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ASYMPTOTIC FORMULAE

RAFAEL JAKIMCZUK

DIVISIÓN MATEMÁTICA
UNIVERSIDAD NACIONAL DE LUJÁN
LUJÁN, BUENOS AIRES
ARGENTINA.
jakimczu@mail.unlu.edu.ar

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ABSTRACT. Let $t_{s,n}$ be the n-th positive integer number which can be written as a power p^t, $t \geq s$, of a prime p ($s \geq 1$ is fixed). Let $\pi_s(x)$ denote the number of prime powers p^t, $t \geq s$, not exceeding x. We study the asymptotic behaviour of the sequence $t_{s,n}$ and of the function $\pi_s(x)$. We prove that the sequence $t_{s,n}$ has an asymptotic expansion comparable to that of p_n (the Cipolla's expansion).

Key words and phrases: Primes, Powers of primes, Cipolla's expansion.

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1. INTRODUCTION

Let p_n be the n-th prime. M. Cipolla [1] proved the following theorem:

There exists a unique sequence $P_j(X)$ $(j \geq 1)$ of polynomials with rational coefficients such that, for every nonnegative integer m,

$$p_n = n \log n + n \log \log n - n + \sum_{j=1}^{m} \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^m n}\right). \tag{1.1}$$

The polynomials $P_j(X)$ have degree j and leading coefficient $\frac{1}{j}$.

$$P_1(X) = X - 2, \quad P_2(X) = \frac{X^2 - 6X + 11}{2}, \quad \ldots.$$

If $m = 0$ equation (1.1) is:

$$p_n = n \log n + n \log \log n - n + o(n). \tag{1.2}$$

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Let $\pi(x)$ denote the number of prime numbers not exceeding x, then

$$\pi(x) = \left(\sum_{i=1}^{m} \frac{(i-1)!x}{\log^i x}\right) + \varepsilon(x)\frac{(m-1)!x}{\log^m x} \qquad (m \geq 1), \tag{1.3}$$

where $\lim_{x\to\infty} \varepsilon(x) = 0$.

Lemma 1.1. *There exists a positive number M such that in the interval $[2,\infty)$, $|\varepsilon(x)| \leq M$.*

Proof. Let us consider the closed interval $[2, a]$. In this interval, $\pi(x) \leq x$, so $\pi(x)$ is bounded. The functions $\frac{(i-1)!x}{\log^i x}$, $i = 1, \dots, m$ and $\frac{\log^m x}{(m-1)!x}$ are continuous on the compact $[2, a]$, so they are also bounded.

As

$$\varepsilon(x) = \left[\pi(x) - \left(\sum_{i=1}^{m} \frac{(i-1)!x}{\log^i x}\right)\right] \frac{\log^m x}{(m-1)!x},$$

$\varepsilon(x)$ is in its turn bounded on $[2, a]$.

Since a is arbitrary and $\lim_{x\to\infty} \varepsilon(x) = 0$, the lemma is proved. □

Let us consider the sequence of positive integer numbers which can be written as a power p^t of a prime p ($t \geq 1$ is fixed). The number of prime powers p^t not exceeding x will be (in view of (1.3))

$$\begin{aligned} \pi(x^{\frac{1}{t}}) &= \left(\sum_{i=1}^{m} \frac{(i-1)!x^{\frac{1}{t}}}{\log^i x^{\frac{1}{t}}}\right) + \varepsilon\left(x^{\frac{1}{t}}\right) \frac{(m-1)!x^{\frac{1}{t}}}{\log^m x^{\frac{1}{t}}} \\ &= \left(\sum_{i=1}^{m} \frac{t^i(i-1)!x^{\frac{1}{t}}}{\log^i x}\right) + +\varepsilon\left(x^{\frac{1}{t}}\right) \frac{t^m(m-1)!x^{\frac{1}{t}}}{\log^m x} \\ &= \left(\sum_{i=1}^{m} \frac{t^i(i-1)!x^{\frac{1}{t}}}{\log^i x}\right) + o\left(\frac{x^{\frac{1}{t}}}{\log^m x}\right). \end{aligned} \tag{1.4}$$

2. THE FUNCTION $\pi_s(x)$

Let $t_{s,n}$ be the n-th positive integer number (in increasing order) which can be written as a power p^t, $t \geq s$, of a prime p ($s \geq 1$ is fixed). Let $\pi_s(x)$ denote the number of prime powers p^t, $t \geq s$, not exceeding x.

Theorem 2.1.

$$\pi_s(x) = \left(\sum_{i=1}^{m} \frac{s^i(i-1)!x^{\frac{1}{s}}}{\log^i x}\right) + o\left(\frac{x^{\frac{1}{s}}}{\log^m x}\right). \tag{2.1}$$

Proof. If $x \in [2^{s+k}, 2^{s+k+1})$ $(k \geq 1)$, then

$$\pi_s(x) = \pi\left(x^{\frac{1}{s}}\right) + \sum_{i=1}^{k} \pi\left(x^{\frac{1}{s+i}}\right).$$

Using (1.4), we obtain

$$\text{(2.2)}\quad \pi_s(x) = \left(\sum_{i=1}^{m} \frac{s^i(i-1)!x^{\frac{1}{s}}}{\log^i x}\right) + \varepsilon\left(x^{\frac{1}{s}}\right)\frac{s^m(m-1)!x^{\frac{1}{s}}}{\log^m x} + \sum_{j=1}^{k}\left(\left(\sum_{i=1}^{m}\frac{(s+j)^i(i-1)!x^{\frac{1}{s+j}}}{\log^i x}\right) + \varepsilon\left(x^{\frac{1}{s+j}}\right)\frac{(s+j)^m(m-1)!x^{\frac{1}{s+j}}}{\log^m x}\right)$$

$$= \left(\sum_{i=1}^{m} \frac{s^i(i-1)!x^{\frac{1}{s}}}{\log^i x}\right) + \varepsilon\left(x^{\frac{1}{s}}\right)\frac{s^m(m-1)!x^{\frac{1}{s}}}{\log^m x} + \sum_{i=1}^{m}\left(\sum_{j=1}^{k}\frac{(s+j)^i(i-1)!x^{\frac{1}{s+j}}}{\log^i x}\right) + \sum_{j=1}^{k}\varepsilon\left(x^{\frac{1}{s+j}}\right)\frac{(s+j)^m(m-1)!x^{\frac{1}{s+j}}}{\log^m x}.$$

In the given conditions, the following inequalities hold for x:

$$\frac{\sum_{j=1}^{k}\frac{(s+j)^i(i-1)!x^{\frac{1}{s+j}}}{\log^i x}}{\frac{s^m(m-1)!x^{\frac{1}{s}}}{\log^m x}} = \frac{\sum_{j=1}^{k}\frac{(s+j)^i}{s^m}\cdot\frac{(i-1)!}{(m-1)!}x^{\frac{1}{s+j}}\log^{m-i}x}{x^{\frac{1}{s}}}$$

$$\leq \sum_{j=1}^{k}\frac{\frac{(s+j)^i}{s^m}\cdot\frac{(i-1)!}{(m-1)!}\log^{m-i}\left(2^{s+k+1}\right)}{2^{\frac{(s+k)j-s}{s(s+j)}}}$$

$$\leq \sum_{j=1}^{k}\frac{(s+k)^i(s+k+1)^{m-i}}{\left(2^{\frac{1}{s(s+1)}}\right)^k}$$

$$= \frac{k(s+k)^i(s+k+1)^{m-i}}{\left(2^{\frac{1}{s(s+1)}}\right)^k} \qquad (i=1,\dots,m).$$

Now, since

$$\lim_{k\to\infty}\frac{k(s+k)^i(s+k+1)^{m-i}}{\left(2^{\frac{1}{s(s+1)}}\right)^k} = 0 \qquad (i=1,\dots,m),$$

we find that

$$\text{(2.3)}\qquad \lim_{x\to\infty}\frac{\sum_{j=1}^{k}\frac{(s+j)^i(i-1)!x^{\frac{1}{s+j}}}{\log^i x}}{\frac{s^m(m-1)!x^{\frac{1}{s}}}{\log^m x}} = 0 \qquad (i=1,\dots,m).$$

On the other hand, from the lemma we have the following inequality

$$\left|\sum_{j=1}^{k}\varepsilon\left(x^{\frac{1}{s+j}}\right)\frac{(s+j)^m(m-1)!x^{\frac{1}{s+j}}}{\log^m x}\right| \leq M\sum_{j=1}^{k}\frac{(s+j)^m(m-1)!x^{\frac{1}{s+j}}}{\log^m x}.$$

This inequality and (2.3) with $i=m$ give

$$\text{(2.4)}\qquad \lim_{x\to\infty}\frac{\sum_{j=1}^{k}\varepsilon\left(x^{\frac{1}{s+j}}\right)\frac{(s+j)^m(m-1)!x^{\frac{1}{s+j}}}{\log^m x}}{\frac{s^m(m-1)!x^{\frac{1}{s}}}{\log^m x}} = 0.$$

Finally, from (2.2), (2.3) and (2.4) we find that

$$\pi_s(x) = \left(\sum_{i=1}^{m} \frac{s^i (i-1)! x^{\frac{1}{s}}}{\log^i x} \right) + \varepsilon'(x) \frac{s^m (m-1)! x^{\frac{1}{s}}}{\log^m x},$$

where $\lim_{x \to \infty} \varepsilon'(x) = 0$. The theorem is proved. □

From (1.4) and (2.1) we obtain the following corollary

Corollary 2.2. *The functions* $\pi_s(x)$ *and* $\pi\left(x^{\frac{1}{s}}\right)$ *have the same asymptotic behaviour* $(s \geq 1)$.

3. THE SEQUENCES $(t_{s,n})^{\frac{1}{s}}$ AND p_n

Theorem 3.1.

$$(t_{s,n})^{\frac{1}{s}} = p_n + o\left(\frac{n}{\log^r n}\right) \qquad (r \geq 0). \tag{3.1}$$

Proof. We proceed by mathematical induction on r.

Equation (2.1) gives $(m = 1)$

$$\lim_{x \to \infty} \frac{\pi_s(x)}{\frac{s x^{\frac{1}{s}}}{\log x}} = 1. \tag{3.2}$$

If we put $x = t_{s,n}$, we get

$$\lim_{n \to \infty} \frac{(t_{s,n})^{\frac{1}{s}}}{n \log (t_{s,n})^{\frac{1}{s}}} = 1. \tag{3.3}$$

From (3.3) we find that

$$\lim_{n \to \infty} \left(\log s + \log (t_{s,n})^{\frac{1}{s}} - \log n - \log \log t_{s,n} \right) = 0. \tag{3.4}$$

Now, since

$$\lim_{n \to \infty} \frac{\log (t_{s,n})^{\frac{1}{s}}}{\log n} = 1, \tag{3.5}$$

we obtain

$$\lim_{n \to \infty} \frac{(t_{s,n})^{\frac{1}{s}}}{n \log (t_{s,n})^{\frac{1}{s}}} = 1$$

if and only if

$$\lim_{n \to \infty} \frac{(t_{s,n})^{\frac{1}{s}}}{n \log n} = 1.$$

We also derive

$$\lim_{n \to \infty} \frac{t_{s,n}}{n^s \log^s n} = 1.$$

From (3.5) we find that

$$\lim_{n \to \infty} \left(-\log s + \log \log t_{s,n} - \log \log n \right) = 0. \tag{3.6}$$

(3.4) and (3.6) give

$$\log (t_{s,n})^{\frac{1}{s}} = \log n + \log \log n + o(1). \tag{3.7}$$

Equation (2.1) gives ($m=2$)

$$\pi_s(x)=\frac{x^{\frac{1}{s}}}{\log x^{\frac{1}{s}}}+\frac{x^{\frac{1}{s}}}{\log^2\left(x^{\frac{1}{s}}\right)}+o\left(\frac{x^{\frac{1}{s}}}{\log^2\left(x^{\frac{1}{s}}\right)}\right),$$

so

$$x^{\frac{1}{s}}=\pi_s(x)\log x^{\frac{1}{s}}-\frac{x^{\frac{1}{s}}}{\log x^{\frac{1}{s}}}+o\left(\frac{x^{\frac{1}{s}}}{\log x^{\frac{1}{s}}}\right).$$

If we put $x=t_{s,n}$, we get

$$(t_{s,n})^{\frac{1}{s}}=n\log\left(t_{s,n}\right)^{\frac{1}{s}}-\frac{(t_{s,n})^{\frac{1}{s}}}{\log\left(t_{s,n}\right)^{\frac{1}{s}}}+o\left(\frac{(t_{s,n})^{\frac{1}{s}}}{\log\left(t_{s,n}\right)^{\frac{1}{s}}}\right). \tag{3.8}$$

Finally, from (3.8), (3.3) and (3.7) we find that

$$(t_{s,n})^{\frac{1}{s}}=n\log n+n\log\log n-n+o\left(n\right). \tag{3.9}$$

Therefore, for $r=0$ the theorem is true because of (1.2) and (3.9).

Let $r\geq 0$ be given, and assume that the theorem holds for r, we will prove it is also true for $r+1$.

From the inductive hypothesis we have (in view of (1.1))

$$p_n=n\log n+n\log\log n-n+\sum_{j=1}^{r}\frac{(-1)^{j-1}nP_j(\log\log n)}{\log^j n}+o\left(\frac{n}{\log^r n}\right), \tag{3.10}$$

and

$$(t_{s,n})^{\frac{1}{s}}=n\log n+n\log\log n-n+\sum_{j=1}^{r}\frac{(-1)^{j-1}nP_j(\log\log n)}{\log^j n}+o\left(\frac{n}{\log^r n}\right). \tag{3.11}$$

From (3.10) we find that

$$\log p_n=\log n+\log\log n+\log\left[1+\frac{\log\log n-1}{\log n}+\sum_{j=1}^{r}\frac{(-1)^{j-1}P_j(\log\log n)}{\log^{j+1} n}+o\left(\frac{1}{\log^{r+1} n}\right)\right]. \tag{3.12}$$

Let us write (1.3) in the form

$$\pi(x)=\left(\sum_{i=1}^{r+3}\frac{(i-1)!x}{\log^i x}\right)+o\left(\frac{x}{\log^{r+3} x}\right). \tag{3.13}$$

If we put $x=p_n$ and use the prime number theorem, we get

$$\frac{n}{p_n}=\left(\sum_{i=1}^{r+3}\frac{(i-1)!}{\log^i p_n}\right)+o\left(\frac{1}{\log^{r+3} n}\right). \tag{3.14}$$

Similarly, from (3.11) we find that

$$\log\left(t_{s,n}\right)^{\frac{1}{s}}=\log n+\log\log n+\log\left[1+\frac{\log\log n-1}{\log n}+\sum_{j=1}^{r}\frac{(-1)^{j-1}P_j(\log\log n)}{\log^{j+1} n}+o\left(\frac{1}{\log^{r+1} n}\right)\right]. \tag{3.15}$$

Let us write (2.1) in the form

$$\pi_s(x) = \left(\sum_{i=1}^{r+3} \frac{(i-1)!x^{\frac{1}{s}}}{\log^i \left(x^{\frac{1}{s}} \right)} \right) + o\left(\frac{x^{\frac{1}{s}}}{\log^{r+3}\left(x^{\frac{1}{s}} \right)} \right). \tag{3.16}$$

If we put $x = t_{s,n}$ and use (3.5), we get

$$\frac{n}{(t_{s,n})^{\frac{1}{s}}} = \left(\sum_{i=1}^{r+3} \frac{(i-1)!}{\log^i \left((t_{s,n})^{\frac{1}{s}} \right)} \right) + o\left(\frac{1}{\log^{r+3} n} \right). \tag{3.17}$$

If $x \geq 1$ and $y \geq 1$, Lagrange's theorem gives us the inequality

$$|\log y - \log x| \leq |y - x|$$

with (3.12) and (3.15), it leads to

$$\log (t_{s,n})^{\frac{1}{s}} - \log p_n = o\left(\frac{1}{\log^{r+1} n} \right). \tag{3.18}$$

From (3.18) we find that

$$\frac{1}{\log^k p_n} - \frac{1}{\log^k (t_{s,n})^{\frac{1}{s}}} = o\left(\frac{1}{\log^{r+k+2} n} \right) = o\left(\frac{1}{\log^{r+3} n} \right) \quad (k = 1, \dots, r+3). \tag{3.19}$$

(3.14), (3.17) and (3.19) give

$$\frac{n}{p_n} - \frac{n}{(t_{s,n})^{\frac{1}{s}}} = o\left(\frac{1}{\log^{r+3} n} \right),$$

that is

$$(t_{s,n})^{\frac{1}{s}} - p_n = (t_{s,n})^{\frac{1}{s}} \frac{1}{\log^{r+2} n} o(1). \tag{3.20}$$

If we write

$$(t_{s,n})^{\frac{1}{s}} = p_n + f(n) \tag{3.21}$$

substituting (3.21) into (3.20) we find that

$$f(n) = \frac{p_n}{\log^{r+2} n + o(1)} o(1),$$

so

$$f(n) = o\left(\frac{n}{\log^{r+1} n} \right) \tag{3.22}$$

(3.21) and (3.22) give

$$(t_{s,n})^{\frac{1}{s}} = p_n + o\left(\frac{n}{\log^{r+1} n} \right).$$

The theorem is thus proved. □

4. THE ASYMPTOTIC BEHAVIOUR OF $t_{s,n}$

Theorem 4.1. *There exists a unique sequence $P_{s,j}(X)$ $(j \geq 1)$ of polynomials with rational coefficients such that, for every nonnegative integer m*

$$t_{s,n} = \sum c_i f_i(n) + \sum_{j=1}^{m} \frac{(-1)^{j-1} n^s P_{s,j}(\log\log n)}{\log^j n} + o\left(\frac{n^s}{\log^m n}\right). \tag{4.1}$$

The polynomials $P_{s,j}(X)$ have degree $j+s-1$ and leading coefficient $\frac{1}{\binom{j+s-1}{s}}$.

The $f_i(n)$ are sequences of the form $n^s \log^r n\,(\log\log n)^u$ and the c_i are constants. $f_1(n) = n^s \log^s n$ and $c_1 = 1$, if $i \neq 1$ then $f_i(n) = o(f_1(n))$.

If $m = 0$ equation (4.1) is

$$t_{s,n} = \sum c_i f_i(n) + o(n^s). \tag{4.2}$$

Proof. From (1.1) and (3.1) we obtain (4.1)

$$\begin{aligned} t_{s,n} &= \left[n\log n + n\log\log n - n + \sum_{j=1}^{m+s-1} \frac{(-1)^{j-1} n P_j(\log\log n)}{\log^j n} + o\left(\frac{n}{\log^{m+s-1} n}\right)\right]^s \\ &= \sum c_i f_i(n) + \sum_{j=1}^{m} \frac{(-1)^{j-1} n^s P_{s,j}(\log\log n)}{\log^j n} + o\left(\frac{n^s}{\log^m n}\right) \end{aligned}$$

if we write

$$P_{s,j}(X) = \sum_{(r,k)} \sum_{j_1+\dots\dots+j_t=j+r} (-1)^{r-t+1} \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j_1}(X) \cdots P_{j_t}(X), \tag{4.3}$$

where $r+k+t = s$.

The first sum runs through the vectors (r,k) ($r \geq 0$, $k \geq 0$, $r+k \in \{0,1,\dots,s-1\}$), such that the set of vectors $(j_1, j_2, \dots, j_t)$ whose coordinates are positive integers which satisfy $j_1 + j_2 + \cdots + j_t = j + r$ is nonempty. The second sum runs through the former nonempty set of vectors $(j_1, j_2, \dots, j_t)$ (this set depends on the vector (r,k)).

If $m = 0$ we obtain (4.2).

Let us consider a vector (r,k). The degree of each polynomial

$$(-1)^{r-t+1} \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j_1}(X) \cdot P_{j_2}(X) \cdots P_{j_t}(X)$$

is $j+r+k$. Hence the degree of the polynomial

$$\sum_{j_1+j_2+\cdots+j_t=j+r} (-1)^{r-t+1} \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j_1}(X) \cdot P_{j_2}(X) \cdots P_{j_t}(X) \tag{4.4}$$

does not exceed $j+r+k$. Since $r+k \in \{0,1,\dots,s-1\}$, the greatest degree of the polynomials (4.4) does not exceed $j+s-1$. On the other hand, in (4.3) there are s polynomials (4.4) of degree $j+s-1$. Since in this case $t=1$, these s polynomials are

$$(-1)^r \binom{s}{r} \binom{s-r}{k} (X-1)^k P_{j+r}(X) \qquad (r+k = s-1)$$

and their sum is

$$\sum_{r=0}^{s-1}(-1)^r\binom{s}{r}\binom{s-r}{s-r-1}(X-1)^{s-r-1}P_{j+r}(X) = \sum_{r=0}^{s-1}(-1)^r\binom{s}{r}(s-r)(X-1)^{s-r-1}P_{j+r}(X). \tag{4.5}$$

Since the leading coefficient of the polynomial $P_{j+r}(X)$ is $\frac{1}{j+r}$, the leading coefficient of the polynomial (4.5) will be

$$\sum_{r=0}^{s-1}(-1)^r\binom{s}{r}\frac{s-r}{j+r}=\frac{1}{\binom{j+s-1}{s}}.$$

Hence the degree of the polynomial (4.3) is $j+s-1$ and its leading coefficient is $\frac{1}{\binom{j+s-1}{s}}$. The theorem is thus proved. □

Examples.

$$t_{1,n}=n\log n+n\log\log n-n+\sum_{j=1}^{m}\frac{(-1)^{j-1}nP_j(\log\log n)}{\log^j n}+o\left(\frac{n}{\log^m n}\right),$$

$$t_{2,n}=n^2\log^2 n+2n^2\log n\log\log n-2n^2\log n+n^2(\log\log n)^2 - 3n^2+\sum_{j=1}^{m}\frac{(-1)^{j-1}n^2P_{2,j}(\log\log n)}{\log^j n}+o\left(\frac{n^2}{\log^m n}\right).$$

Corollary 4.2. *The sequences $t_{s,n}$ and p_n^s $(s\geq 1)$ have the same asymptotic expansion, namely (4.1).*

Note. G. Mincu [2] proved Theorem 3.1 and Theorem 4.1 when $s=2$.

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