

Journal of Inequalities in Pure and Applied Mathematics

CERTAIN SECOND ORDER LINEAR DIFFERENTIAL SUBORDINATIONS

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©2000 Victoria University
ISSN (electronic): 1443-5756
003-04



volume 5, issue 3, article 59,
2004.

*Received 29 December, 2003;
accepted 03 April, 2004.*

Communicated by: N.E. Cho

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Abstract

In this present investigation, we obtain some results for certain second order linear differential subordination. We also discuss some applications of our results.

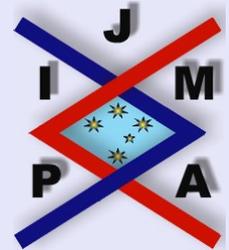
2000 Mathematics Subject Classification: Primary 30C80, Secondary 30C45.

Key words: Analytic functions, Hadamard product (or convolution), differential subordination, Ruscheweyh derivatives, univalent functions, convex functions.

The author is thankful to the referee for his comments.

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1. Introduction

Let \mathcal{H} denote the class of all *analytic* functions in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] := \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}) \right\}$$

and

$$\mathcal{A}(p, n) := \left\{ f \in \mathcal{H} : f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbb{N}) \right\}.$$

Set

$$\mathcal{A}_p := \mathcal{A}(p, 1), \quad \mathcal{A} := \mathcal{A}_1.$$

For two functions $f, g \in \mathcal{H}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in Δ and write

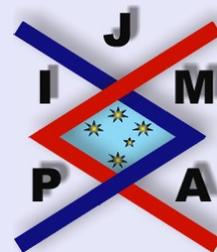
$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function $w(z) \in \mathcal{H}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$(1.1) \quad f(z) = g(w(z)) \quad (z \in \Delta).$$



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In particular, if the function g is univalent in Δ , the above subordination (1.1) is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Miller and Mocanu [2] considered the *second order linear differential subordination*

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

where A, B, C and D are complex-valued functions defined on Δ and $h(z)$ is any convex function and in particular $h(z) = (1+z)/(1-z)$. In fact, they have proved the following:

Theorem 1.1 (Miller and Mocanu [2, Theorem 4.1a, p.188]). *Let n be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B(z), C(z), D(z) : \Delta \rightarrow \mathbb{C}$ satisfy $\Re B(z) \geq A$ and*

$$(1.2) \quad [\Im C(z)]^2 \leq n[\Re B(z) - A]\Re(nB(z) - nA - 2D(z)).$$

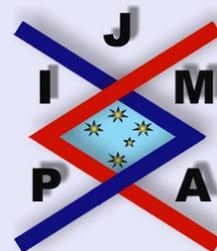
If $p \in \mathcal{H}[1, n]$ and if

$$(1.3) \quad \Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} > 0,$$

then

$$\Re p(z) > 0.$$

Also Miller and Mocanu [2] have proved the following:



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Theorem 1.2 (Miller and Mocanu [2, Theorem 4.1e, p.195]). Let h be convex univalent in Δ with $h(0) = 0$ and let $A \geq 0$. Suppose that $k > 4/|h'(0)|$ and that $B(z)$, $C(z)$ and $D(z)$ are analytic in Δ and satisfy

$$\Re B(z) \geq A + |C(z) - 1| - \Re(C(z) - 1) + k|D(z)|.$$

If $p \in \mathcal{H}[0, 1]$ satisfies the differential subordination

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z)$$

then $p \prec h$.

In this paper, we extend Theorem 1.1 by assuming

$$\Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} > \alpha, \quad (0 \leq \alpha < 1)$$

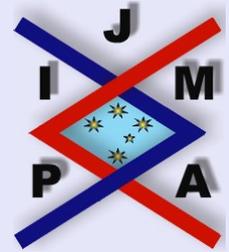
and Theorem 1.2 by assuming that the function $h(z)$ is convex of order α . Certain results of Karunakaran and Ponnusamy [6], Juneja and Ponnusamy [7] and Owa and Srivastava [8] are obtained as special cases. Also we give application of our results to certain functions defined by the familiar Ruscheweyh derivatives.

For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (n, p \in \mathbb{N}),$$

the *Hadamard product* (or *convolution*) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k =: (g * f)(z).$$



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The *Ruscheweyh derivative* of $f(z)$ of order $\delta + p - 1$ is defined by

$$(1.4) \quad D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (f \in \mathcal{A}(p, n); \delta \in \mathbb{R} \setminus (-\infty, -p])$$

or, equivalently, by

$$(1.5) \quad D^{\delta+p-1} f(z) := z^p + \sum_{k=p+1}^{\infty} \binom{\delta+k-1}{k-p} a_k z^k$$

$$(f \in \mathcal{A}(p, n); \delta \in \mathbb{R} \setminus (-\infty, -p]).$$

In particular, if $\delta = l$ ($l + p \in \mathbb{N}$), we find from the definition (1.4) or (1.5) that

$$D^{l+p-1} f(z) = \frac{z^p}{(l+p-1)!} \frac{d^{l+p-1}}{dz^{l+p-1}} \{z^{l-1} f(z)\}$$

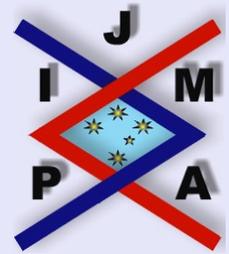
$$(f \in \mathcal{A}(p, n); l + p \in \mathbb{N}).$$

In our present investigation of the second order linear differential subordination, we need the following definitions and results:

Definition 1.1 (Miller and Mocanu [2, Definition 2.2b, p. 21]). Let Q be the set of functions q that are analytic and univalent on $\overline{\Delta} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} q(z) = \infty\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \setminus E(q)$, where $\partial\Delta := \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\Delta} := \Delta \cup \partial\Delta$.



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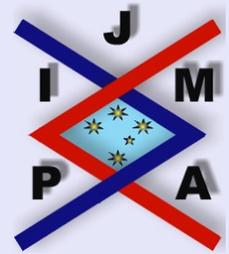


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Theorem 1.3 (Miller and Mocanu [2, Lemma 2.2d, p. 24]). Let $q \in Q$, with $q(0) = a$. Let $p(z) = a + p_n z^n + \dots$ be analytic in Δ with $p(z) \not\equiv a$ and $n \geq 1$. If $p(z)$ is not subordinate to $q(z)$, then there exist points $z_0 = r_0 e^{i\theta_0} \in \Delta$ and $\zeta_0 \in \partial\Delta - E(q)$, and an $m \geq n \geq 1$ for which $p(\Delta_{r_0}) \subset q(\Delta)$,

$$(1.6) \quad \begin{aligned} & (i) \quad p(z_0) = q(\zeta_0) \\ & (ii) \quad z_0 p'(z_0) = m \zeta_0 q'(\zeta_0), \text{ and} \\ & (iii) \quad \Re[z_0 p''(z_0)/p'(z_0) + 1] \geq m \Re[z_0 q''(\zeta_0)/q'(\zeta_0) + 1], \end{aligned}$$

where $\Delta_r := \{z \in \mathbb{C} : |z| < r\}$.

Theorem 1.4 (cf. Miller and Mocanu [2, Theorem 2.3i (i), p. 35]). Let Ω be a simply connected domain and $\psi : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}$ satisfies the condition

$$\psi(i\sigma, \zeta, \mu + i\eta; z) \notin \Omega$$

for $z \in \Delta$ and for real σ, ζ, μ, η satisfying $\zeta \leq -n(1 + \sigma^2)/2$ and $\zeta + \mu \leq 0$. Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be analytic in Δ . If

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $\Re p(z) > 0$.

2. Differential Subordination with Convex Functions of Order α

By appealing to Theorem 1.3, we first prove the following:

Theorem 2.1. *Let h be a convex univalent function of order α , $0 \leq \alpha < 1$, in Δ with $h(0) = 0$ and let $A \geq 0$. Suppose that*

$$k > 2^{2(1-\alpha)} / |h'(0)|$$

and that $B(z)$, $C(z)$ and $D(z)$ are analytic in Δ and satisfy

$$(2.1) \quad n\Re B(z) \geq n(1 - \alpha n)A + \frac{1}{2\beta(\alpha)} [|C(z) - 1| - \Re(C(z) - 1)] + k|D(z)|,$$

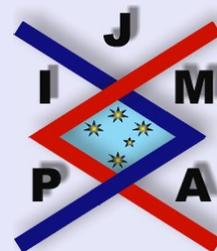
where

$$(2.2) \quad \beta(\alpha) := \begin{cases} \frac{4^\alpha(1 - 2\alpha)}{4 - 2^{2\alpha+1}} & \alpha \neq \frac{1}{2} \\ (\log 4)^{-1} & \alpha = \frac{1}{2}. \end{cases}$$

If $p \in \mathcal{H}[0, n]$ satisfies the differential subordination

$$(2.3) \quad Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

then $p \prec h$.



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Proof. Our proof of Theorem 2.1 is essentially similar to Theorem 1.2 of Miller and Mocanu [2]. Let the subordination in (2.3) be satisfied so that $D(0) = 0$. Since

$$k|h'(0)| > 2^{2(1-\alpha)},$$

there is an r_0 , $0 < r_0 < 1$ such that

$$\frac{(1+r_0)^{2(1-\alpha)}}{r_0} = k|h'(0)| \quad \text{and} \quad 2^{2(1-\alpha)} < \frac{(1+r)^{2(1-\alpha)}}{r} < k|h'(0)|$$

for $r_0 < r < 1$. Since h is convex of order α in Δ , the function $h_r(z) = h(rz)$ is convex of order α in $\bar{\Delta}$ ($r_0 < r < 1$). By setting $p_r(z) = p(rz)$ for $r_0 < r < 1$, we see that the subordination (2.3) becomes

$$(2.4) \quad u_r(z) := Az^2p_r''(z) + B(rz)zp_r'(z) + C(rz)p_r(z) + D(rz) \prec h_r(z) \\ (z \in \Delta; r_0 < r < 1).$$

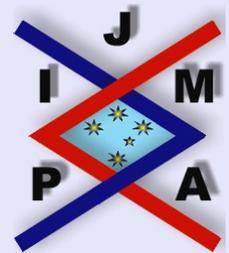
Assume that p_r is not subordinate to h_r , for some r in $(r_0, 1)$. Then by Theorem 1.3 there exist points $z_0 \in \Delta$, $w_0 \in \partial\Delta$ and an $m \geq n \geq 1$ such that

$$(2.5) \quad p_r(z_0) = h_r(w_0), z_0p_r'(z_0) = mw_0h_r'(w_0),$$

$$(2.6) \quad \Re \left(1 + \frac{z_0p_r''(z_0)}{p_r'(z_0)} \right) \geq m\Re \left(1 + \frac{w_0h_r''(w_0)}{h_r'(w_0)} \right).$$

Therefore we have

$$(2.7) \quad \Re \left(1 + \frac{z_0^2p_r''(z_0)}{mw_0h_r'(w_0)} \right) \geq m\alpha.$$



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From Equations (2.5), (2.6) and (2.7), it follows that

$$(2.8) \quad \Re \left(\frac{z_0^2 p_r''(z_0)}{w_0 h_r'(w_0)} \right) \geq m(m\alpha - 1).$$

Since $h_r(z)$ is convex of order α or equivalently

$$\Re \left(1 + \frac{z h_r''(z)}{h_r'(z)} \right) > \alpha \quad (z \in \bar{\Delta}),$$

by [2, Theorem 3.3f, p.115], we have

$$\Re \frac{z h_r'(z)}{h_r(z)} > \beta(\alpha) \quad (z \in \bar{\Delta})$$

where $\beta(\alpha)$ is given by Equation (2.2) and this condition is equivalent to

$$\left| \frac{h_r(z)}{z h_r'(z)} - \frac{1}{2\beta(\alpha)} \right| \leq \frac{1}{2\beta(\alpha)} \quad (z \in \bar{\Delta}).$$

Therefore,

$$(2.9) \quad \Re \left[(C(rz_0) - 1) \frac{h_r(w_0)}{w_0 h_r'(w_0)} \right] \geq \frac{1}{2\beta} \{ \Re[C(rz_0) - 1] - |C(rz_0) - 1| \}.$$

Since h is convex of order α , we have the following well-known estimate:

$$|h'(z)| \geq \frac{|h'(0)|}{(1+r)^{2(1-\alpha)}} \quad (|z| = r < 1).$$



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By setting $z = rw_0$, we see that

$$(2.10) \quad |w_0 h'_r(w_0)| \geq \frac{r|h'(0)|}{(1+r)^{2(1-\alpha)}} \quad (|w_0| = 1).$$

By setting

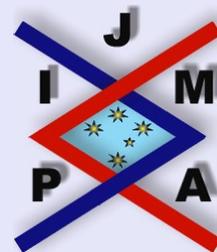
$$(2.11) \quad V := \frac{Az_0^2 p''_r(z_0)}{w_0 h'_r(w_0)} + \frac{B(rz_0)z_0 p'_r(z_0)}{w_0 h'_r(w_0)} + (C(rz_0) - 1) \frac{p_r(z_0)}{w_0 h'_r(w_0)} + \frac{D(rz_0)}{w_0 h'_r(w_0)},$$

we see that

$$(2.12) \quad u_r(z_0) = h_r(w_0) + V w_0 h'_r(w_0).$$

From (2.8), (2.9), (2.10) and (2.11), we have

$$\begin{aligned} \Re V &\geq m(m\alpha - 1)A + m\Re B(rz_0) + \frac{1}{2\beta(\alpha)} [\Re(C(rz_0) - 1) - |C(rz_0) - 1|] \\ &\quad - \frac{(1+r)^{2(1-\alpha)}}{r|h'(0)|} |D(rz_0)| \\ &\geq m[(n\alpha - 1)A + \Re B(rz_0)] \\ &\quad + \frac{1}{2\beta(\alpha)} [\Re(C(rz_0) - 1) - |C(rz_0) - 1|] - k|D(rz_0)| \\ &\geq n[(n\alpha - 1)A + \Re B(rz_0)] \\ &\quad - \frac{1}{2\beta(\alpha)} [|C(rz_0) - 1| - \Re(C(rz_0) - 1)] - k|D(rz_0)| \geq 0, \end{aligned}$$



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it follows that $u_r(z_0) \notin h_r(\Delta)$, a contradiction. Therefore, $p_r \prec h_r$ for $r \in (r_0, 1)$. By letting $r \rightarrow 1^-$, we obtain the desired conclusion $p \prec h$. \square

Remark 2.1. When $\alpha = 0, n = 1$, Theorem 2.1 reduces to Theorem 1.2 of Miller and Mocanu [2].

From the proof of Theorem 2.1, it is clear that the condition $h(0) = 0$ is not necessary when $C(z) = 1$ and hence the following:

Corollary 2.2. Let h be a convex univalent function of order α , $0 \leq \alpha < 1$, in Δ , $h(0) = a$ and let $A \geq 0$. Suppose that

$$k > 2^{2(1-\alpha)} / |h'(0)|$$

and that $B(z)$ and $D(z)$ are analytic in Δ with $D(0) = 0$ and

$$(2.13) \quad n \Re B(z) \geq n(1 - \alpha n)A + k|D(z)|$$

for all $z \in \Delta$. If $p \in \mathcal{H}[a, n]$, $p(0) = h(0)$, satisfies the differential subordination

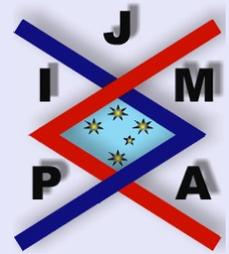
$$(2.14) \quad Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z),$$

then $p \prec h$.

By taking $A = 0$ and $D(z) = 0$ in Theorem 2.1, we obtain the following:

Corollary 2.3. Let h be a convex univalent function of order α , $0 \leq \alpha < 1$, in Δ with $h(0) = 0$. Let $B(z)$ and $C(z)$ be analytic functions on Δ satisfying

$$\Re B(z) \geq \frac{1}{2n\beta(\alpha)} [|C(z) - 1| - \Re(C(z) - 1)],$$



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where $\beta(\alpha)$ is as given in Theorem 2.1. If $p \in \mathcal{H}[0, n]$ satisfies the subordination

$$B(z)zp'(z) + C(z)p(z) \prec h(z),$$

then $p(z) \prec h(z)$.

By taking $B(z) = 1$, $\alpha = 0$, $n = 1$, in Corollary 2.3, we have the following:

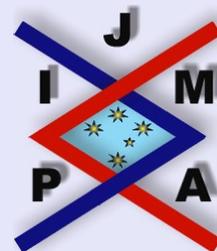
Corollary 2.4. Let h be a convex univalent function in Δ with $h(0) = 0$. Let $C(z)$ be analytic functions on Δ satisfying

$$\Re C(z) > |C(z) - 1|.$$

If the analytic function $p(z)$ satisfies the subordination

$$zp'(z) + C(z)p(z) \prec h(z),$$

then $p(z) \prec h(z)$.



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3. Differential Subordination with Caratheodory Functions of Order α

By appealing to Theorem 1.4, we now prove the following:

Theorem 3.1. Let n be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B(z), C(z), D(z) : \Delta \rightarrow \mathbb{C}$ satisfy $\Re B(z) \geq A$ and

$$(3.1) \quad [\Im C(z)]^2 \leq n [\Re B(z) - A] \\ \times \left[n(\Re B(z) - A) - \frac{\delta + 2\alpha}{1 - \alpha} \Re C(z) - \frac{2 + \delta}{1 - \alpha} \Re(D(z) - \alpha) \right].$$

If $p \in \mathcal{H}[1, n]$ and

$$(3.2) \quad \Re \{ Az^2 p''(z) + B(z) z p'(z) + C(z) p(z) + D(z) \} > \alpha \quad (\alpha < 1),$$

then

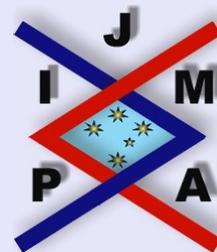
$$\Re p(z) > \frac{\delta + 2\alpha}{\delta + 2}.$$

Proof. Define the function $P(z)$ by

$$P(z) := \frac{p(z) - \gamma}{1 - \gamma} \quad \text{where} \quad \gamma := \frac{\delta + 2\alpha}{\delta + 2}.$$

Then inequality (3.2) can be written as

$$\Re \{ \psi(P(z), zP'(z), z^2P''(z); z) \} > 0,$$



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where

$$\psi(r, s, t; z) = At + B(z)s + C(z)r + \frac{\gamma C(z) + D(z) - \alpha}{1 - \gamma}.$$

In view of Theorem 1.4, it is enough to show that

$$\Re\psi(i\sigma, \zeta, \mu + i\eta; z) \leq 0$$

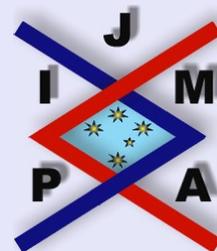
for all real numbers σ, ζ, μ and η with $\zeta \leq \frac{-n(1+\sigma^2)}{2}, \zeta + \mu \leq 0$ and for all $z \in \Delta$. Now,

$$\begin{aligned} & \Re\psi(i\sigma, \zeta, \mu + i\eta; z) \\ &= \mu A + \zeta \Re B(z) - \sigma \Im C(z) + \Re \left[\frac{\gamma C(z) + D(z) - \alpha}{1 - \gamma} \right] \\ &\leq \zeta (\Re B(z) - A) - \sigma \Im C(z) + \Re \left[\frac{\gamma C(z) + D(z) - \alpha}{1 - \gamma} \right] \\ &\leq -\frac{1}{2} \left\{ n[\Re B(z) - A]\sigma^2 + 2\Im C(z)\sigma \right. \\ &\quad \left. + n[\Re B(z) - A] - 2\Re \left[\frac{\gamma C(z) + D(z) - \alpha}{1 - \gamma} \right] \right\} \leq 0, \end{aligned}$$

provided (3.1) holds. This completes the proof of our Theorem 3.1. \square

For $\alpha = \delta = 0$, Theorem 3.1 reduces to Theorem 1.1.

By taking $D = 0$ and $C(z) = 1$ in Theorem 3.1, we have the following:



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Corollary 3.2. Let $A \geq 0$ and $\Re B(z) - A > \delta > 0$. If $p \in \mathcal{H}[1, n]$ satisfies

$$\Re\{Az^2p''(z) + B(z)zp'(z) + p(z)\} > \alpha \quad (\alpha < 1)$$

then

$$\Re p(z) > \frac{n\delta + 2\alpha}{n\delta + 2}.$$

Corollary 3.3. Let $\lambda(z)$ and $R(z)$ be functions defined on Δ and

$$\Re \lambda(z) > \delta + \frac{2 + \delta}{(1 - \alpha)n} \Re R(z) \geq 0.$$

If $p \in \mathcal{H}[1, n]$ satisfies

$$\Re\{\lambda(z)zp'(z) + p(z) + R(z)\} > \alpha \quad (\alpha < 1),$$

then

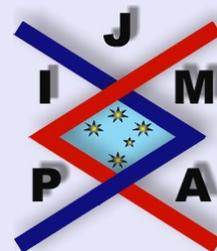
$$\Re p(z) > \frac{2\alpha + \delta n}{2 + \delta n}.$$

A special case of Corollary 3.3 is obtained by Owa and Srivastava [8, Lemma 2, p. 254].

The proof of the following theorem is similar and hence it is omitted.

Theorem 3.4. Let n be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B(z), C(z), D(z) : \Delta \rightarrow \mathbb{C}$ satisfy $\Re B(z) \geq A$ and

$$(3.3) \quad [\Im C(z)]^2 \leq n[\Re B(z) - A] \left[n(\Re B(z) - A) - \frac{\delta + 2\alpha}{1 - \alpha} \Re C(z) - \frac{2 + \delta}{1 - \alpha} \Re(D(z) - \alpha) \right].$$



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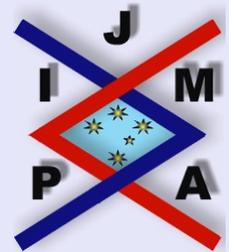
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If $p \in \mathcal{H}[1, n]$ satisfies

$$(3.4) \quad \Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} < \alpha \quad (\alpha > 1),$$

then

$$\Re p(z) < \frac{\delta + 2\alpha}{\delta + 2}.$$



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4. Applications

We now give certain applications of our results obtained in Section 2 and 3.

Theorem 4.1. Let $\gamma \in \mathbb{C}$ with $\gamma \neq -1, -2, -3, \dots$ and let ϕ, Φ be analytic functions on Δ with $\phi(z)\Phi(z) \neq 0$ for $z \in \Delta$. If

$$\Re C(z) - |C(z) - 1| > 1 - 2n\beta(\alpha)\Re B(z),$$

where

$$B(z) := \frac{\Phi(z)}{\phi(z)} \text{ and } C(z) := \frac{\gamma\Phi(z) + z\Phi'(z)}{\phi(z)},$$

then the integral operator defined by

$$I(f)(z) := \frac{1}{z^\gamma\Phi(z)} \int_0^z t^{\gamma-1} f(t)\phi(t)dt$$

satisfies $I(f)(z) \prec h(z)$ for every function $f(z) \prec h(z)$ where $h(z)$ is a convex function of order α .

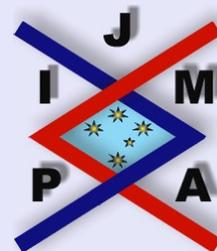
Proof. The result follows immediately from Corollary 2.3. □

Theorem 4.2. Let h be a convex univalent function of order α in Δ , $0 \leq \alpha < 1$ and $h(0) = 1$. Let M, N, R be analytic in Δ with $R(0) = 0$ and

$$M(z) = z^n + \dots, \text{ and } N(z) = z^n + \dots$$

Let

$$\Re \frac{\beta N(z)}{zN'(z)} > k|R(z)| \quad \left(k > \frac{2^{2(1-\alpha)}}{|h'(0)|} \right).$$



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If

$$(4.1) \quad \beta \frac{M'(z)}{N'(z)} + (1 - \beta) \frac{M(z)}{N(z)} + R(z) \prec h(z),$$

then

$$\frac{M(z)}{N(z)} \prec h(z).$$

Proof. Let the function $p(z)$ be defined by

$$p(z) = M(z)/N(z).$$

Then $p(0) = 1 = h(0)$ and it follows that

$$p(z) + \frac{N(z)}{zN'(z)} zp'(z) = \frac{M'(z)}{N'(z)}.$$

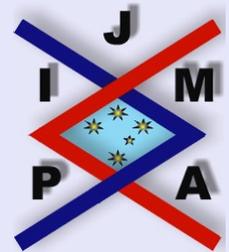
Also, a computation shows that the subordination in (4.1) is equivalent to

$$p(z) + \frac{\beta N(z)}{zN'(z)} zp'(z) + R(z) \prec h(z).$$

The result now follows by an application of Corollary 2.2 □

Remark 4.1. When $\beta = 1, \alpha = 0$, Theorem 4.2 reduces to [2, Theorem 4.1h, p. 199] of Miller and Mocanu. If $\alpha = 0$ and $R(z) = 0$, then Theorem 4.2 reduces to a result of Juneja and Ponnusamy [7, Corollary 1, p. 290].

More generally, we have the following:



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Theorem 4.3. Let $\delta > -p$ be any real number, $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 0$. Let $R(z)$ be a function defined on Δ with $R(0) = 0$ and $h(z)$ a convex function of order α , $0 \leq \alpha < 1$, $h(0) = 1$. Let $g \in \mathcal{A}_p$ satisfy

$$\Re \left\{ \lambda \frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)} \right\} \geq \mu(\delta + p)|R(z)|, \quad \left(k > \frac{2^{2(1-\alpha)}}{|h'(0)|} \right).$$

If $f \in \mathcal{A}_p$ satisfies

$$(1 - \lambda) \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^\mu + \lambda \frac{D^{\delta+p}f(z)}{D^{\delta+p}g(z)} \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^{\mu-1} + R(z) \prec h(z),$$

then

$$\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^\mu \prec h(z).$$

Proof. Let the function $p(z)$ be defined by

$$p(z) := \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^\mu.$$

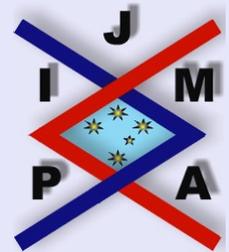
Then a computation shows that the following subordination holds:

$$B(z)zp'(z) + p(z) + R(z) \prec h(z),$$

where

$$B(z) := \frac{\lambda}{\mu(\delta + p)} \frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)}.$$

The result follows by an application of Corollary 2.2. □



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When $R(z) = 0$ and $\mu = 1$, the Theorem 4.3 reduces to Juneja and Ponusamy [7, Theorem 1, p. 289].

Theorem 4.4. Let α be a complex number $\Re\alpha > 0$ and $\beta < 1$. Let M, N, R be analytic in Δ with $R(0) = 0$ and

$$M(z) := z^n + c_1 z^{n+k} + \dots, \quad N(z) := z^n + d_1 z^{n+k} + \dots.$$

Let

$$\Re \frac{\alpha N(z)}{z N'(z)} > \delta + \frac{2 + \delta k}{(1 - \beta)k} \Re R(z).$$

If

$$(4.2) \quad \Re \left[\alpha \frac{M'(z)}{N'(z)} + (1 - \alpha) \frac{M(z)}{N(z)} + R(z) \right] > \beta,$$

then

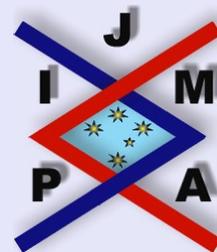
$$\Re \frac{M(z)}{N(z)} > \frac{2\beta + k\delta}{2 + k\delta}.$$

Proof. Let $p(z) := M(z)/N(z)$. Then $p(0) = 1 = h(0)$. It follows that

$$p(z) + \frac{N(z)}{z N'(z)} z p'(z) = \frac{M'(z)}{N'(z)}.$$

Then

$$\begin{aligned} \Re p(z) + \frac{\alpha N(z)}{z N'(z)} z p'(z) + R(z) &= \Re \left[\alpha \frac{M'(z)}{N'(z)} + (1 - \alpha) \frac{M(z)}{N(z)} + R(z) \right] \\ &> \beta. \end{aligned}$$



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If $B(z)$ is defined by $B(z) := \alpha N(z)/[zN'(z)]$, then it follows that

$$\Re B(z) > \delta + \frac{2 + \delta k}{(1 - \beta)k} \Re R(z).$$

The result now follows by an application of Corollary 3.3 □

Remark 4.2. For $R(z) = 0$, $\beta = 0$, Theorem 4.4 is due to Karunakaran and Ponnusamy [6, Theorem B, p. 562].

Theorem 4.5. Let $\delta > -p$ be any real number, $\lambda \in \mathbb{C}$ with $\Re \lambda \geq 0$. Let $R(z)$ be a function defined on Δ with $R(0) = 0$, $0 \leq \alpha < 1$. Let $g \in \mathcal{A}_p$ satisfies

$$\Re \left\{ \lambda \frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)} \right\} > \mu(\delta + p)\delta + \frac{\mu(\delta + p)(2 + \delta)}{1 - \alpha} \Re R(z) \geq 0.$$

If $f \in \mathcal{A}_p$ satisfies

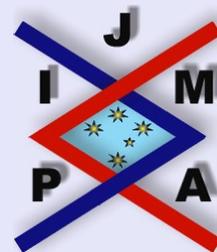
$$\Re \left\{ (1 - \lambda) \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^\mu + \lambda \frac{D^{\delta+p}f(z)}{D^{\delta+p}g(z)} \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^{\mu-1} + R(z) \right\} > \alpha,$$

then

$$\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^\mu \geq \frac{2\alpha + \delta}{2 + \delta}.$$

Proof. Let

$$p(z) := \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)} \right]^\mu.$$



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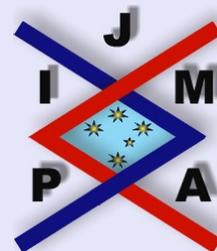
Then a computation shows that

$$\Re\{B(z)zp'(z) + p(z) + R(z)\} > \alpha,$$

where

$$B(z) := \frac{\lambda}{\mu(\delta + p)} \frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)}.$$

The result follows easily. □



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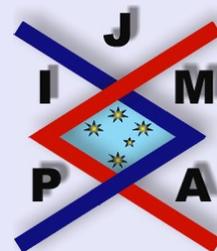
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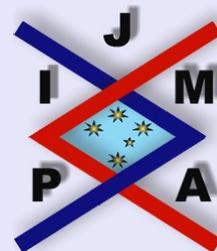
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