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COMMENTS ON SOME ANALYTIC INEQUALITIES

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Abstract

Some interesting inequalities proved by Dragomir and van der Hoek are generalized with some remarks on the results.

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Contents

1. Comments and Remarks on the Results of Dragomir and van der Hoek

The aim of this paper is to discuss and improve some inequalities proved in [1] and [2]. Dragomir and van der Hoek proved the following inequality in [1]:

Theorem 1.1 ([1], Theorem 2.1.(ii)). *Let n be a positive integer and $p \geq 1$ be a real number. Let us define $G(n,p) = \sum_{i=1}^{n} i^p / n^{p+1}$, then $G(n+1,p) \leq G(n,p)$ for each $p \geq 1$ and for each positive integer n.*

The most general result obtained in [1] as a consequence of Theorem 1.1 is the following:

Theorem 1.2 ([1], Theorem 2.8.). *Let n be a positive integer, $p \geq 1$ and x_i, $i = 1, \ldots, n$ real numbers such that $m \leq x_i \leq M$, with $m \neq M$. Let $G(n,p) = \sum_{i=1}^{n} i^p / n^{p+1}$, then the following inequalities hold*

$$G(n,p)\left(mn^{p+1} + \frac{1}{(M-m)^p}\left(\sum_{i=1}^{n} x_i - mn\right)^{p+1}\right) \leq \sum_{i=1}^{n} i^p x_i \leq G(n,p)\left(Mn^{p+1} - \frac{1}{(M-m)^p}\left(Mn - \sum_{i=1}^{n} x_i\right)^{p+1}\right). \tag{1.1}$$

The inequality (1.1) is sharp in the sense that $G(n,p)$, depending on n and

p, cannot be replaced by a bigger constant so that (1.1) would remain true for each $x_i \in [0,1]$.

For $M = 1$ and $m = 0$, from (1.1), it follows that (with assumptions listed in Theorem 1.2)

$$G(n,p)\left(\sum_{i=1}^{n} x_i\right)^{p+1} \le \sum_{i=1}^{n} i^p x_i \le G(n,p)\left(n^{p+1} - \left(n - \sum_{i=1}^{n} x_i\right)^{p+1}\right).$$

Let us also mention the inequalities obtained for the special case $p = 1$:

$$\frac{1}{2}\left(1+\frac{1}{n}\right)\left(\sum_{i=1}^{n} x_i\right)^2 \le \sum_{i=1}^{n} i x_i \le \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2n\sum_{i=1}^{n} x_i - \left(\sum_{i=1}^{n} x_i\right)^2\right). \tag{1.2}$$

The sharpness of inequalities (1.2) could be proven directly by putting $x_i = 1$ for every $i = 1, \dots, n$.

For $\sum_{i=1}^{n} x_i = 1$, from (1.2), the estimates of expectation of a guessing function are obtained in [1]:

$$\frac{1}{2}\left(1+\frac{1}{n}\right) \le \sum_{i=1}^{n} i x_i \le \frac{1}{2}\left(1+\frac{1}{n}\right)(2n-1). \tag{1.3}$$

Similar inequalities for the moments of second and third order are also derived in [1].

Inequalities (1.3) are obviously not sharp, since for $n \geq 2$

$$\sum_{i=1}^{n} i x_i > \sum_{i=1}^{n} x_i = 1 > \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

and

$$\sum_{i=1}^{n} i x_i < n \sum_{i=1}^{n} x_i = n < \frac{1}{2}\left(1 + \frac{1}{n}\right)(2n-1).$$

More generally, for $S = \sum_{i=1}^{n} x_i,\ n \geq 2$, the obvious inequalities

$$\sum_{i=1}^{n} i x_i > \sum_{i=1}^{n} x_i = S, \qquad \sum_{i=1}^{n} i x_i < n \sum_{i=1}^{n} x_i = nS \tag{1.4}$$

give better estimates than (1.2) for $S \leq 1$.

We improve the inequality (1.2) with a constant depending not only on n, but on $\sum_{i=1}^{n} x_i$. Our first result is a generalization of Theorem 1.1.

2. Main Results

We generalize Theorem 1.1 by taking

$$F(n,p,a)=\frac{\sum_{i=1}^{n} f(i)}{nf(n)}, \qquad f(i)=(i+a)^p$$

instead of $G(n,p)$. Obviously, we have $F(n,p,0)=G(n,p)$. By obtaining the same result as that mentioned in Theorem 1.1 with F instead of G, we can find a for which we obtain the best estimates for inequalities of type (1.2).

Theorem 2.1. *Let $n\geq 2$ be an integer and $p\geq 1$, $a\geq -1$ be real numbers. Let us define $F(n,p,a)=\sum_{i=1}^{n}(i+a)^p/\,n(n+a)^p$, then $F(n+1,p,a)\leq F(n,p,a)$ for each $p\geq 1$, $a\geq -1$ and for each integer $n\geq 2$.*

Proof. We compute

$$\begin{aligned}
&F(n,p,a)-F(n+1,p,a)\\
&\qquad=\frac{\sum_{i=1}^{n}(i+a)^p}{n(n+a)^p}-\frac{\sum_{i=1}^{n+1}(i+a)^p}{(n+1)(n+1+a)^p}\\
&\qquad=\sum_{i=1}^{n}(i+a)^p\left(\frac{1}{n(n+a)^p}-\frac{1}{(n+1)(n+1+a)^p}\right)-\frac{1}{n+1}\\
&\qquad=\frac{1}{n+1}\left(F(n,p,a)\frac{(n+1)(n+1+a)^p-n(n+a)^p}{(n+1+a)^p}-1\right).
\end{aligned}$$

So, we have to prove

$$F(n,p,a)\geq\frac{(n+1+a)^p}{(n+1)(n+1+a)^p-n(n+a)^p},$$

or equivalently, (for $n \geq 2$),

$$\sum_{i=1}^{n}(i+a)^p \geq \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p}. \tag{2.1}$$

We prove inequality (2.1) for each positive integer n by induction. For $n = 1$ we have

$$1 \geq \frac{(2+a)^p}{2(2+a)^p - (1+a)^p},$$

which is obviously true.

Let us suppose that for some n the inequality

$$\sum_{i=1}^{n}(i+a)^p \geq \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p}$$

holds.

We have

$$\begin{aligned}\sum_{i=1}^{n+1}(i+a)^p &= \sum_{i=1}^{n}(i+a)^p + (n+1+a)^p \\ &\geq \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p} + (n+1+a)^p \\ &= \frac{(n+1)(n+1+a)^{2p}}{(n+1)(n+1+a)^p - n(n+a)^p}.\end{aligned}$$

In order to show

$$\sum_{i=1}^{n+1}(i+a)^p \geq \frac{(n+1)(n+1+a)^p(n+2+a)^p}{(n+2)(n+2+a)^p-(n+1)(n+1+a)^p}$$

we need to prove the following inequality

$$\frac{(n+1+a)^p}{(n+1)(n+1+a)^p-n(n+a)^p} \geq \frac{(n+2+a)^p}{(n+2)(n+2+a)^p-(n+1)(n+1+a)^p},$$

i.e.

$$(n+2+a)^p\frac{(n+1+a)^p+n(n+a)^p}{n+1} \geq (n+1+a)^{2p}.$$

or

$$\frac{\big((n+2+a)(n+1+a)\big)^p+n\big((n+2+a)(n+a)\big)^p}{n+1} \geq (n+1+a)^{2p}. \tag{2.2}$$

Since $f(x)=(x+a)^p$ is convex for $p \geq 1$ and $x \geq -a$, applying Jensen's inequality we have

$$L \geq \left(\frac{(n+2+a)(n+1+a)+n(n+2+a)(n+a)}{n+1}\right)^p,$$

where L denotes the left hand side in (2.2). To prove (2.2) it is sufficient to prove the inequality

$$(n+2+a)(n+1+a)+n(n+2+a)(n+a) \geq (n+1)(n+1+a)^2,$$

which is true for $a \geq -1$. □

Remark 2.1. *We did not allow $n = 1$, since $F(1, p, -1)$ is not defined.*

Following the same idea given in [1], we can derive the following results:

Theorem 2.2. *Let $F(n, p, a)$ be defined as in Theorem 2.1, $x_i \in [0, 1]$ for $i = 1, \dots, n$ and $S = \sum_{i=1}^{n} x_i$, then*

$$F(n, p, a) \cdot S \cdot f(S) \leq \sum_{i=1}^{n} f(i) x_i \leq F(n, p, a) \cdot (n f(n) - (n - S) f(n - S)), \tag{2.3}$$

where $f(n) = (n + a)^p$.

Proof. The first inequality can be proved in exactly the same way as was done in [1] (Th.2.3). The second inequality follows from the first by putting $a_i = 1 - x_i \in [0, 1]$, and then $x_i = a_i$. □

The special case of this result improves the inequality (1.2):

Corollary 2.3. *Let $n \geq 2$ be an integer, $x_i \in [0, 1]$ for $i = 1, \dots, n$ and $S = \sum_{i=1}^{n} x_i$, then*

$$\frac{1}{2}\left(1 + \frac{1}{S}\right) \leq \frac{\sum_{i=1}^{n} i x_i}{S^2} \leq \frac{1}{2}\left(\frac{2n + 1}{S} - 1\right). \tag{2.4}$$

Proof. Let $a = -1$ and $p = 1$. We compute $F(n, 1, -1) = \frac{1}{2}$. Inequality (2.4) now follows from (2.3) after some computation. □

We can now compare inequalities (2.4) and (1.2); the estimates in (2.4) are obviously better.

In comparing with obvious inequalities (1.4), the estimates in (2.4) are better for $S > 1$ (they coincide for $S = 1$).

References

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