

Journal of Inequalities in Pure and Applied Mathematics

COMMENTS ON SOME ANALYTIC INEQUALITIES

ILKO BRNETIĆ AND JOSIP PEČARIĆ

Faculty of Electrical Engineering and Computing,
University of Zagreb
Unska 3, Zagreb, CROATIA.
EMail: ilko.brnetic@fer.hr

Faculty of Textile Technology,
University of Zagreb
Pierottijeva 6, Zagreb, CROATIA.
EMail: pecaric@mahazu.hazu.hr
URL: <http://mahazu.hazu.hr/DepMPCS/indexJP.html>



volume 4, issue 1, article 20,
2003.

*Received 5 January, 2003;
accepted 3 February, 2003.*

Communicated by: S.S. Dragomir

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

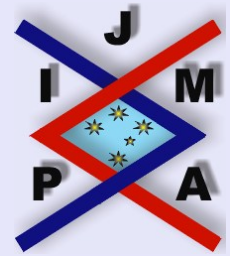
Some interesting inequalities proved by Dragomir and van der Hoek are generalized with some remarks on the results.

2000 Mathematics Subject Classification: 26D15.

Key words: Convex functions.

Contents

1	Comments and Remarks on the Results of Dragomir and van der Hoek	3
2	Main Results	6
	References	



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 2 of 11

1. Comments and Remarks on the Results of Dragomir and van der Hoek

The aim of this paper is to discuss and improve some inequalities proved in [1] and [2]. Dragomir and van der Hoek proved the following inequality in [1]:

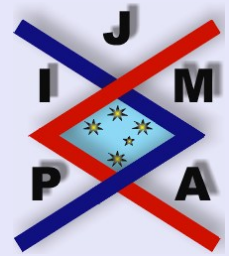
Theorem 1.1 ([1], Theorem 2.1.(ii)). *Let n be a positive integer and $p \geq 1$ be a real number. Let us define $G(n, p) = \sum_{i=1}^n i^p / n^{p+1}$, then $G(n+1, p) \leq G(n, p)$ for each $p \geq 1$ and for each positive integer n .*

The most general result obtained in [1] as a consequence of Theorem 1.1 is the following:

Theorem 1.2 ([1], Theorem 2.8.). *Let n be a positive integer, $p \geq 1$ and $x_i, i = 1, \dots, n$ real numbers such that $m \leq x_i \leq M$, with $m \neq M$. Let $G(n, p) = \sum_{i=1}^n i^p / n^{p+1}$, then the following inequalities hold*

$$(1.1) \quad G(n, p) \left(mn^{p+1} + \frac{1}{(M-m)^p} \left(\sum_{i=1}^n x_i - mn \right)^{p+1} \right) \\ \leq \sum_{i=1}^n i^p x_i \\ \leq G(n, p) \left(Mn^{p+1} - \frac{1}{(M-m)^p} \left(Mn - \sum_{i=1}^n x_i \right)^{p+1} \right).$$

The inequality (1.1) is sharp in the sense that $G(n, p)$, depending on n and



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 3 of 11

p , cannot be replaced by a bigger constant so that (1.1) would remain true for each $x_i \in [0, 1]$.

For $M = 1$ and $m = 0$, from (1.1), it follows that (with assumptions listed in Theorem 1.2)

$$G(n, p) \left(\sum_{i=1}^n x_i \right)^{p+1} \leq \sum_{i=1}^n i^p x_i \leq G(n, p) \left(n^{p+1} - \left(n - \sum_{i=1}^n x_i \right)^{p+1} \right).$$

Let us also mention the inequalities obtained for the special case $p = 1$:

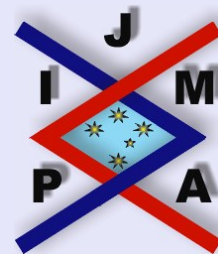
$$(1.2) \quad \frac{1}{2} \left(1 + \frac{1}{n} \right) \left(\sum_{i=1}^n x_i \right)^2 \leq \sum_{i=1}^n i x_i \leq \frac{1}{2} \left(1 + \frac{1}{n} \right) \left(2n \sum_{i=1}^n x_i - \left(\sum_{i=1}^n x_i \right)^2 \right).$$

The sharpness of inequalities (1.2) could be proven directly by putting $x_i = 1$ for every $i = 1, \dots, n$.

For $\sum_{i=1}^n x_i = 1$, from (1.2), the estimates of expectation of a guessing function are obtained in [1]:

$$(1.3) \quad \frac{1}{2} \left(1 + \frac{1}{n} \right) \leq \sum_{i=1}^n i x_i \leq \frac{1}{2} \left(1 + \frac{1}{n} \right) (2n - 1).$$

Similar inequalities for the moments of second and third order are also derived in [1].



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 4 of 11

Inequalities (1.3) are obviously not sharp, since for $n \geq 2$

$$\sum_{i=1}^n ix_i > \sum_{i=1}^n x_i = 1 > \frac{1}{2} \left(1 + \frac{1}{n} \right),$$

and

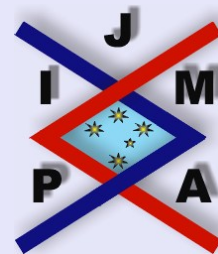
$$\sum_{i=1}^n ix_i < n \sum_{i=1}^n x_i = n < \frac{1}{2} \left(1 + \frac{1}{n} \right) (2n - 1).$$

More generally, for $S = \sum_{i=1}^n x_i$, $n \geq 2$, the obvious inequalities

$$(1.4) \quad \sum_{i=1}^n ix_i > \sum_{i=1}^n x_i = S, \quad \sum_{i=1}^n ix_i < n \sum_{i=1}^n x_i = nS$$

give better estimates than (1.2) for $S \leq 1$.

We improve the inequality (1.2) with a constant depending not only on n , but on $\sum_{i=1}^n x_i$. Our first result is a generalization of Theorem 1.1.



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 5 of 11

2. Main Results

We generalize Theorem 1.1 by taking

$$F(n, p, a) = \frac{\sum_{i=1}^n f(i)}{nf(n)}, \quad f(i) = (i + a)^p$$

instead of $G(n, p)$. Obviously, we have $F(n, p, 0) = G(n, p)$. By obtaining the same result as that mentioned in Theorem 1.1 with F instead of G , we can find a for which we obtain the best estimates for inequalities of type (1.2).

Theorem 2.1. *Let $n \geq 2$ be an integer and $p \geq 1$, $a \geq -1$ be real numbers. Let us define $F(n, p, a) = \sum_{i=1}^n (i + a)^p / n(n + a)^p$, then $F(n + 1, p, a) \leq F(n, p, a)$ for each $p \geq 1$, $a \geq -1$ and for each integer $n \geq 2$.*

Proof. We compute

$$\begin{aligned} & F(n, p, a) - F(n + 1, p, a) \\ &= \frac{\sum_{i=1}^n (i + a)^p}{n(n + a)^p} - \frac{\sum_{i=1}^{n+1} (i + a)^p}{(n + 1)(n + 1 + a)^p} \\ &= \sum_{i=1}^n (i + a)^p \left(\frac{1}{n(n + a)^p} - \frac{1}{(n + 1)(n + 1 + a)^p} \right) - \frac{1}{n + 1} \\ &= \frac{1}{n + 1} \left(F(n, p, a) \frac{(n + 1)(n + 1 + a)^p - n(n + a)^p}{(n + 1 + a)^p} - 1 \right). \end{aligned}$$

So, we have to prove

$$F(n, p, a) \geq \frac{(n + 1 + a)^p}{(n + 1)(n + 1 + a)^p - n(n + a)^p},$$



Comments on Some Analytic
Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 6 of 11

or equivalently, (for $n \geq 2$),

$$(2.1) \quad \sum_{i=1}^n (i+a)^p \geq \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p}.$$

We prove inequality (2.1) for each positive integer n by induction. For $n = 1$ we have

$$1 \geq \frac{(2+a)^p}{2(2+a)^p - (1+a)^p},$$

which is obviously true.

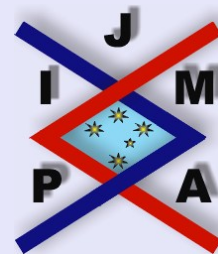
Let us suppose that for some n the inequality

$$\sum_{i=1}^n (i+a)^p \geq \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p}$$

holds.

We have

$$\begin{aligned} \sum_{i=1}^{n+1} (i+a)^p &= \sum_{i=1}^n (i+a)^p + (n+1+a)^p \\ &\geq \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p} + (n+1+a)^p \\ &= \frac{(n+1)(n+1+a)^{2p}}{(n+1)(n+1+a)^p - n(n+a)^p}. \end{aligned}$$



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 7 of 11

In order to show

$$\sum_{i=1}^{n+1} (i+a)^p \geq \frac{(n+1)(n+1+a)^p(n+2+a)^p}{(n+2)(n+2+a)^p - (n+1)(n+1+a)^p}$$

we need to prove the following inequality

$$\frac{(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p} \geq \frac{(n+2+a)^p}{(n+2)(n+2+a)^p - (n+1)(n+1+a)^p},$$

i.e.

$$(n+2+a)^p \frac{(n+1+a)^p + n(n+a)^p}{n+1} \geq (n+1+a)^{2p}.$$

or

$$(2.2) \quad \frac{((n+2+a)(n+1+a))^p + n((n+2+a)(n+a))^p}{n+1} \geq (n+1+a)^{2p}.$$

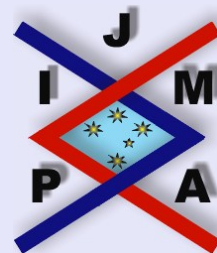
Since $f(x) = (x+a)^p$ is convex for $p \geq 1$ and $x \geq -a$, applying Jensen's inequality we have

$$L \geq \left(\frac{(n+2+a)(n+1+a) + n(n+2+a)(n+a)}{n+1} \right)^p,$$

where L denotes the left hand side in (2.2). To prove (2.2) it is sufficient to prove the inequality

$$(n+2+a)(n+1+a) + n(n+2+a)(n+a) \geq (n+1)(n+1+a)^2,$$

which is true for $a \geq -1$. □



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 8 of 11

Remark 2.1. We did not allow $n = 1$, since $F(1, p, -1)$ is not defined.

Following the same idea given in [1], we can derive the following results:

Theorem 2.2. Let $F(n, p, a)$ be defined as in Theorem 2.1, $x_i \in [0, 1]$ for $i = 1, \dots, n$ and $S = \sum_{i=1}^n x_i$, then

$$(2.3) \quad F(n, p, a) \cdot S \cdot f(S) \leq \sum_{i=1}^n f(i)x_i \leq F(n, p, a) \cdot (nf(n) - (n - S)f(n - S)),$$

where $f(n) = (n + a)^p$.

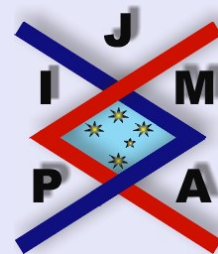
Proof. The first inequality can be proved in exactly the same way as was done in [1] (Th.2.3). The second inequality follows from the first by putting $a_i = 1 - x_i \in [0, 1]$, and then $x_i = a_i$. \square

The special case of this result improves the inequality (1.2):

Corollary 2.3. Let $n \geq 2$ be an integer, $x_i \in [0, 1]$ for $i = 1, \dots, n$ and $S = \sum_{i=1}^n x_i$, then

$$(2.4) \quad \frac{1}{2} \left(1 + \frac{1}{S} \right) \leq \frac{\sum_{i=1}^n ix_i}{S^2} \leq \frac{1}{2} \left(\frac{2n+1}{S} - 1 \right).$$

Proof. Let $a = -1$ and $p = 1$. We compute $F(n, 1, -1) = \frac{1}{2}$. Inequality (2.4) now follows from (2.3) after some computation. \square



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

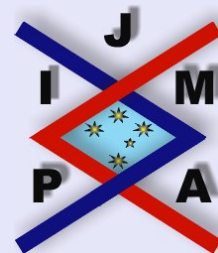
Close

Quit

Page 9 of 11

We can now compare inequalities (2.4) and (1.2); the estimates in (2.4) are obviously better.

In comparing with obvious inequalities (1.4), the estimates in (2.4) are better for $S > 1$ (they coincide for $S = 1$).



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 10 of 11

References

- [1] S.S. DRAGOMIR AND J. VAN DER HOEK, Some new analytic inequalities and their applications in guessing theory *JMAA*, **225** (1998), 542–556.
- [2] S.S. DRAGOMIR AND J. VAN DER HOEK, Some new inequalities for the average number of guesses, *Kyungpook Math. J.*, **39**(1) (1999), 11–17.



Comments on Some Analytic Inequalities

Ilko Brnetić and Josip Pečarić

Title Page

Contents



Go Back

Close

Quit

Page 11 of 11