## APPROXIMATION BY MODIFIED SZÁSZ-MIRAKYAN OPERATORS

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Received: 23 December, 2008

Accepted: 28 May, 2009

Communicated by: I. Gavrea

2000 AMS Sub. Class.: 41A36, 41A25.

Key words: Szász-Mirakyan operator, Polynomial weight space, Order of approximation,

Voronovskaya type theorem.

Abstract: We introduce the modified Szász-Mirakyan operators  $S_{n;r}$  related to the Borel

methods  $B_r$  of summability of sequences. We give theorems on approximation

properties of these operators in the polynomial weight spaces.

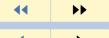


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### 1. Introduction

The approximation of functions by Szász-Mirakyan operators

(1.1) 
$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}_0, \ n \in \mathbb{N},$$

 $(\mathbb{R}_0 = [0, \infty), \mathbb{N} = \{1, 2, ...\})$  has been examined in many papers and monographs (e.g. [11], [1], [2], [4], [5]).

The above operators were modified by several authors (e.g. [3], [6], [9], [10], [12]) which showed that new operators have similar or better approximation properties than  $S_n$ . M. Becker in the paper [1] studied approximation problems for the operators  $S_n$  in the polynomial weight space  $C_p$ ,  $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , connected with the weight function  $w_p$ ,

(1.2) 
$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \quad \text{if} \quad p \in \mathbb{N},$$

for  $x \in \mathbb{R}_0$ . The  $C_p$  is the set of all functions  $f : \mathbb{R}_0 \to \mathbb{R}$   $(\mathbb{R} = (-\infty, \infty))$  for which  $fw_p$  is uniformly continuous and bounded on  $\mathbb{R}_0$  and the norm is defined by

(1.3) 
$$||f||_p \equiv ||f(\cdot)||_p := \sup_{x \in \mathbb{R}_0} w_p(x)|f(x)|.$$

The space  $C_p^m$ ,  $m \in \mathbb{N}$ ,  $p \in \mathbb{N}_0$ , of m-times differentiable functions  $f \in C_p$  with derivatives  $f^{(k)} \in C_p$ ,  $1 \le k \le m$ , and the norm (1.3) was considered also in [1].

In [1] it was proved that  $S_n$  is a positive linear operator acting from the space  $C_p$  to  $C_p$  for every  $p \in \mathbb{N}_0$ . Moreover, for a fixed  $p \in \mathbb{N}_0$  there exist  $M_k(p) = const. > 0$ , k = 1, 2, depending only on p such that for every  $f \in C_p$  there hold the inequalities:

(1.4) 
$$||S_n(f)||_p \le M_1(p)||f||_p$$
 for  $n \in \mathbb{N}$ ,



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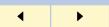
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and

$$(1.5) w_p(x) |S_n(f;x) - f(x)| \le M_2(p)\omega_2\left(f; C_p; \sqrt{\frac{x}{n}}\right), x \in \mathbb{R}_0, n \in \mathbb{N},$$

where  $\omega_2(f; C_p; \cdot)$  is the second modulus of continuity of f.

In this paper we introduce the following modified Szász-Mirakyan operators

(1.6) 
$$S_{n,r}(f;x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n}\right), \quad x \in \mathbb{R}_0, n \in \mathbb{N},$$

for  $f \in C_p$  and every fixed  $r \in \mathbb{N}$ , where

(1.7) 
$$A_r(t) := \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} \quad \text{for} \quad t \in \mathbb{R}_0.$$

Clearly  $A_1(t) = e^t$ ,  $A_2(t) = \cosh t \equiv \frac{1}{2} (e^t + e^{-t})$  and  $S_{n;1}(f;x) \equiv S_n(f;x)$  for  $f \in C_p$ ,  $x \in \mathbb{R}_0$  and  $n \in \mathbb{N}$ . (The operators  $S_{n;2}$  were investigated in [9] for functions belonging to exponential weight spaces.)

We mention that the definition of  $S_{n;r}$  is related to the Borel method of summability of sequences. It is well known ([7]) that a sequence  $(a_n)_0^\infty$ ,  $a_n \in \mathbb{R}$ , is summable to g by the Borel method  $B_r$ ,  $r \in \mathbb{N}$ , if the series  $\sum_{k=0}^\infty \frac{x^{rk}}{(rk)!} a_k$  is convergent on  $\mathbb{R}$  and

$$\lim_{x \to \infty} r e^{-x} \sum_{k=0}^{\infty} \frac{x^{rk}}{(rk)!} a_k = g.$$

In Section 2 we shall give some elementary properties of  $S_{n;r}$ . The approximation theorems will be given in Section 3.



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## 2. Auxiliary Results

It is known ([1]) that for  $e_k(x)=x^k$ , k=0,1,2, there holds:  $S_n(e_0;x)=1$ ,  $S_n(e_1;x)=x$  and  $S_n(e_2;x)=x^2+\frac{x}{n}$ , which imply that

(2.1) 
$$S_n\left((e_1(t) - e_1(x))^2; x\right) = \frac{x}{n} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}.$$

Moreover, for every fixed  $q \in \mathbb{N}$ , there exists a polynomial  $\mathcal{P}_q(x) = \sum_{k=0}^q a_k x^k$  with real coefficients  $a_k$ ,  $a_q \neq 0$ , depending only on q such that

$$(2.2) S_n\left((e_1(t) - e_1(x))^{2q}; x\right) \le \mathcal{P}_q(x) n^{-q} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}.$$

From (1.1) – (1.4), (1.6) and (1.7) we deduce that  $S_{n;r}$  is a positive linear operator well defined on every space  $C_p$ ,  $p \in \mathbb{N}_0$ , and

$$(2.3) S_{n:r}(e_0; x) = 1,$$

(2.4) 
$$S_{n,r}(e_1; x) = \frac{x}{n} \frac{A'_r(nx)}{A_r(nx)},$$

(2.5) 
$$S_{n,r}(e_2;x) = \frac{x^2}{n^2} \frac{A_r''(nx)}{A_r(nx)} + \frac{x}{n^2} \frac{A_r'(nx)}{A_r(nx)},$$

for  $x \in \mathbb{R}_0$  and  $n, r \in \mathbb{N}$ , and

(2.6) 
$$S_{n:r}(f;0) = f(0) \text{ for } f \in C_p, n, r \in \mathbb{N}.$$

Here we derive a simpler formula for  $A_r$ .

**Lemma 2.1.** Let  $r \in \mathbb{N}$  be a fixed number. Then  $A_r$  defined by (1.7) can be rewritten in the form:  $A_1(t) = e^t$ ,  $A_2(t) = \cosh t$ ,

(2.7) 
$$A_{2m}(t) = \frac{1}{m} \left[ \cosh t + \sum_{k=1}^{m-1} \exp\left(t \cos \frac{k\pi}{m}\right) \cos\left(t \sin \frac{k\pi}{m}\right) \right],$$



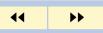
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for  $2 \leq m \in \mathbb{N}$ , and

(2.8)  $A_{2m+1}(t)$ 

$$= \frac{1}{2m+1} \left[ e^t + 2\sum_{k=1}^m \exp\left(t\cos\frac{2k\pi}{2m+1}\right) \cos\left(t\sin\frac{2k\pi}{2m+1}\right) \right],$$

for  $m \in \mathbb{N}$  and  $t \in \mathbb{R}_0$ .

*Proof.* The formulas for  $A_1$  and  $A_2$  are obvious by (1.7). For  $r \geq 3$  and  $t \in \mathbb{R}_0$  we have

$$e^{t} = \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!} + \sum_{k=0}^{\infty} \frac{t^{rk+1}}{(rk+1)!} + \dots + \sum_{k=0}^{\infty} \frac{t^{rk+r-1}}{(rk+r-1)!}$$

which by (1.7) can be written in the form

$$e^{t} = A_{r}(t) + \int_{0}^{t} A_{r}(u) du + \int_{0}^{t} \int_{0}^{v_{1}} A_{r}(u) du dv_{1} + \dots + \int_{0}^{t} \int_{0}^{v_{1}} \dots \int_{0}^{v_{r-2}} A_{r}(u) du dv_{r-2} \dots dv_{1}.$$

By (r-1)-times differentiation we get the equality

$$e^t = A_r^{(r-1)}(t) + A_r^{(r-2)}(t) + \dots + A_r'(t) + A_r(t)$$
 for  $t \in \mathbb{R}_0$ ,

which shows that  $y = A_r(t)$  is the solution of the differential equation

(2.9) 
$$y^{(r-1)} + y^{(r-2)} + \dots + y' + y = e^t$$

satisfying the initial conditions

(2.10) 
$$y(0) = 1, \quad y'(0) = y''(0) = \dots = y^{(r-2)}(0) = 0.$$

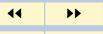


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Using now the Laplace transformation

$$\mathcal{L}[y(t)] = Y(s) := \int_0^\infty y(t)e^{-st}dt, \quad s = x + iy,$$

we have by (2.10)

$$\mathcal{L}[y^{(k)}(t)] = s^k Y(s) - s^{k-1}$$
 for  $k = 1, \dots, r-1$ ,

and consequently we get from (2.7)

$$(s^{r-1} + s^{r-2} + \dots + s + 1) Y(s) = \frac{1}{s-1} + s^{r-2} + s^{r-3} + \dots + s + 1,$$

and

(2.11) 
$$Y(s) = \frac{s^{r-1}}{s^r - 1}.$$

By the inverse Laplace transformation we get

(2.12) 
$$y(t) = \mathcal{L}^{-1} \left[ \frac{s^{r-1}}{s^r - 1} \right] \quad \text{for} \quad t \in \mathbb{R}_0,$$

and this  $\mathcal{L}^{-1}$  transform can be calculated by the residues of Y.

It is known that the inverse transform of a rational function  $\frac{P(s)}{Q(s)}$  with the simple poles  $s_k$  can be written as follows

(2.13) 
$$\mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_{s_k} {}^* \frac{P(s_k)e^{s_k t}}{Q'(s_k)} + 2re \sum_{s_k} {}^{**} \frac{P(s_k)e^{s_k t}}{Q'(s_k)},$$

where  $\sum^*$  denotes the sum for all real  $s_k$  and  $\sum^{**}$  denotes the sum for all complex  $s_k = x_k + iy_k$  with a positive  $y_k$ .



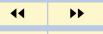
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The function Y defined by (2.11) has the simple poles  $s_k = \sqrt[r]{1} = e^{2k\pi i/r}$  for k = 0, 1, ..., r - 1. From this and (2.12) and (2.13) for  $r = 2m, 2 \le m \in \mathbb{N}$ , we get

$$y(t) = \frac{1}{2m} \left( \sum_{s_k}^* e^{s_k t} + 2re \sum_{s_k}^{**} e^{s_k t} \right)$$
$$= \frac{1}{m} \left[ \cosh t + \sum_{k=1}^{m-1} \exp\left(t \cos \frac{k\pi}{m}\right) \cos\left(t \sin \frac{k\pi}{m}\right) \right].$$

This shows that the formula (2.7) is proved.

Analogously by (2.12) and (2.13) we obtain (2.8).

From (2.7) and (2.8) we have that

$$A_3(t) = \frac{1}{3} \left( e^t + 2e^{-t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) \right),$$

$$A_4(t) = \frac{1}{2} (\cosh t + \cos t),$$

$$A_6(t) = \frac{1}{3} \left( \cosh t + 2\cosh \frac{t}{2} \cos \left( \frac{\sqrt{3}}{2} t \right) \right), \quad \text{for} \quad t \in \mathbb{R}_0.$$

Applying the formula (1.7) and Lemma 2.1, we immediately obtain the following:

**Lemma 2.2.** For every fixed  $r \in \mathbb{N}$  there exists a positive constant  $M_3(r)$  depending only on r such that

(2.14) 
$$1 \le \frac{e^{nx}}{A_n(nx)} \le M_3(r) \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N}.$$



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**Lemma 2.3.** Let  $r \in \mathbb{N}$ . Then for  $e_1(x) = x$  there holds

(2.15) 
$$\lim_{n \to \infty} n S_{n;r} \left( e_1(t) - e_1(x); x \right) = 0$$

and

$$\lim_{n \to \infty} n S_{n;r} \left( (e_1(t) - e_1(x))^2; x \right) = x,$$

at every  $x \in \mathbb{R}_0$ . Moreover, we have

$$(2.16) S_{n;r}\left(\left(e_1(t) - e_1(x)\right)^{2q}; x\right) \le M_3(r) S_n\left(\left(e_1(t) - e_1(x)\right)^{2q}; x\right)$$

for  $x \in \mathbb{R}_0$ ,  $n \in \mathbb{N}$  and every fixed  $q \in \mathbb{N}$ .

*Proof.* The inequality (2.16) is obvious by (1.1), (1.6) and (2.14).

We shall prove only (2.15) for  $r = 2m, m \in \mathbb{N}$ .

If r = 2 then  $A_2(t) = \cosh t$  and by (2.4) we have

$$S_{n;2}(e_1(t) - e_1(x); x) = x \left( \frac{\sinh nx}{\cosh nx} - 1 \right)$$
$$= \frac{-2x}{e^{2nx} + 1} \quad \text{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N},$$

which implies (2.15).

If r = 2m with  $2 \le m \in \mathbb{N}$ , then by (2.4), (2.7) and (2.14) we get

$$|S_{n;2m}(e_1(t) - e_1(x); x)| = \frac{x}{A_{2m}(nx)} \left| \frac{1}{n} A'_{2m}(nx) - A_{2m}(nx) \right|$$



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$$= \frac{x}{mA_{2m}(nx)} \left| \sinh nx - \cosh nx \right|$$

$$+ \sum_{k=1}^{m-1} \exp\left(nx \cos \frac{k\pi}{m}\right) \left[\cos \frac{k\pi}{m} \cos\left(nx \sin \frac{k\pi}{m}\right) - \sin \frac{k\pi}{m} \sin\left(nx \sin \frac{k\pi}{m}\right) - \cos\left(nx \sin \frac{k\pi}{m}\right)\right] \right|$$

$$\leq M_3(2m) \frac{x}{m} \left[e^{-2nx} + 3\sum_{k=1}^{m-1} \exp\left(-2nx \sin^2 \frac{k\pi}{m}\right)\right]$$

and from this we immediately obtain (2.15).

From (1.6), (1.1) - (1.4) and (2.14) the following lemma results.

**Lemma 2.4.** The operator  $S_{n;r}$ ,  $n, r \in \mathbb{N}$ , is linear and positive, and acts from the space  $C_p$  to  $C_p$  for every  $p \in \mathbb{N}_0$ . For  $f \in C_p$ 

$$||S_{n,r}(f)||_p \le ||f||_p ||S_{n,r}(1/w_p)||_p \le M_3(r)||f||_p \cdot ||S_n(1/w_p)||_p \le M_4(p,r)||f||_p \quad for \quad n,r,\in\mathbb{N},$$

where  $M_4(p,r) = M_1(p)M_3(r)$  and  $M_1(p)$ ,  $M_3(r)$  are positive constants given in (1.4) and (2.14).



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### 3. Theorems

First we shall prove two theorems on the order of approximation of  $f \in C_p$  by  $S_{n;r}$ , r > 2.

**Theorem 3.1.** Let  $p \in \mathbb{N}_0$  and  $2 \le r \in \mathbb{N}$  be fixed numbers. Then there exists  $M_5(p,r) = const. > 0$  (depending only on p and r) such that for every  $f \in C_p^1$  there holds the inequality

(3.1) 
$$w_p(x) |S_{n;r}(f;x) - f(x)| \le M_5(p,r) ||f'||_p \sqrt{\frac{x}{n}},$$

for  $x \in \mathbb{R}_0$  and  $n \in \mathbb{N}$ .

*Proof.* Let  $f \in C_p^1$ . Then by (1.6), (1.7) and (2.14) it follows that

$$|S_{n,r}(f;x) - f(x)| \le S_{n,r}(|f(t) - f(x)|;x)$$
  
  $\le M_3(r)S_n(|f(t) - f(x)|;x) \text{ for } x \in \mathbb{R}_0, n \in \mathbb{N},$ 

and for  $t, x \in \mathbb{R}_0$ 

$$|f(t) - f(x)| = \left| \int_x^t f'(u) \, du \right| \le ||f'||_p \left( \frac{1}{w_p(t)} + \frac{1}{w_p(x)} \right) |t - x|.$$

Using now the operator  $S_n$ , (1.1) - (1.4) and (2.1), we get

$$w_{p}(x)S_{n}(|f(t) - f(x)|; x) \leq ||f'||_{p} \left\{ w_{p}(x)S_{n} \left( \frac{|t - x|}{w_{p}(t)}; x \right) + S_{n}(|t - x|; x) \right\}$$

$$\leq ||f'||_{p} \left( S_{n} \left( (t - x)^{2}; x \right) \right)^{1/2} \left\{ 2 ||S_{n}(1/w_{2p})||_{2p}^{1/2} + 1 \right\}$$

$$\leq \left( 2\sqrt{M_{1}(2p)} + 1 \right) ||f'||_{p} \sqrt{\frac{x}{n}} \quad \text{for} \quad x \in \mathbb{R}_{0}, n \in \mathbb{N}.$$



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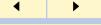
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Combining the above, we obtain the estimation (3.1).

**Theorem 3.2.** Let  $p \in \mathbb{N}_0$  and  $2 \le r \in \mathbb{N}$  be fixed. Then there exists  $M_6(p,r) = const. > 0$  (depending only on p and r) such that for every  $f \in C_p$ ,  $x \in \mathbb{R}_0$  and  $n \in \mathbb{N}$  there holds

(3.2) 
$$w_p(x) |S_{n;r}(f;x) - f(x)| \le M_6(p,r)\omega_1\left(f;C_p;\sqrt{\frac{x}{n}}\right),$$

where  $\omega_1(f; C_p; \cdot)$  is the modulus of continuity of  $f \in C_p$ , i.e.

(3.3) 
$$\omega_1(f; C_p; t) := \sup_{0 \le u \le t} \|\Delta_u f(\cdot)\|_p \quad \text{for} \quad t \ge 0,$$

and 
$$\Delta_u f(x) = f(x+u) - f(x)$$
.

*Proof.* The inequality (3.2) for x = 0 follows by (1.2), (2.6) and (3.3).

Let  $f \in C_p$  and x > 0. We use the Steklov function  $f_h$ ,

$$f_h(x) := \frac{1}{h} \int_0^h f(x+t) dt$$
 for  $x \in \mathbb{R}_0, h > 0$ .

This  $f_h$  belongs to the space  $C_p^1$  and by (3.3) it follows that

$$(3.4) ||f - f_h||_p \le \omega_1(f; C_p; h)$$

and

(3.5) 
$$||f_h'||_p \le h^{-1}\omega_1(f; C_p; h), \text{ for } h > 0.$$

By the above properties of  $f_h$  and (2.3) we can write

$$\begin{aligned}
& \left| S_{n,r} \big( f(t); x \big) - f(x) \right| \\
& \leq \left| S_{n,r} \left( f(t) - f_h(t); x \right) \right| + \left| S_{n,r} \big( f_h(t); x \big) - f_h(x) \right| + \left| f_h(x) - f(x) \right|, \\
\end{aligned}$$



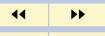
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for  $n \in \mathbb{N}$  and h > 0. Next, by Lemma 2.4 and (3.4) we get

$$w_p(x) |S_{n,r}(f(t) - f_h(t); x)| \le M_4(p,r) ||f - f_h||_p \le M_4(p,r) \omega_1(f; C_p; h).$$

In view of Theorem 3.1 and (3.5) we have

$$w_p(x) |S_{n;r}(f_h;x) - f_h(x)| \le M_5(p,r) ||f_h'||_p \sqrt{\frac{x}{n}} \le M_5(p,r) h^{-1} \sqrt{\frac{x}{n}} \omega_1(f;C_p;h).$$

Consequently,

(3.6) 
$$w_p(x) |S_{n;r}(f;x) - f(x)|$$
  

$$\leq \omega_1(f;C_p;h) \left( M_4(p,r) + M_5(p,r)h^{-1}\sqrt{\frac{x}{n}} + 1 \right),$$

for x > 0,  $n \in \mathbb{N}$  and h > 0. Putting  $h = \sqrt{x/n}$  in (3.6) for given x and n, we obtain the desired estimation (3.2).

Theorem 3.2 implies the following:

**Corollary 3.3.** If  $f \in C_p$ ,  $p \in \mathbb{N}_0$ , and  $2 \le r \in \mathbb{N}$ , then

$$\lim_{n\to\infty} S_{n;r}(f;x) = f(x) \quad \text{at every} \quad x \in \mathbb{R}_0.$$

This convergence is uniform on every interval  $[x_1, x_2]$ ,  $x_1 \ge 0$ .

The Voronovskaya type theorem given in [1] for the operators  $S_n$  can be extended to  $S_{n;r}$  with  $r \geq 2$ .

**Theorem 3.4.** Suppose that  $f \in C_p^2$ ,  $p \in \mathbb{N}_0$ , and  $2 \le r \in \mathbb{N}$ . Then

(3.7) 
$$\lim_{n \to \infty} n \left( S_{n,r}(f;x) - f(x) \right) = \frac{x}{2} f''(x)$$

at every  $x \in \mathbb{R}_0$ .



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*Proof.* The statement (3.7) for x=0 is obvious by (2.6). Choosing x>0, we can write the Taylor formula for  $f\in C_p^2$ :

$$f(t) = f(x) + f'(x) + \frac{1}{2}f''(x)(t-x)^2 + \varphi(t,x)(t-x)^2$$
 for  $t \in \mathbb{R}_0$ ,

where  $\varphi(t) \equiv \varphi(t, x)$  is a function belonging to  $C_p$  and  $\lim_{t \to x} \varphi(t) = \varphi(x) = 0$ . Using now the operator  $S_{n:r}$  and (2.3), we get

$$S_{n,r}(f(t);x) = f(x) + f'(x)S_{n,r}(t-x;x) + \frac{1}{2}f''(x)S_{n,r}((t-x)^2;x) + S_{n,r}(\varphi(t)(t-x)^2;x),$$

for  $n \in \mathbb{N}$ , which by Lemma 2.3 implies that

(3.8) 
$$\lim_{n \to \infty} n \left( S_{n;r}(f(t); x) - f(x) \right) = \frac{x}{2} f''(x) + \lim_{n \to \infty} n S_{n;r} \left( \varphi(t) (t - x)^2; x \right).$$

It is clear that

(3.9) 
$$\left| S_{n,r} \left( \varphi(t)(t-x)^2; x \right) \right| \le \left( S_{n,r} (\varphi^2(t); x) S_{n,r} ((t-x)^4; x) \right)^{1/2},$$
 and by Corollary 3.3

(3.10) 
$$\lim_{n \to \infty} S_{n;r}(\varphi^2(t); x) = \varphi^2(x) = 0.$$

Moreover, by (2.16) and (2.2) we deduce that the sequence  $(n^2S_{n;r}((t-x)^4;x))_1^{\infty}$  is bounded at every fixed  $x \in \mathbb{R}_0$ . From this and (3.9) and (3.10) we get

$$\lim_{n \to \infty} n S_{n;r} \left( \varphi(t)(t-x)^2; x \right) = 0$$

which with (3.8) yields the statement (3.7).



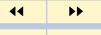
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### 4. Remarks

Remark 1. We observe that the estimation (1.5) for the operators  $S_n$  is better than (3.2) obtained for  $S_{n;r}$  with  $r \geq 2$ . It is generated by formulas (2.3) – (2.5) and Lemma 2.1 which show that the operators  $S_{n;r}$ ,  $r \geq 2$ , preserve only the function  $e_0(x) = 1$ . The operators  $S_n$  preserve the function  $e_k(x) = x^k$ , k = 0, 1.

Remark 2. In the paper [2], the approximation properties of the Szász-Mirakyan operators  $S_n$  in the exponential weight spaces  $C_q^*$ , q>0, with the weight function  $v_q(x)=e^{-qx}$ ,  $x\in\mathbb{R}_0$  were examined. Obviously the operators  $S_{n;r}$ ,  $r\geq 2$ , can be investigated also in these spaces.

Remark 3. G. Kirov in [8] defined the new Bernstein polynomials for m-times differentiable functions and showed that these operators have better approximation properties than classical Bernstein polynomials.

The Kirov idea was applied to the operators  $S_n$  in [10].

We mention that the Kirov method can be extended to the operators  $S_{n;r}$  with  $r \geq 2$ , i.e. for functions  $f \in C_p^m$ ,  $m \in \mathbb{N}$ ,  $p \in \mathbb{N}_0$ , and a fixed  $2 \leq r \in \mathbb{N}$  we can consider the operators

$$S_{n;r}^*(f;x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} \sum_{j=0}^m \frac{f^{(j)}\left(\frac{rk}{n}\right)}{j!} \left(\frac{rk}{n} - x\right)^j,$$

for  $x \in \mathbb{R}_0$  and  $n \in \mathbb{N}$ .

In [10] it was proved that the  $S_{n;1}^*$  have better approximation properties for  $f \in C_n^m$ ,  $m \ge 2$ , than  $S_{n;1}$ .



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