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IMPROVEMENT OF AN OSTROWSKI TYPE INEQUALITY FOR MONOTONIC MAPPINGS AND ITS APPLICATION FOR SOME SPECIAL MEANS

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Abstract

We first improve two Ostrowski type inequalities for monotonic functions, then provide its application for special means.

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Contents

Improvement of An Ostrowski Type Inequality for Monotonic Mappings and Its Application for Some Special Means

S.S. Dragomir and M.L. Fang

1. Introduction

In [1], Dragomir established the following Ostrowski's inequality for monotonic mappings.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$. Then for all $x \in [a,b]$, we have the following inequality*

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| &\le \frac{1}{b-a}\left\{ [2x-(a+b)]f(x) + \int_a^b sgn(t-x)f(t)dt \right\} \\ &\le \frac{1}{b-a}[(x-a)(f(x)-f(a)) + (b-x)(f(b)-f(x))] \\ &\le \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a} \right](f(b)-f(a)). \end{aligned} \tag{1.1}$$

The constant $\frac{1}{2}$ is the best possible one.

In [2], Dragomir, Pečarić and Wang generalized Theorem 1.1 and proved

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$ and $t_1, t_2, t_3 \in (a,b)$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\left| \int_a^b f(x)dx - [(t_1-a)f(a) + (t_3-t_1)f(t_2) + (b-t_3)f(b)] \right|$$

$$
\begin{aligned}
&\leq (b-t_3)f(b) + (2t_2 - t_1 - t_3)f(t_2) - (t_1 - a)f(a) \\
&\qquad + \int_a^b T(x)f(x)dx \\
&\leq (b-t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
&\qquad + (t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(a)) \\
&\leq \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)), \qquad (1.2)
\end{aligned}
$$

where $T(x) = sgn(t_1 - x)$, *for* $x \in [a, t_2]$, *and* $T(x) = sgn(t_3 - x)$, *for* $x \in [t_2, b]$.

In the present paper, we firstly improve the above results, and then provide its application for some special means.

2. Main Result

We shall start with the following result.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing mapping on $[a,b]$ and let $t_1, t_2, t_3 \in [a,b]$ be such that $t_1 \le t_2 \le t_3$. Then*

$$\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right|$$

$$\le \max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \tag{2.1}$$
$$(t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\}$$

$$\le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)). \tag{2.2}$$

Proof. Since $f(x)$ is a monotonic nondecreasing mapping on $[a,b]$, we have

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (t_3 - t_1)f(t_2) + (b - t_3)f(b)] \right| \\
&= \left| \int_a^{t_1} (f(x) - f(a))dx + \int_{t_1}^{t_3} (f(x) - f(t_2))dx + \int_{t_3}^{b} (f(x) - f(b))dx \right| \\
&= \left| \left[\int_a^{t_1} (f(x) - f(a))dx + \int_{t_2}^{t_3} (f(x) - f(t_2))dx \right] \right. \\
&\qquad \left. - \left[\int_{t_1}^{t_2} (f(t_2) - f(x))dx + \int_{t_3}^{b} (f(b) - f(x))dx \right] \right| \\
&\le \max\{(b - t_3)(f(b) - f(t_3)) + (t_2 - t_1)(f(t_2) - f(t_1)), \\
&\qquad (t_3 - t_2)(f(t_3) - f(t_2)) + (t_1 - a)(f(t_1) - f(a))\} \\
&\le \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a)).
\end{aligned}$$

Thus (2.1) and (2.2) is proved. □

For $t_1 = t_2 = t_3 = x$, Theorem 2.1 becomes the following corollary.

Corollary 2.2. *Let f be defined as in Theorem 2.1. Then*

$$\begin{aligned}\left|\int_a^b f(x)dx - [(x-a)f(a) + (b-x)f(b)]\right| \\ &\leq \max\{(b-x)(f(b)-f(x)),(x-a)(f(x)-f(a))\} \\ &\leq \max\{x-a, b-x\}\cdot\max\{(f(x)-f(a)),(f(b)-f(x))\} \\ &\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](f(b)-f(a)).\end{aligned}$$

For $x = \frac{a+b}{2}$, we get trapezoid inequality.

Corollary 2.3. *Let f be defined as in Theorem 2.1. Then*

$$\begin{aligned}\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \\ &\leq \frac{b-a}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right) - f(a)\right), \left(f(b) - f\left(\frac{a+b}{2}\right)\right)\right\} \quad (2.3)\\ &\leq \frac{1}{2}(b-a)(f(b)-f(a)).\end{aligned}$$

For $t_1 = a$, $t_2 = x$, $t_3 = b$, we get Theorem 1.1.

3. Application for Special Means

In this section, we shall give application of Corollary 2.3. Let us recall the following means.

1. The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \quad a,b \geq 0.$$

2. The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \quad a,b \geq 0.$$

3. The harmonic mean:

$$H = H(a,b) := \frac{2}{1/a + 1/b}, \quad a,b \geq 0.$$

4. The logarithmic mean:

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, \quad a,b \geq 0, a \neq b; \text{ If } a = b, \text{ then } L(a,b) = a.$$

5. The identric mean:

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \quad a,b \geq 0, a \neq b; \text{ If } a = b, \text{ then } I(a,b) = a.$$

6. The p-logarithmic mean:

$$L_p = L_p(a,b) := \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \quad a \neq b; \text{ If } a=b, \text{ then } L_p(a,b)=a,$$

where $p \neq -1, 0$ and $a, b > 0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

We are going to use inequality (2.3) in the following equivalent version:

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \tag{3.1}$$
$$\leq \frac{1}{2}\max\left\{\left(f\left(\frac{a+b}{2}\right)-f(a)\right),\left(f(b)-f\left(\frac{a+b}{2}\right)\right)\right\}$$
$$\leq \frac{1}{2}(f(b)-f(a)),$$

where $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$.

3.1. Mapping $f(x)=x^p$

Consider the mapping $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f(x)=x^p, p>0$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = L_p^p(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^p, b^p),$$

$$f(b) - f(a) = p(b-a)L_{p-1}^{p-1}.$$

Then by (3.1), we get

$$\begin{aligned}
\left|L_p^p(a,b) - A(a^p,b^p)\right| &\le \frac{1}{2}\max\left\{\left(\frac{a+b}{2}\right)^p - a^p, b^p - \left(\frac{a+b}{2}\right)^p\right\} \\
&= \frac{1}{2}\left[b^p - \left(\frac{a+b}{2}\right)^p\right] \\
&= \frac{1}{2}(b^p - a^p) - \frac{1}{2}\left(\left(\frac{a+b}{2}\right)^p - a^p\right) \\
&\le \frac{1}{2}p(b-a)L_{p-1}^{p-1} - \frac{p(b-a)a^{p-1}}{4}. \qquad (3.2)
\end{aligned}$$

Remark 3.1. The following result was proved in [2].

$$\left|L_p^p(a,b) - A(a^p,b^p)\right| \le \frac{1}{2}p(b-a)L_{p-1}^{p-1}.$$

3.2. Mapping $f(x) = -1/x$

Consider the mapping $f : [a,b] \subset (0,\infty) \to \mathbb{R},\ f(x) = -\frac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = -L^{-1}(a,b),$$

$$\frac{f(a)+f(b)}{2} = -\frac{A(a,b)}{G^2(a,b)},$$

$$f(b)-f(a) = \frac{b-a}{G^2(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}\left|\frac{A(a,b)}{G^2(a,b)} - L^{-1}(a,b)\right| &\le \frac{1}{2}\max\left\{\frac{1}{a}-\frac{2}{a+b}, \frac{2}{a+b}-\frac{1}{b}\right\} \\ &= \frac{1}{2}\cdot\frac{b-a}{a(a+b)} = \frac{1}{2}\cdot\frac{b-a}{ab} - \frac{1}{2}\cdot\frac{b-a}{b(a+b)} \\ &\le \frac{1}{2}\cdot\frac{b-a}{G^2(a,b)} - \frac{1}{2}\cdot\frac{b-a}{b(a+b)}.\end{aligned}$$

Thus we get

$$0 \le AL - G^2 \le \frac{1}{2}\frac{b}{a+b}(b-a)L. \tag{3.3}$$

Remark 3.2. The following result was proved in [2].

$$0 \le AG - G^2 \le \frac{1}{2}(b-a)L.$$

3.3. Mapping $f(x) = \ln x$

Consider the mapping $f : [a,b] \subset (0,\infty) \to \mathbb{R},\ f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(t)dt = \ln I(a,b),$$

$$\frac{f(a)+f(b)}{2} = \ln G(a,b),$$
$$f(b)-f(a) = \frac{b-a}{L(a,b)}.$$

Then by (3.1), we get

$$\begin{aligned}|\ln I(a,b) - \ln G(a,b)| &\le \frac{1}{2}\max\left\{\ln\frac{a+b}{2} - \ln a, \ln b - \ln\frac{a+b}{2}\right\}\\ &= \frac{1}{2}\ln\frac{a+b}{2a} = \frac{1}{2}\frac{b-a}{L(a,b)} - \frac{1}{2}\ln\frac{2b}{a+b}.\end{aligned}$$

Thus we get

$$1 \le \frac{I}{G} \le \sqrt{\frac{a+b}{2b}}e^{\frac{1}{2}\cdot\frac{b-a}{L(a,b)}}. \tag{3.4}$$

Remark 3.3. The following result was proved in [2].

$$1 \le \frac{I}{G} \le e^{\frac{1}{2}\cdot\frac{b-a}{L(a,b)}}.$$

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