Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 1, Article 5, 2002

# L'HOSPITAL TYPE RULES FOR MONOTONICITY, WITH APPLICATIONS

IOSIF PINELIS

DEPARTMENT OF MATHEMATICAL SCIENCES
MICHIGAN TECHNOLOGICAL UNIVERSITY
HOUGHTON, MI 49931, USA
ipinelis@mtu.edu

*Received 29 January, 2001; accepted 11 August, 2001.*
*Communicated by Feng Qi*

ABSTRACT. Let $f$ and $g$ be differentiable functions on an interval $(a,b)$, and let the derivative $g'$ be positive on $(a,b)$. The main result of the paper implies that, if $f(a+) = g(a+) = 0$ and $\frac{f'}{g'}$ is increasing on $(a,b)$, then $\frac{f}{g}$ is increasing on $(a,b)$.

*Key words and phrases:* L'Hospital's Rule, Monotonicity, Information inequalities.

2000 *Mathematics Subject Classification.* Primary: 26A48, 26D10; Secondary: 26D07, 60E15, 62B10, 94A17.

## 1. L'HOSPITAL TYPE RULE FOR MONOTONICITY

Let $-\infty \le a < b \le \infty$. Let $f$ and $g$ be differentiable functions on the interval $(a,b)$. Assume also that the derivative $g'$ is nonzero and does not change sign on $(a,b)$; in other words, either $g' > 0$ everywhere on $(a,b)$ or $g' < 0$ on $(a,b)$. The following statement reminds one of the l'Hospital rule for computing limits and turns out to be useful in a number of contexts.

**Proposition 1.1.** *Suppose that* $f(a+) = g(a+) = 0$ *or* $f(b-) = g(b-) = 0$. *(Then* $g$ *is nonzero and does not change sign on* $(a,b)$*, since* $g'$ *is so.)*

(1) *If* $\frac{f'}{g'}$ *is increasing on* $(a,b)$*, then* $\left(\frac{f}{g}\right)' > 0$ *on* $(a,b)$.
(2) *If* $\frac{f'}{g'}$ *is decreasing on* $(a,b)$*, then* $\left(\frac{f}{g}\right)' < 0$ *on* $(a,b)$.

*Proof.* Assume first that $f(a+) = g(a+) = 0$. Assume also that $\frac{f'}{g'}$ is increasing on $(a,b)$, as in part 1 of the proposition. Fix any $x \in (a,b)$ and consider the function

$$h_x(y) := f'(x)g(y) - g'(x)f(y), \quad y \in (a,b).$$

ISSN (electronic): 1443-5756

I am pleased to thank Flemming Topsøe for encouraging me to submit above results to JIPAM and for making his preprint [9] available.
010-01

This function is differentiable and hence continuous on $(a,b)$. Moreover, for all $y \in (a,x)$,

$$\frac{d}{dy}h_x(y) = f'(x)g'(y) - g'(x)f'(y) = g'(x)g'(y)\left(\frac{f'(x)}{g'(x)} - \frac{f'(y)}{g'(y)}\right) > 0,$$

because $g'$ is nonzero and does not change sign on $(a,b)$ and $\frac{f'}{g'}$ is increasing on $(a,b)$. Hence, the function $h_x$ is increasing on $(a,x)$; moreover, being continuous, $h_x$ is increasing on $(a,x]$. Note also that $h_x\,(a+) = 0$. It follows that $h_x(x) > 0$, and so,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2} = \frac{h_x(x)}{g(x)^2} > 0.$$

This proves part 1 of the proposition under the assumption that $f(a+) = g(a+) = 0$. The proof under the assumption that $f(b-) = g(b-) = 0$ is similar; alternatively, one may replace here $f(x)$ and $g(x)$ for all $x \in (a,b)$ by $f(a+b-x)$ and $g(a+b-x)$, respectively. Thus, part 1 is proved.

Part 2 follows from part 1: replace $f$ by $-f$. □

**Remark 1.2.** Instead of the requirement that $f$ and $g$ be differentiable on $(a,b)$, it would be enough to assume, for instance, only that $f$ and $g$ are continuous and both have finite right derivatives $f'_+$ and $g'_+$ (or finite left derivatives $f'_-$ and $g'_-$) on $(a,b)$ and then use $\frac{f'_+}{g'_+}$ (or, respectively, $\frac{f'_-}{g'_-}$) in place of $\frac{f'}{g'}$. In such a case, one would need to use the fact that, if a function $h$ is continuous on $(a,b)$ and $h'_+ > 0$ on $(a,b)$ or $h'_- > 0$ on $(a,b)$, then $h$ is increasing on $(a,b)$; cf. e.g. Theorem 3.4.4 in [1].

The following corollary is immediate from Proposition 1.1.

**Corollary 1.3.** *Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$.*

(1) *If $\frac{f'}{g'}$ is increasing on $(a,b)$, then $\frac{f}{g}$ is increasing on $(a,b)$.*
(2) *If $\frac{f'}{g'}$ is decreasing on $(a,b)$, then $\frac{f}{g}$ is decreasing on $(a,b)$.*

**Remark 1.4.** The related result that any family of probability distributions with a monotone likelihood ratio is stochastically monotone is well known in statistics; see e.g. [2] for this and many other similar statements. For the case when $f$ and $g$ are probability tail functions, a proof of Corollary 1.3 may be found in [3]. In fact, essentially the same proof remains valid for the general setting, at least when the double integrals below exist and possess the usual properties; we are reproducing that proof now, for the readers' convenience: if $f(a+) = g(a+) = 0$, $\frac{f'}{g'}$ is increasing on $(a,b)$, $g'$ does not change sign on $(a,b)$, and $a < x < y < b$, then

$$\begin{aligned} f(x)\cdot(g(y)-g(x)) &= \iint\limits_{\substack{u\in(a,x)\\ v\in(x,y)}} f'(u)g'(v)\,du\,dv \\ &< \iint\limits_{\substack{u\in(a,x)\\ v\in(x,y)}} g'(u)f'(v)\,du\,dv \qquad (1.1)\\ &= g(x)\cdot(f(y)-f(x)), \end{aligned}$$

whence $f(x)g(y) < g(x)f(y)$, and so, $\frac{f(x)}{g(x)} < \frac{f(y)}{g(y)}$; inequality (1.1) takes place because $u < v$ implies $\frac{f'(u)}{g'(u)} < \frac{f'(v)}{g'(v)}$, and so, $f'(u)g'(v) < g'(u)f'(v)$. The proof in the case when one has $f(b-) = g(b-) = 0$ instead of $f(a+) = g(a+) = 0$ is quite similar.

Ideas similar to the ones discussed above were also present, albeit implicitly, in [5].

**Remark 1.5.** Corollary 1.3 will hold if the terms "increasing" and "decreasing" are replaced everywhere by "non-decreasing" and "non-increasing", respectively.

## 2. APPLICATIONS TO INFORMATION INEQUALITIES

In this section, applications of the above l'Hospital type rule to information inequalities are given. Other applications, as well as extensions and refinements of this rule, will be given in a series of papers following this one: in [7], extensions to non-monotonic ratios of functions, with applications to certain probability inequalities arising in bioequivalence studies and to problems of convexity; in [6], applications to monotonicity of the relative error of a Padé approximation for the complementary error function; in [8], applications to probability inequalities for sums of bounded random variables.

With all these applications, apparently we have only "scratched the surface". Yet, even the diversity of the cited results suggests that the monotonicity counterparts of the l'Hospital Rule may have as wide a range of application as the l'Hospital Rule itself.

Consider now the entropy function

$$H(p,q) := -p\ln p - q\ln q,$$

where $q := 1 - p$ and $p \in (0,1)$. In effect, it is a function of one variable, say $p$.

Topsøe [9] proved the inequalities

$$\ln p \cdot \ln q \le H(p,q) \le \frac{\ln p \cdot \ln q}{\ln 2} \tag{2.1}$$

and

$$\ln 2 \cdot 4pq \le H(p,q) \le \ln 2 \cdot (4pq)^{1/\ln 4} \tag{2.2}$$

for all $p \in (0,1)$ and also showed that these bounds on the entropy are exact; namely, they are attained when $p \downarrow 0$ or $p = 1/2$. Topsøe also indicated promising applications of bounds (2.2) in statistics. He noticed that the bounds in (2.1) and (2.2), as well as their exactness, would naturally be obtained from the monotonicity properties stated below, using also the symmetry of the entropy function: $H(p,q) = H(q,p)$.

**Conjecture 2.1.** [9] *The ratio*

$$r(p) := \frac{\ln p \ln q}{H(p,q)}$$

*is decreasing in* $p \in (0, 1/2]$, *from* $r(0+) = 1$ *to* $r(1/2) = \ln 2$.

**Conjecture 2.2.** [9] *The ratio*

$$R(p) := \frac{\ln\left(\frac{H(p,q)}{\ln 2}\right)}{\ln(4pq)}$$

*is decreasing on* $(0, 1/2)$, *from* $R(0+) = 1$ *to* $R\left(\frac{1}{2}-\right) = \frac{1}{\ln 4}$.

We shall now prove these conjectures, based on Proposition 1.1 of the previous section.

*Proof of Conjecture 2.1.* On $(0,1)$,

$$r = \frac{f}{g},$$

where $f(p) := \ln p \ln q$ and $g(p) := H(p,q)$. Consider, for $p \in (0,1)$,

$$r_1(p) := \frac{f'(p)}{g'(p)} = \frac{\frac{1}{p}\ln q - \frac{1}{q}\ln p}{\ln q - \ln p}; \quad r_2(p) := \frac{f''(p)}{g''(p)} = \frac{f_2(p)}{g_2(p)},$$

where

$$f_2(p) := -(pq)^2 f''(p) = p^2 \ln p + q^2 \ln q + 2pq \quad \text{and} \quad g_2(p) := -(pq)^2 g''(p) = pq;$$

$$r_3(p) := \frac{f_2'(p)}{g_2'(p)} = \frac{2p\ln p - 2q\ln q + q - p}{q-p}; \quad r_4(p) := \frac{f_2''(p)}{g_2''(p)} = -1 - \ln pq.$$

Now we apply Proposition 1.1 repeatedly, four times. First, note that $r_4$ is decreasing on $(0,1/2)$ and $f_2'(1/2) = g_2'(1/2) = 0$; hence, $r_3$ is decreasing on $(0,1/2)$. This and $f_2(0+) = g_2(0+) = 0$ imply that $r_2$ is decreasing on $(0,1/2)$. This and $f'(1/2) = g'(1/2) = 0$ imply that $r_1$ is decreasing on $(0,1/2)$. Finally, this and $f(0+) = g(0+) = 0$ imply that $r$ is decreasing on $(0,1/2)$. □

*Proof of Conjecture 2.2.* On $(0,1/2)$,

$$R = \frac{F}{G},$$

where $F(p) := \ln\left(\dfrac{H(p,q)}{\ln 2}\right)$ and $G(p) := \ln(4pq)$. Next,

$$\frac{F'}{G'} = \frac{F_1}{G_1}, \tag{2.3}$$

where $F_1(p) := \ln q - \ln p$ and $G_1(p) := \left(\dfrac{1}{p} - \dfrac{1}{q}\right) H(p,q)$. Further, $\dfrac{F_1'}{G_1'} = \dfrac{1}{2-r_2}$, where $r_2$ is the same as in the proof of Conjecture 2.1, and $r_2$ is decreasing on $(0,1/2)$, as was shown. In addition, $r_2 < 2$ on $(0,1)$. Hence, $\dfrac{F_1'}{G_1'} = \dfrac{1}{2-r_2}$ is decreasing on $(0,1/2)$. Also, $F_1(1/2) = G_1(1/2) = 0$. Now Proposition 1.1 implies that $\dfrac{F_1}{G_1}$ is decreasing on $(0,1/2)$; hence, by (2.3), $\dfrac{F'}{G'}$ is decreasing on $(0,1/2)$. It remains to notice that $F(1/2) = G(1/2) = 0$ and use once again Proposition 1.1. □

It might seem surprising that these proofs uncover a connection between the two seemingly unrelated conjectures – via the ratio $r_2$.

Concerning other proofs of Conjecture 2.1, see the final version of [9]. Concerning another conjecture by Topsøe [9], related to Conjecture 2.2, see [7].

## REFERENCES

[1] R. KANNAN AND C.K. KRUEGER, *Advanced Analysis on the Real Line*, Springer, New York, 1996.

[2] J. KEILSON AND U. SUMITA, Uniform stochastic ordering and related inequalities, *Canad. J. Statist.*, **10** (1982), 181–198.

[3] I. PINELIS, Extremal probabilistic problems and Hotelling's $T^2$ test under symmetry condition, *Preprint* (1991).

[4] I. PINELIS, Extremal probabilistic problems and Hotelling's $T^2$ test under a symmetry condition, *Ann. Stat.*, **22** (1994), 357–368.

[5] I. PINELIS, On the Yao-Iyer inequality in bioequivalence studies. *Math. Inequal. Appl.* (2001), 161–162.

[6] I. PINELIS, Monotonicity Properties of the Relative Error of a Padé Approximation for Mills' Ratio, *J. Ineq. Pure & Appl. Math.*, **3**(2) (2002), Article 20. (http://jipam.vu.edu.au/v3n2/012_01.html).

[7] I. PINELIS, L'Hospital type rules for oscillation, with applications, *J. Ineq. Pure & Appl. Math.*, **2**(3) (2001), Article 33. (http://jipam.vu.edu.au/v2n3/011_01.html).

[8] I. PINELIS, L'Hospital type rules for monotonicity: an application to probability inequalities for sums of bounded random variables, *J. Ineq. Pure & Appl. Math.*, **3**(1) (2002), Article 7. (http://jipam.vu.edu.au/v3n1/013_01.html).

[9] F. TOPSØE, Bounds for entropy and divergence for distributions over a two-element set, *J. Ineq. Pure & Appl. Math.*, **2**(2) (2001), Article 25. (http://jipam.vu.edu.au/v2n2/044_00.html).