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ON A BOJANIĆ–STANOJEVIĆ TYPE INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. An extension of the Bojanić–Stanojević type inequality [1] is made by considering the r-th derivate of the Dirichlet kernel $D_k^{(r)}$ instead of D_k. Namely, the following inequality is proved

$$\left\| \sum_{k=1}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \leq M_p n^{r+1} \left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p},$$

where $\|\cdot\|_1$ is the L^1-norm, $\{\alpha_k\}$ is a sequence of real numbers, $1 < p \leq 2$, $r = 0, 1, 2, \dots$ and M_p is an absolute constant dependent only on p. As an application of this inequality, it is shown that the class $\mathcal{F}_{pr}$ is a subclass of $\mathcal{BV} \cap \mathcal{C}_r$, where $\mathcal{F}_{pr}$ is the extension of the Fomin's class, $\mathcal{C}_r$ is the extension of the Garrett–Stanojević class [8] and $\mathcal{BV}$ is the class of all null sequences of bounded variation.

Key words and phrases: Bojanić–Stanojević inequality, Sidon–Fomin's inequality, Bernstein's inequality. L^1-convergence cosine series.

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1. INTRODUCTION

In 1939, Sidon [5] proved his namesake inequality, which is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications, for instance in L^1-convergence problems and summation methods with respect to trigonometric series, newer and newer improvements of the original inequality have been proved by several authors.

Fomin [2], by applying the linear method for summing of Fourier series, gave another proof of the inequality and thus it is known as Sidon-Fomin's inequality. In addition, S. A. Telyakovskii in [7] has given an elegant proof of Sidon-Fomin's inequality.

Lemma 1.1. *(Sidon-Fomin). Let $\{\alpha_k\}_{k=0}^n$ be a sequence of real numbers such that $|\alpha_k| \leq 1$ for all k. Then there exists a positive constant M such that for any $n \geq 0$,*

$$\left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\|_1 \leq M(n+1)\,. \tag{1.1}$$

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In [9] we extended this result and we gave two different proofs of the following lemma.

Lemma 1.2. [9]. *Let $\{\alpha_j\}_{j=0}^k$ be a sequence of real numbers such that $|\alpha_k| \le 1$ for all k. Then there exists a positive constant M, such that for any $n \ge 0$,*

$$\left\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \le M(n+1)^{r+1}. \tag{1.2}$$

However, Bojanić and Stanojević [1] proved the following more general inequality of (1.1).

Lemma 1.3. [1]. *Let $\{\alpha_i\}_{i=0}^n$ be a sequence of real numbers. Then for any $1 < p \le 2$ and $n \ge 0$*

$$\left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\|_1 \le M_p(n+1) \left(\frac{1}{n+1} \sum_{k=0}^{n} |\alpha_k|^p \right)^{1/p}, \tag{1.3}$$

where the constant M_p depends only on p.

We note that this estimate is essentially contained (case $p = 2$) in Fomin [2]. Sidon-Fomin's inequality is a special case of the Bojanić-Stanojević inequality, i.e., it can easily be deduced from Lemma 1.3.

It is easy to see that the Bojanić-Stanojević inequality is not valid for $p = 1$. Indeed, if $\alpha_n = 1$ and $\alpha_k = 0$ ($k \ne n$, $k \in \mathbb{N}$) then the left side is of order $\log n/n$ while the right side is of order $1/n$ as $n \to \infty$.

In order to prove our new results we need the following lemma.

Lemma 1.4. [10]. *If $T_n(x)$ is a trigonometrical polynomial of order n, then*

$$\|T_n^{(r)}\| \le n^r \|T_n\|.$$

This is S. Bernstein's inequality in the $L^1(0,\pi)$-metric (see [10, Vol. 2, p.11]).

2. MAIN RESULT

Now we will prove a counterpart of inequality (1.3) in the case where the r-th derivate of the Dirichlet's kernel $D_k^{(r)}$ is used instead of $D(x)$.

Theorem 2.1. *Let $\{\alpha_k\}_{k=1}^n$ be a sequence of real numbers. Then for any $1 < p \le 2$ and $r = 0, 1, 2, \dots, n \in \mathbb{N}$ the following inequality holds:*

$$\left\| \sum_{k=1}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \le M_p n^{r+1} \left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p}, \tag{2.1}$$

where the constant M_p depends only on p.

Proof. Applying first the Bernstein inequality and then the Bojanić-Stanojević inequality, we have

$$\left\| \sum_{k=1}^{n} \alpha_k D_k^{(r)}(x) \right\| \le n^r \left\| \sum_{k=1}^{n} \alpha_k D_k^{(r)}(x) \right\| \le M_p n^{r+1} \left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^p \right)^{1/p}.$$

It is easy to see that the inequality (1.2) is a special case of the inequality (2.1), i.e. it can easily be deduced from Theorem 2.1. □

3. APPLICATION

The problem of L^1-convergence via Fourier coefficients consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for $\|S_n - f\| = o(1)$, $n \to \infty$ is given in the form $a_n \lg n = o(1)$, $n \to \infty$. Here S_n denotes the partial sums of the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, .$$

The Sidon-Telyakovskii class $\mathcal{S}$ [7] is a classical example for which the condition $a_n \lg n = o(1)$, $n \to \infty$ is equivalent to $\|S_n - f\| = o(1)$, $n \to \infty$. Later Fomin [3] extended the Sidon-Telyakovskii class by defining a class $\mathcal{F}_p$, $p > 1$ of Fourier coefficients as follows: a sequence $\{a_k\}$ belongs to $\mathcal{F}_p$, $p > 1$ if $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=k}^{\infty} |\Delta a_i|^p \right)^{1/p} < \infty \, . \tag{3.1}$$

We note that Fomin [3] has given an equivalent form of the condition (3.1). Namely, he proved that $\{a_n\} \in \mathcal{F}_p$, $p > 1$ iff $\sum_{s=1}^{\infty} 2^s \, \Delta_s^{(p)} < \infty$, where

$$\Delta_s^{(p)} = \left\{ \frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p \right\}^{1/p} .$$

Let $\mathcal{BV}$ denote the class of null sequence $\{a_n\}$ of bounded variation, i.e. $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$.

The class $\mathcal{C}$ was defined by Garrett and Stanojević [4] as follows: a null sequence of real numbers satisfy the condition $\mathcal{C}$ if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ independent of n, such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon \, , \quad \text{for every } n \, .$$

On the other hand, Stanojević [6] proved the following inclusion between the classes $\mathcal{F}_p$, $\mathcal{C}$ and $\mathcal{BV}$.

Lemma 3.1. [6]. *For all $1 < p \le 2$ the following inclusion holds: $\mathcal{F}_p \subset \mathcal{BV} \cap \mathcal{C}$.*

In [8] we defined an extension $\mathcal{C}_r$, $r = 0, 1, 2, \ldots$, of the Garrett-Stanojević class. Namely, a null sequence $\{a_k\}$ belongs to the class $\mathcal{C}_r$, $r = 0, 1, 2, \ldots$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| < \varepsilon \, , \quad \text{for all } n \, .$$

When $r = 0$, we denote $\mathcal{C}_r = \mathcal{C}$.

Denote by I_m the dyadic interval $[2^{m-1}, 2^m)$, for $m \ge 1$. A null sequence $\{a_n\}$ belongs to the class F_{pr}, $p > 1$, $r = 0, 1, 2, \ldots$ if

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} < \infty, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1 \, .$$

It is obvious that $F_{pr} \subset F_p$. For $r = 0$, we obtain the Fomin's class F_p.

Theorem 3.2. *For all $1 < p \le 2$ and $r = 0, 1, 2, \ldots$ the following inclusion holds $F_{pr} \subset BV \cap C_r$.*

Proof. By Lemma 3.1, it is clear that $F_{pr} \subset BV$. It suffices to show that

$$\left\| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right\| = o(1)\,, \quad n \to \infty\,.$$

Since

$$\sum_{m=1}^{\infty} 2^{m(1/q+r)} \left(\sum_{k \in I_m} |\Delta a_k|^p \right)^{1/p} = 2 \sum_{m=1}^{\infty} \left\{ 2^{(m-1)[(r+1)p-1]} \sum_{k \in I_m} |\Delta a_k|^p \right\}^{1/p},$$

we have

$$\sum_{k=1}^{\infty} k^{(r+1)p-1} |\Delta a_k|^p < \infty\,.$$

Applying the Theorem 2.1, we obtain

$$\left\| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right\| \le M_p \left(\sum_{k=n}^{\infty} k^{(r+1)p-1} |\Delta a_k|^p \right)^{1/p} = o(1), \quad n \to \infty\,.$$

□

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