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## IMPROVEMENT OF AN OSTROWSKI TYPE INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATION FOR SOME SPECIAL MEANS

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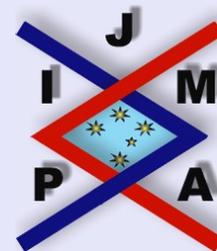
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Abstract

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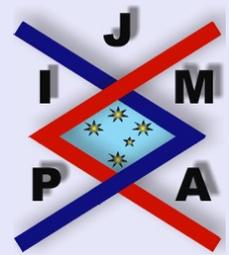


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## Abstract

We first improve two Ostrowski type inequalities for monotonic functions, then provide its application for special means.

*2000 Mathematics Subject Classification:* 26D15, 26D10.

*Key words:* Ostrowski's inequality, Trapezoid inequality, Special means.

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# 1. Introduction

In [1], Dragomir established the following Ostrowski's inequality for monotonic mappings.

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a monotonic nondecreasing mapping on  $[a, b]$ . Then for all  $x \in [a, b]$ , we have the following inequality*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left\{ [2x - (a+b)]f(x) + \int_a^b \operatorname{sgn}(t-x)f(t) dt \right\} \\ & \leq \frac{1}{b-a} [(x-a)(f(x) - f(a)) + (b-x)(f(b) - f(x))] \\ (1.1) \quad & \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] (f(b) - f(a)). \end{aligned}$$

The constant  $\frac{1}{2}$  is the best possible one.

In [2], Dragomir, Pečarić and Wang generalized Theorem 1.1 and proved

**Theorem 1.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a monotonic nondecreasing mapping on  $[a, b]$  and  $t_1, t_2, t_3 \in (a, b)$  be such that  $t_1 \leq t_2 \leq t_3$ . Then*

$$\left| \int_a^b f(x) dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right|$$



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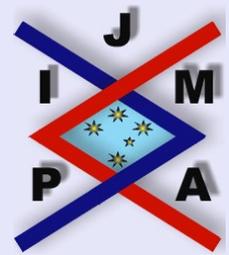


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$$\begin{aligned}
 &\leq (b - t_3)f(b) + (2t_2 - t_1 - t_3)f(t_2) - (t_1 - a)f(a) \\
 &\quad + \int_a^b T(x)f(x)dx \\
 &\leq (b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
 &\quad + (t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(a)) \\
 (1.2) \quad &\leq \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)),
 \end{aligned}$$

where  $T(x) = \text{sgn}(t_1 - x)$ , for  $x \in [a, t_2]$ , and  $T(x) = \text{sgn}(t_3 - x)$ , for  $x \in [t_2, b]$ .

In the present paper, we firstly improve the above results, and then provide its application for some special means.

## 2. Main Result

We shall start with the following result.

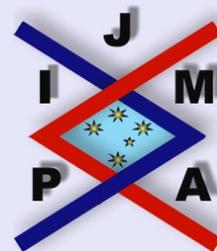
**Theorem 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a monotonic nondecreasing mapping on  $[a, b]$  and let  $t_1, t_2, t_3 \in [a, b]$  be such that  $t_1 \leq t_2 \leq t_3$ . Then*

$$\begin{aligned} & \left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\ (2.1) \quad & \leq \max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\ & (t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \end{aligned}$$

$$(2.2) \quad \leq \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).$$

*Proof.* Since  $f(x)$  is a monotonic nondecreasing mapping on  $[a, b]$ , we have

$$\begin{aligned} & \left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\ & = \left| \int_a^{t_1} (f(x) - f(a))dx + \int_{t_1}^{t_3} (f(x) - f(t_2))dx + \int_{t_3}^b (f(x) - f(b))dx \right| \\ & = \left| \left[ \int_a^{t_1} (f(x) - f(a))dx + \int_{t_2}^{t_3} (f(x) - f(t_2))dx \right] \right. \\ & \quad \left. - \left[ \int_{t_1}^{t_2} (f(t_2) - f(x))dx + \int_{t_3}^b (f(b) - f(x))dx \right] \right| \\ & \leq \max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\ & \quad (t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \\ & \leq \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)). \end{aligned}$$



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Thus (2.1) and (2.2) is proved. □

For  $t_1 = t_2 = t_3 = x$ , Theorem 2.1 becomes the following corollary.

**Corollary 2.2.** *Let  $f$  be defined as in Theorem 2.1. Then*

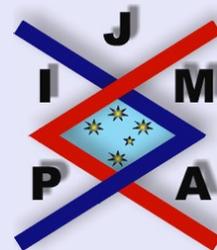
$$\begin{aligned} & \left| \int_a^b f(x)dx - [(x-a)f(a) + (b-x)f(b)] \right| \\ & \leq \max\{(b-x)(f(b) - f(x)), (x-a)(f(x) - f(a))\} \\ & \leq \max\{x-a, b-x\} \cdot \max\{(f(x) - f(a)), (f(b) - f(x))\} \\ & \leq \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] (f(b) - f(a)). \end{aligned}$$

For  $x = \frac{a+b}{2}$ , we get trapezoid inequality.

**Corollary 2.3.** *Let  $f$  be defined as in Theorem 2.1. Then*

$$\begin{aligned} & \left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b-a) \right| \\ (2.3) \quad & \leq \frac{b-a}{2} \max \left\{ \left( f \left( \frac{a+b}{2} \right) - f(a) \right), \left( f(b) - f \left( \frac{a+b}{2} \right) \right) \right\} \\ & \leq \frac{1}{2}(b-a)(f(b) - f(a)). \end{aligned}$$

For  $t_1 = a, t_2 = x, t_3 = b$ , we get Theorem 1.1.




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### 3. Application for Special Means

In this section, we shall give application of Corollary 2.3. Let us recall the following means.

1. The arithmetic mean:

$$A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0.$$

2. The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0.$$

3. The harmonic mean:

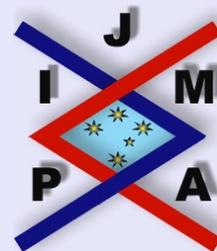
$$H = H(a, b) := \frac{2}{1/a + 1/b}, \quad a, b \geq 0.$$

4. The logarithmic mean:

$$L = L(a, b) := \frac{b - a}{\ln b - \ln a}, \quad a, b \geq 0, a \neq b; \text{ If } a = b, \text{ then } L(a, b) = a.$$

5. The identric mean:

$$I = I(a, b) := \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, \quad a, b \geq 0, a \neq b; \text{ If } a = b, \text{ then } I(a, b) = a.$$



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6. The  $p$ -logarithmic mean:

$$L_p = L_p(a, b) := \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, \quad a \neq b; \text{ If } a = b, \text{ then } L_p(a, b) = a,$$

where  $p \neq -1, 0$  and  $a, b > 0$ .

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

We are going to use inequality (2.3) in the following equivalent version:

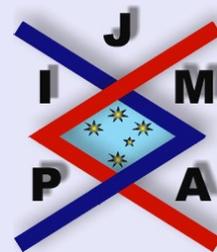
$$(3.1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} \max \left\{ \left( f \left( \frac{a+b}{2} \right) - f(a) \right), \left( f(b) - f \left( \frac{a+b}{2} \right) \right) \right\} \leq \frac{1}{2} (f(b) - f(a)),$$

where  $f : [a, b] \rightarrow \mathbb{R}$  is monotonic nondecreasing on  $[a, b]$ .

### 3.1. Mapping $f(x) = x^p$

Consider the mapping  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = x^p, p > 0$ . Then

$$\frac{1}{b-a} \int_a^b f(t) dt = L_p^p(a, b),$$



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$$\frac{f(a) + f(b)}{2} = A(a^p, b^p),$$

$$f(b) - f(a) = p(b - a)L_{p-1}^{p-1}.$$

Then by (3.1), we get

$$\begin{aligned} |L_p^p(a, b) - A(a^p, b^p)| &\leq \frac{1}{2} \max \left\{ \left( \frac{a+b}{2} \right)^p - a^p, b^p - \left( \frac{a+b}{2} \right)^p \right\} \\ &= \frac{1}{2} \left[ b^p - \left( \frac{a+b}{2} \right)^p \right] \\ &= \frac{1}{2} (b^p - a^p) - \frac{1}{2} \left( \left( \frac{a+b}{2} \right)^p - a^p \right) \\ (3.2) \quad &\leq \frac{1}{2} p(b-a)L_{p-1}^{p-1} - \frac{p(b-a)a^{p-1}}{4}. \end{aligned}$$

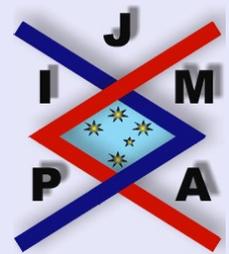
*Remark 3.1.* The following result was proved in [2].

$$|L_p^p(a, b) - A(a^p, b^p)| \leq \frac{1}{2} p(b-a)L_{p-1}^{p-1}.$$

### 3.2. Mapping $f(x) = -1/x$

Consider the mapping  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = -\frac{1}{x}$ . Then

$$\frac{1}{b-a} \int_a^b f(t) dt = -L^{-1}(a, b),$$



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$$\frac{f(a) + f(b)}{2} = -\frac{A(a, b)}{G^2(a, b)},$$

$$f(b) - f(a) = \frac{b - a}{G^2(a, b)}.$$

Then by (3.1), we get

$$\begin{aligned} \left| \frac{A(a, b)}{G^2(a, b)} - L^{-1}(a, b) \right| &\leq \frac{1}{2} \max \left\{ \frac{1}{a} - \frac{2}{a+b}, \frac{2}{a+b} - \frac{1}{b} \right\} \\ &= \frac{1}{2} \cdot \frac{b-a}{a(a+b)} = \frac{1}{2} \cdot \frac{b-a}{ab} - \frac{1}{2} \cdot \frac{b-a}{b(a+b)} \\ &\leq \frac{1}{2} \cdot \frac{b-a}{G^2(a, b)} - \frac{1}{2} \cdot \frac{b-a}{b(a+b)}. \end{aligned}$$

Thus we get

$$(3.3) \quad 0 \leq AL - G^2 \leq \frac{1}{2} \frac{b}{a+b} (b-a)L.$$

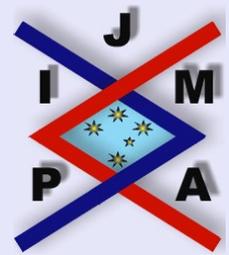
*Remark 3.2.* The following result was proved in [2].

$$0 \leq AG - G^2 \leq \frac{1}{2} (b-a)L.$$

### 3.3. Mapping $f(x) = \ln x$

Consider the mapping  $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = \ln x$ . Then

$$\frac{1}{b-a} \int_a^b f(t) dt = \ln I(a, b),$$




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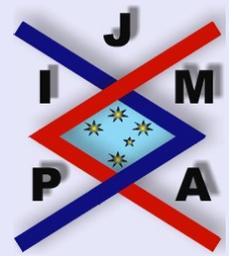


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$$\frac{f(a) + f(b)}{2} = \ln G(a, b),$$
$$f(b) - f(a) = \frac{b - a}{L(a, b)}.$$

Then by (3.1), we get

$$|\ln I(a, b) - \ln G(a, b)| \leq \frac{1}{2} \max \left\{ \ln \frac{a+b}{2} - \ln a, \ln b - \ln \frac{a+b}{2} \right\}$$
$$= \frac{1}{2} \ln \frac{a+b}{2a} = \frac{1}{2} \frac{b-a}{L(a, b)} - \frac{1}{2} \ln \frac{2b}{a+b}.$$

Thus we get

$$(3.4) \quad 1 \leq \frac{I}{G} \leq \sqrt{\frac{a+b}{2b}} e^{\frac{1}{2} \cdot \frac{b-a}{L(a, b)}}.$$

*Remark 3.3.* The following result was proved in [2].

$$1 \leq \frac{I}{G} \leq e^{\frac{1}{2} \cdot \frac{b-a}{L(a, b)}}.$$

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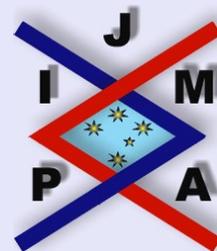
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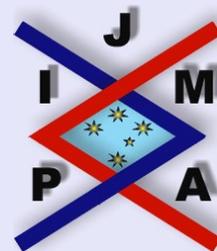
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