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RATIONAL IDENTITIES AND INEQUALITIES

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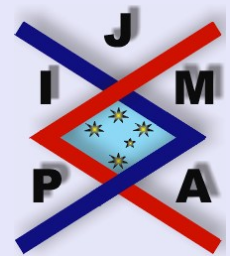


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Abstract

Recently, in [4] the author studied some rational identities and inequalities involving Fibonacci and Lucas numbers. In this paper we generalize these rational identities and inequalities to involve a wide class of sequences.

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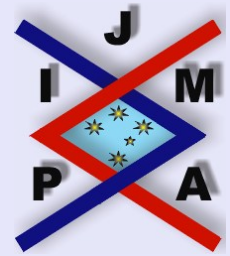
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1. Introduction

The Fibonacci and Lucas sequences are a source of many interesting identities and inequalities. For example, Benjamin and Quinn [1], and Vajda [5] gave combinatorial proofs for many such identities and inequalities. Recently, Díaz-Barrero [4] (see also [2, 3]) introduced some rational identities and inequalities involving Fibonacci and Lucas numbers. A sequence $(a_n)_{n \geq 0}$ is said to be *positive increasing* if $0 < a_n < a_{n+1}$ for all $n \geq 1$, and *complex increasing* if $0 < |a_n| \leq |a_{n+1}|$ for all $n \geq 1$. In this paper, we generalize the identities and inequalities which are given in [4] to obtain several rational identities and inequalities involving positive increasing sequences or complex sequences.



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2. Identities

In this section we present several rational identities and inequalities by using results on contour integrals.

Theorem 2.1. *Let $(a_n)_{n \geq 0}$ be any complex increasing sequence such that $a_p \neq a_q$ for all $p \neq q$. For all positive integers r ,*

$$\sum_{k=1}^n \left(\frac{1 + a_{r+k}^\ell}{a_{r+k}} \prod_{j=1, j \neq k}^n (a_{r+k} - a_{r_j})^{-1} \right) = \frac{(-1)^{n+1}}{\prod_{j=1}^n a_{r+j}}$$

holds, with $0 \leq \ell \leq n - 1$.

Proof. Let us consider the integral

$$I = \frac{1}{2\pi i} \oint_{\gamma} \frac{1 + z^\ell}{z A_n(z)} dz,$$

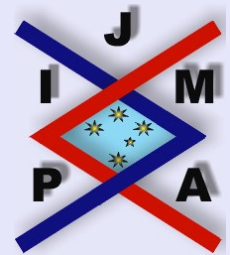
where $\gamma = \{z \in \mathbb{C} : |z| < |a_{r+1}|\}$ and $A_n(z) = \prod_{j=1}^n (z - a_{r+j})$. Evaluating the integral I in the exterior of the γ contour, we get $I_1 = \sum_{k=1}^n R_k$ where

$$R_k = \lim_{z \rightarrow a_{r+k}} \left(\frac{1 + z^\ell}{z} \prod_{j=1, j \neq k}^n (z - a_{r_j})^{-1} \right) = \frac{1 + a_{r+k}^\ell}{a_{r+k}} \prod_{j=1, j \neq k}^n (a_{r+k} - a_{r_j})^{-1}.$$

On the other hand, evaluating I in the interior of the γ contour, we obtain

$$I_2 = \lim_{z \rightarrow 0} \frac{1 + z}{A_n(z)} = \frac{1}{A_n(0)} = \frac{(-1)^n}{\prod_{j=1}^n a_{r+j}}.$$

Using Cauchy's theorem on contour integrals we get that $I_1 + I_2 = 0$, as claimed. \square



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Theorem 2.1 for $a_n = F_n$ the n Fibonacci number ($F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$) gives [4, Theorem 2.1], and for $a_n = L_n$ the n Lucas number ($L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$) gives [4, Theorem 2.2]. As another example, Theorem 2.1 for $a_n = P_n$ the n th Pell number ($P_0 = 0$, $P_1 = 1$, and $P_{n+2} = P_{n+1} + P_n$ for all $n \geq 0$) we get that

$$\sum_{k=1}^n \left(\frac{1 + P_{r+k}^\ell}{P_{r+k}} \prod_{j=1, j \neq k}^n (P_{r+k} - P_{r_j})^{-1} \right) = \frac{(-1)^{n+1}}{\prod_{j=1}^n P_{r+j}}$$

holds, with $0 \leq \ell \leq n - 1$. In particular, we obtain

Corollary 2.2. For all $n \geq 2$,

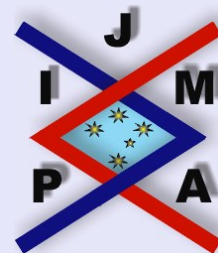
$$\begin{aligned} \frac{(P_n^2 + 1)P_{n+1}P_{n+2}}{(P_{n+1} - P_n)(P_{n+2} - P_n)} + \frac{P_n(P_{n+1}^2 + 1)P_{n+2}}{(P_n - P_{n+1})(P_{n+2} - P_{n+1})} \\ + \frac{P_nP_{n+1}(P_{n+2}^2 + 1)}{(P_n - P_{n+2})(P_{n+1} - P_{n+2})} = 1. \end{aligned}$$

Theorem 2.3. Let $(a_n)_{n \geq 0}$ be any complex increasing sequence such that $a_p \neq a_q$ for all $p \neq q$. For all $n \geq 2$,

$$\sum_{k=1}^n \frac{1}{a_k^{n-2}} \prod_{j=1, j \neq k}^n \left(1 - \frac{a_j}{a_k} \right) = 0.$$

Proof. Let us consider the integral

$$I = \frac{1}{2\pi i} \oint_{\gamma} \frac{z}{A_n(z)} dz,$$



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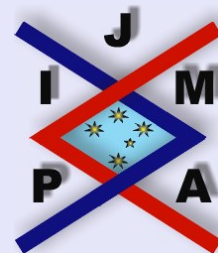
where $\gamma = \{z \in \mathbb{C} : |z| < |a_{n+1}|\}$ and $A_n(z) = \prod_{j=1}^n (z - a_{r+j})$. Evaluating the integral I in the exterior of the γ contour, we get $I_1 = 0$. Evaluating I in the interior of the γ contour, we obtain

$$\begin{aligned} I_2 &= \sum_{k=1}^n \operatorname{Res}(z/A_n(z); z = a_k) \\ &= \sum_{k=1}^n \prod_{j=1, j \neq k}^n \frac{a_k}{a_k - a_j} \\ &= \sum_{k=1}^n \frac{1}{a_k^{n-2}} \prod_{j=1, j \neq k}^n \left(1 - \frac{a_j}{a_k}\right). \end{aligned}$$

Using Cauchy's theorem on contour integrals we get that $I_1 + I_2 = 0$, as claimed. \square

For example, Theorem 2.3 for $a_n = L_n$ the n th Lucas number gives [4, Theorem 2.5]. As another example, Theorem 2.3 for $a_n = P_n$ the n th Pell number obtains, for all $n \geq 2$,

$$\sum_{k=1}^n \frac{1}{P_k^{n-2}} \prod_{j=1, j \neq k}^n \left(1 - \frac{P_j}{P_k}\right) = 0.$$



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3. Inequalities

In this section we suggest some inequalities on positive increasing sequences.

Theorem 3.1. Let $(a_n)_{n \geq 0}$ be any positive increasing sequence such that $a_1 \geq 1$. For all $n \geq 1$,

$$(3.1) \quad a_n^{a_{n+1}} + a_{n+1}^{a_n} < a_n^{a_n} + a_{n+1}^{a_{n+1}}.$$

and

$$(3.2) \quad a_{n+1}^{a_{n+2}} - a_{n+1}^{a_n} < a_{n+2}^{a_{n+2}} - a_{n+2}^{a_n}.$$

Proof. To prove (3.1) we consider the integral

$$I = \int_{a_n}^{a_{n+1}} (a_{n+1}^x \log a_{n+1} - a_n^x \log a_n) dx.$$

Since a_n satisfies $1 \leq a_n < a_{n+1}$ for all $n \geq 1$, so for all x , $a_n \leq x \leq a_{n+1}$ we have that

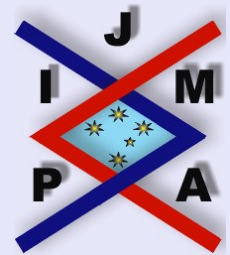
$$a_n^x \log a_n < a_{n+1}^x \log a_n < a_{n+1}^x \log a_{n+1},$$

hence $I > 0$. On the other hand, evaluating the integral I directly, we get that

$$I = (a_{n+1}^{a_{n+1}} - a_n^{a_{n+1}}) - (a_{n+1}^{a_n} - a_n^{a_n}),$$

hence

$$a_n^{a_{n+1}} + a_{n+1}^{a_n} < a_n^{a_n} + a_{n+1}^{a_{n+1}}$$



as claimed in (3.1). To prove (3.2) we consider the integral

$$J = \int_{a_n}^{a_{n+2}} (a_{n+2}^x \log a_{n+2} - a_{n+1}^x \log a_{n+1}) dx.$$

Since a_n satisfies $1 \leq a_{n+1} < a_{n+2}$ for all $n \geq 0$, so for all x , $a_{n+1} \leq x \leq a_{n+2}$ we have that

$$a_{n+1}^x \log a_{n+1} < a_{n+2}^x \log a_{n+2},$$

hence $J > 0$. On the other hand, evaluating the integral J directly, we get that

$$I = (a_{n+2}^{a_{n+2}} - a_{n+2}^{a_n}) - (a_{n+1}^{a_{n+2}} - a_{n+1}^{a_n}),$$

hence

$$a_{n+1}^{a_{n+2}} - a_{n+1}^{a_n} < a_{n+2}^{a_{n+2}} - a_{n+2}^{a_n}$$

as claimed in (3.2). □

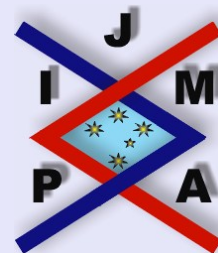
For example, Theorem 3.1 for $a_n = L_n$ the n th Lucas number gives [4, Theorem 3.1]. As another example, Theorem 3.1 for $a_n = P_n$ the n th Pell number obtains, for all $n \geq 1$,

$$P_n^{P_{n+1}} + P_{n+1}^{P_n} < P_n^{P_n} + P_{n+1}^{P_{n+1}},$$

where P_n is the n th Pell number.

Theorem 3.2. *Let $(a_n)_{n \geq 0}$ be any positive increasing sequence such that $a_1 \geq 1$. For all $n, m \geq 1$,*

$$a_{n+m}^{a_n} \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j+1}} < \prod_{j=0}^m a_{n+j}^{a_{n+j}}.$$



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Proof. Let us prove this theorem by induction on m . Since $1 \leq a_n < a_{n+1}$ for all $n \geq 1$ then $a_n^{a_{n+1}-a_n} < a_{n+1}^{a_{n+1}-a_n}$, equivalently, $a_n^{a_{n+1}} a_{n+1}^{a_n} < a_n^{a_n} a_{n+1}^{a_{n+1}}$, so the theorem holds for $m = 1$. Now, assume for all $n \geq 1$

$$a_{n+m-1}^{a_n} \prod_{j=0}^{m-2} a_{n+j}^{a_{n+j+1}} < \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j}}.$$

On the other hand, similarly as in the case $m = 1$, for all $n \geq 1$,

$$a_{n+m-1}^{a_{n+m}-a_n} < a_{n+m}^{a_{n+m}-a_n}.$$

Hence,

$$a_{n+m-1}^{a_{n+m}-a_n} a_{n+m-1}^{a_n} \prod_{j=0}^{m-2} a_{n+j}^{a_{n+j+1}} < a_{n+m}^{a_{n+m}-a_n} \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j}},$$

equivalently,

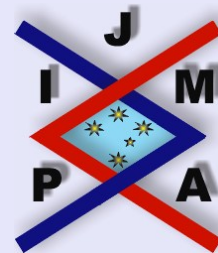
$$a_{n+m}^{a_n} \prod_{j=0}^{m-1} a_{n+j}^{a_{n+j+1}} < \prod_{j=0}^m a_{n+j}^{a_{n+j}},$$

as claimed. \square

Theorem 3.2 for $a_n = L_n$ the n th Lucas number and $m = 3$ gives [4, Theorem 3.3].

Theorem 3.3. *Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be any two sequences such that $0 < a_n < b_n$ for all $n \geq 1$. Then for all $n \geq 1$,*

$$\sum_{i=1}^n (b_j + a_j) \geq \frac{2n^{n+1}}{(n+1)^n} \prod_{i=1}^n \frac{b_j^{1+1/n} - a_j^{1+1/n}}{b_j - a_j}.$$



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Proof. Using the AM-GM inequality, namely

$$\frac{1}{n} \sum_{i=1}^n x_i \geq \prod_{i=1}^n x_i^{1/n},$$

where $x_i > 0$ for all $i = 1, 2, \dots, n$, we get that

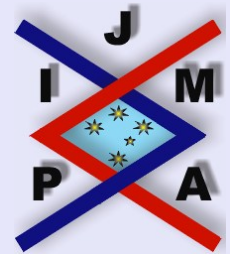
$$\int_{b_1}^{a_1} \cdots \int_{b_n}^{a_n} \frac{1}{n} \sum_{i=1}^n x_i dx_1 \cdots dx_n \geq \int_{b_1}^{a_1} \cdots \int_{b_n}^{a_n} \prod_{i=1}^n x_i^{1/n} dx_1 \cdots dx_n,$$

equivalently,

$$\frac{1}{2n} \sum_{i=1}^n (b_i^2 - a_i^2) \prod_{j=1, j \neq i}^n (b_j - a_j) \geq \prod_{i=1}^n \left(\frac{n}{n+1} (b_i^{1+1/n} - a_i^{1+1/n}) \right),$$

hence, on simplifying the above inequality we get the desired result. \square

Theorem 3.3 for $a_n = L_n^{-1}$ where L_n is the n th Lucas number and $b_n = F_n^{-1}$ where F_n is the n th Fibonacci number gives [4, Theorem 3.4].



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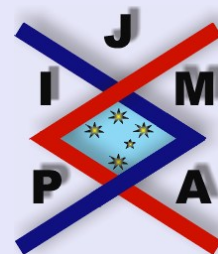
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