$. Then*

$$\int_0^1 \big[f(x)\big]^3 \,\mathrm{d}x \leqslant \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2. \tag{1.2}$$

*Equality in (1.2) holds if and only if $f(x) = 0$ or $f(x) = x$.*

**Proposition 1.3.** *Suppose $f(x)$ has continuous derivative of the $n$-th order on the interval $[a,b]$, $f^{(i)}(a) \geqslant 0$ and $f^{(n)}(x) \geqslant n!$, where $0 \leqslant i \leqslant n-1$, then*

$$\int_a^b \big[f(x)\big]^{n+2} \,\mathrm{d}x \geqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^{n+1}. \tag{1.3}$$

*Proof.* Let

$$H(t) = \int_a^t \big[f(x)\big]^{n+2} \,\mathrm{d}x - \left[\int_a^t f(x)\,\mathrm{d}x\right]^{n+1}, \quad t \in [a,b]. \tag{1.4}$$

Direct calculation produces

$$\begin{aligned}
H'(t) =& \left\{ \big[f(x)\big]^{n+1} - (n+1)\left[\int_a^t f(x)\,\mathrm{d}x\right]^n \right\} f(t) \triangleq h_1(t)f(t),\\
h_1'(t) =& (n+1)\left\{ \big[f(x)\big]^{n-1} f'(t) - n\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n-1} \right\} f(t) \triangleq (n+1)h_2(t)f(t),\\
h_2'(t) =& \left\{ \big[f(x)\big]^{n-2} f''(t) + (n-1)\big[f(t)\big]^{n-3}\big[f'(t)\big]^2 \right.\\
&\left. - n(n-1)\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n-2} \right\} f(t) \triangleq h_3(t)f(t).
\end{aligned}$$

By induction, we obtain

$$h_i'(t) = \left\{ f^{(i)}(t)\big[f(t)\big]^{n-i} + p_i(t) - \frac{n!}{(n-i)!}\left[\int_a^t f(x)\,\mathrm{d}x\right]^{n-i} \right\} f(t) \triangleq h_{i+1}(t)f(t), \tag{1.5}$$

where $2 \leqslant i \leqslant n$ and

$$\begin{aligned}
p_2(t) &= (n-1)\big[f(t)\big]^{n-3}\big[f'(t)\big]^2,\\
p_{i+1}(t)f(t) &= p_i'(t) + (n-i)f^{(i)}(t)\big[f(t)\big]^{n-i-1} f'(t).
\end{aligned} \tag{1.6}$$

From $f^{(n)}(t) \geqslant n!$ and $f^{(i)}(a) \geqslant 0$ for $0 \leqslant i \leqslant n-1$, it follows that $f^{(i)}(t) \geqslant 0$ and are increasing for $0 \leqslant i \leqslant n-1$.

Using mathematical induction, it is easy to see that

$$p_i(t) = \sum_{j_0+\sum\limits_{k=1}^{i-1} k\cdot j_k = n-1} C(j_0, j_1, \dots, j_{i-1}) \prod_{k=0}^{i-1} \left[f^{(k)}(t)\right]^{j_k},$$

where $j_k$ and $C(j_0, j_1, \dots, j_{i-1})$ are nonnegative integers, $0 \leqslant k \leqslant i-1$.

Therefore, we obtain $p_k'(t) \geqslant 0$ and $p_{k+1}(t) \geqslant 0$, then $p_{k-1}'(t)$ and $p_k(t)$ are increasing for $2 \leqslant k \leqslant n$. Straightforward computation yields

$$h_{n+1}(t) = f^{(n)}(t) + p_n(t) - n!.$$

Considering $f^{(n)}(t) \geqslant n!$, we get $h_{n+1}(t) \geqslant 0$, and $h_n'(t) \geqslant 0$, then $h_n(t)$ increases.

By our definitions of $h_i(t)$, we have, for $1 \leqslant i \leqslant n-1$,

$$h_{i+1}(a) = f^{(i)}(a)\left[f(a)\right]^{n-i} + p_i(a) \geqslant 0.$$

Therefore, using induction on $i$, we obtain $h_i'(t) \geqslant 0$, $h_i(t) \geqslant 0$, and $h_i(t)$ are increasing for $1 \leqslant i \leqslant n$. Then $H'(t) \geqslant 0$ and increases, and $H(t) \geqslant 0$. The inequality (1.3) follows from $H(b) \geqslant 0$. Thus, Proposition 1.3 is proved. □

**Corollary 1.4.** *Let $f(x)$ be $n$-times differentiable on $[a,b]$, $f^{(i)}(a) \geqslant 0$ and $f^{(n)}(x) \geqslant n!$ for $0 \leqslant i \leqslant n-1$. Then the functions $H(t)$, $h_j(t)$ and $p_k(t)$ defined by the formulae* (1.4), (1.5) *and* (1.6) *are increasing and convex, where $1 \leqslant j \leqslant n-1$ and $2 \leqslant k \leqslant n-2$.*

**Remark 1.5.** The inequality (1.3) is not found in [1, 2, 4, 5]. So maybe it is a new inequality.

Lastly, we propose the following open problem:

**Theorem 1.6** (Open Problem)**.** *Under what conditions does the inequality*

$$\int_a^b \left[f(x)\right]^t \,\mathrm{d}x \geqslant \left(\int_a^b f(x)\,\mathrm{d}x\right)^{t-1} \tag{1.7}$$

*hold for $t > 1$* ?

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