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A SURVEY ON CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE DISCRETE INEQUALITIES

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ABSTRACT. The main purpose of this survey is to identify and highlight the discrete inequalities that are connected with $(CBS)-$ inequality and provide refinements and reverse results as well as to study some functional properties of certain mappings that can be naturally associated with this inequality such as superadditivity, supermultiplicity, the strong versions of these and the corresponding monotonicity properties. Many companion, reverse and related results both for real and complex numbers are also presented.

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CONTENTS

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1. INTRODUCTION

The Cauchy-Bunyakovsky-Schwarz inequality, or for short, the $(CBS)-$ inequality, plays an important role in different branches of Modern Mathematics including Hilbert Spaces Theory, Probability & Statistics, Classical Real and Complex Analysis, Numerical Analysis, Qualitative Theory of Differential Equations and their applications.

The main purpose of this survey is to identify and highlight the discrete inequalities that are connected with $(CBS)-$ inequality and provide refinements and reverse results as well as to study some functional properties of certain mappings that can be naturally associated with this inequality such as superadditivity, supermultiplicity, the strong versions of these and the corresponding monotonicity properties. Many companions and related results both for real and complex numbers are also presented.

The first section is devoted to a number of $(CBS)-$ type inequalities that provides not only natural generalizations but also several extensions for different classes of analytic functions of a real variable. A generalization of the Wagner inequality for complex numbers is obtained. Several results discovered by the author in the late eighties and published in different journals of lesser circulation are also surveyed.

The second section contains different refinements of the $(CBS)-$ inequality including de Bruijn's inequality, McLaughlin's inequality, the Daykin-Eliezer-Carlitz result in the version presented by Mitrinović-Pečarić and Fink as well as the refinements of a particular version obtained by Alzer and Zheng. A number of new results obtained by the author, which are connected with the above ones, are also presented.

Section 4 is devoted to the study of functional properties of different mappings naturally associated to the $(CBS)-$ inequality. Properties such as superadditivity, strong superadditivity, monotonicity and supermultiplicity and the corresponding inequalities are mentioned.

In the next section, Section 5, reverse results for the $(CBS)-$ inequality are surveyed. The results of Cassels, Pólya-Szegö, Greub-Rheinbold, Shisha-Mond and Zagier are presented with their original proofs. New results and versions for complex numbers are also obtained. Reverse results in terms of $p-$norms of the forward difference recently discovered by the author and some refinements of Cassels and Pólya-Szegö results obtained via Andrica-Badea inequality are mentioned. Some new facts derived from Grüss type inequalities are also pointed out.

Section 6 is devoted to various inequalities related to the $(CBS)-$ inequality. The two inequalities obtained by Ostrowski and Fan-Todd results are presented. New inequalities obtained via Jensen type inequality for convex functions are derived, some inequalities for the Čebyşev functionals are pointed out. Versions for complex numbers that generalize Ostrowski results are also emphasised.

It was one of the main aims of the survey to provide complete proofs for the results considered. We also note that in most cases only the original references are mentioned. Each section concludes with a list of the references utilized and thus may be read independently.

Being self contained, the survey may be used by both postgraduate students and researchers interested in Theory of Inequalities & Applications as well as by Mathematicians and other Scientists dealing with numerical computations, bounds and estimates where the $(CBS)-$ inequality may be used as a powerful tool.

The author intends to continue this survey with another one devoted to the functional and integral versions of the $(CBS)-$ inequality. The corresponding results holding in inner-product and normed spaces will be considered as well.

2. (CBS) – TYPE INEQUALITIES

2.1. (CBS) −**Inequality for Real Numbers.** The following inequality is known in the literature as *Cauchy's* or *Cauchy-Schwarz's* or *Cauchy-Bunyakovsky-Schwarz's* inequality. For simplicity, we shall refer to it throughout this work as the (CBS) −inequality.

Theorem 2.1. *If* $\bar{\mathrm{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathrm{b}} = (b_1, \dots, b_n)$ *are sequences of real numbers, then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \tag{2.1}$$

with equality if and only if the sequences $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *are proportional, i.e., there is a* $r \in \mathbb{R}$ *such that* $a_k = r b_k$ *for each* $k \in \{1, \dots, n\}$.

Proof. (1) Consider the quadratic polynomial $P : \mathbb{R} \to \mathbb{R}$,

$$P(t) = \sum_{k=1}^{n} (a_k t - b_k)^2 . \tag{2.2}$$

It is obvious that

$$P(t) = \left(\sum_{k=1}^{n} a_k^2\right) t^2 - 2\left(\sum_{k=1}^{n} a_k b_k\right) t + \sum_{k=1}^{n} b_k^2$$

for any $t \in \mathbb{R}$.

Since $P(t) \ge 0$ for any $t \in \mathbb{R}$ it follows that the discriminant Δ of P is negative, i.e.,

$$0 \ge \frac{1}{4}\Delta = \left(\sum_{k=1}^{n} a_k b_k\right)^2 - \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

and the inequality (2.1) is proved.

(2) If we use Lagrange's identity

$$\begin{aligned} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 &= \frac{1}{2}\sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2 \\ &= \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i)^2 \end{aligned} \tag{2.3}$$

then (2.1) obviously holds.

The equality holds in (2.1) iff

$$(a_i b_j - a_j b_i)^2 = 0$$

for any $i, j \in \{1, \dots, n\}$ which is equivalent with the fact that $\bar{\mathrm{a}}$ and $\bar{\mathrm{b}}$ are proportional. □

Remark 2.2. The inequality (2.1) apparently was firstly mentioned in the work [2] of A.L. Cauchy in 1821. The integral form was obtained in 1859 by V.Y. Bunyakovsky [1]. The corresponding version for inner-product spaces obtained by H.A. Schwartz is mainly known as Schwarz's inequality. For a short history of this inequality see [3]. In what follows we use the spelling adopted in the paper [3]. For other spellings of Bunyakovsky's name, see MathSciNet.

2.2. $(CBS)-$**Inequality for Complex Numbers.** The following version of the $(CBS)-$inequality for complex numbers holds [4, p. 84].

Theorem 2.3. *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are sequences of complex numbers, then*

$$\left| \sum_{k=1}^{n} a_k b_k \right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2, \tag{2.4}$$

with equality if and only if there is a complex number $c \in \mathbb{C}$ *such that* $a_k = c\bar{b}_k$ *for any* $k \in \{1, \dots, n\}$.

Proof. (1) For any complex number $\lambda \in \mathbb{C}$ one has the equality

$$\begin{aligned} \sum_{k=1}^{n} \left| a_k - \lambda \bar{b}_k \right|^2 &= \sum_{k=1}^{n} \left(a_k - \lambda \bar{b}_k \right) \left(\bar{a}_k - \bar{\lambda} b_k \right) \\ &= \sum_{k=1}^{n} |a_k|^2 + |\lambda|^2 \sum_{k=1}^{n} |b_k|^2 - 2 \operatorname{Re} \left(\bar{\lambda} \sum_{k=1}^{n} a_k b_k \right). \end{aligned} \tag{2.5}$$

If in (2.5) we choose $\lambda_0 \in \mathbb{C}$,

$$\lambda_0 := \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} |b_k|^2}, \qquad \bar{\mathbf{b}} \neq \mathbf{0}$$

then we get the identity

$$0 \leq \sum_{k=1}^{n} \left| a_k - \lambda_0 \bar{b}_k \right|^2 = \sum_{k=1}^{n} |a_k|^2 - \frac{\left| \sum_{k=1}^{n} a_k b_k \right|^2}{\sum_{k=1}^{n} |b_k|^2}, \tag{2.6}$$

which proves (2.4).

By virtue of (2.6), we conclude that equality holds in (2.4) if and only if $a_k = \lambda_0 \bar{b}_k$ for any $k \in \{1, \dots, n\}$.

(2) Using Binet-Cauchy's identity for complex numbers

$$\begin{aligned} \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} z_i t_i - \sum_{i=1}^{n} x_i t_i \sum_{i=1}^{n} z_i y_i &= \frac{1}{2} \sum_{i,j=1}^{n} (x_i z_j - x_j z_i)(y_i t_j - y_j t_i) \\ &= \sum_{1 \leq i < j \leq n} (x_i z_j - x_j z_i)(y_i t_j - y_j t_i) \end{aligned} \tag{2.7}$$

for the choices $x_i = \bar{a}_i$, $z_i = b_i$, $y_i = a_i$, $t_i = \bar{b}_i$, $i = \{1, \dots, n\}$, we get

$$\begin{aligned} \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} a_i b_i \right|^2 &= \frac{1}{2} \sum_{i,j=1}^{n} \left| \bar{a}_i b_j - \bar{a}_j b_i \right|^2 \\ &= \sum_{1 \leq i < j \leq n} \left| \bar{a}_i b_j - \bar{a}_j b_i \right|^2. \end{aligned} \tag{2.8}$$

Now the inequality (2.4) is a simple consequence of (2.8).

The case of equality is obvious by the identity (2.8) as well.

□

Remark 2.4. By the (CBS) −inequality for real numbers and the generalised triangle inequality for complex numbers

$$\sum_{i=1}^{n}|z_i| \geq \left|\sum_{i=1}^{n} z_i\right|, \quad z_i \in \mathbb{C}, \ \ i \in \{1,\dots,n\}$$

we also have

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n}|a_k b_k|\right)^2 \leq \sum_{k=1}^{n}|a_k|^2 \sum_{k=1}^{n}|b_k|^2 .$$

Remark 2.5. The Lagrange identity for complex numbers stated in [4, p. 85] is wrong. It should be corrected as in (2.8).

2.3. **An Additive Generalisation.** The following generalisation of the (CBS) −inequality was obtained in [5, p. 5].

Theorem 2.6. *If* $\bar{\mathbf{a}} = (a_1,\dots,a_n)$, $\bar{\mathbf{b}} = (b_1,\dots,b_n)$, $\bar{\mathbf{c}} = (c_1,\dots,c_n)$ *and* $\bar{\mathbf{d}} = (d_1,\dots,d_n)$ *are sequences of real numbers and* $\bar{\mathbf{p}} = (p_1,\dots,p_n)$, $\bar{\mathbf{q}} = (q_1,\dots,q_n)$ *are nonnegative, then*

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} q_i b_i^2 + \sum_{i=1}^{n} p_i c_i^2 \sum_{i=1}^{n} q_i d_i^2 \geq 2\sum_{i=1}^{n} p_i a_i c_i \sum_{i=1}^{n} q_i b_i d_i. \tag{2.9}$$

If $\bar{\mathbf{p}}$ *and* $\bar{\mathbf{q}}$ *are sequences of positive numbers, then the equality holds in (2.9) iff* $a_i b_j = c_i d_j$ *for any* $i,j \in \{1,\dots,n\}$.

Proof. We will follow the proof from [5].

From the elementary inequality

$$a^2 + b^2 \geq 2ab \text{ for any } a,b \in \mathbb{R} \tag{2.10}$$

with equality iff $a = b$, we have

$$a_i^2 b_j^2 + c_i^2 d_j^2 \geq 2a_i c_i b_j d_j \ \text{ for any } \ i,j \in \{1,\dots,n\}. \tag{2.11}$$

Multiplying (2.11) by $p_i q_j \geq 0$, $i,j \in \{1,\dots,n\}$ and summing over i and j from 1 to n, we deduce (2.9).

If $p_i, q_j > 0$ $(i = 1,\dots,n)$, then the equality holds in (2.9) iff $a_i b_j = c_i d_j$ for any $i,j \in \{1,\dots,n\}$. □

Remark 2.7. The condition $a_i b_j = c_i d_j$ for $c_i \neq 0, b_j \neq 0$ $(i,j = 1,\dots,n)$ is equivalent with $\frac{a_i}{c_i} = \frac{d_j}{b_j}$ $(i,j = 1,\dots,n)$, i.e., $\bar{\mathbf{a}}$, $\bar{\mathbf{c}}$ and $\bar{\mathbf{b}}$, $\bar{\mathbf{d}}$ are proportional with the same constant k.

Remark 2.8. If in (2.9) we choose $p_i = q_i = 1$ $(i = 1,\dots,n)$, $c_i = b_i$, and $d_i = a_i$ $(i = 1,\dots,n)$, then we recapture the (CBS) −inequality.

The following corollary holds [5, p. 6].

Corollary 2.9. *If* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ *and* $\bar{\mathbf{d}}$ *are nonnegative, then*

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i^3 c_i \sum_{i=1}^{n} b_i^3 d_i + \sum_{i=1}^{n} c_i^3 a_i \sum_{i=1}^{n} d_i^3 b_i\right] \geq \sum_{i=1}^{n} a_i^2 c_i^2 \sum_{i=1}^{n} b_i^2 d_i^2, \tag{2.12}$$

$$\frac{1}{2}\left[\sum_{i=1}^{n} a_i^2 b_i d_i \cdot \sum_{i=1}^{n} b_i^2 a_i c_i + \sum_{i=1}^{n} c_i^2 b_i d_i \cdot \sum_{i=1}^{n} d_i^2 a_i c_i\right] \geq \left(\sum_{i=1}^{n} a_i b_i c_i d_i\right)^2 . \tag{2.13}$$

Another result is embodied in the following corollary [5, p. 6].

Corollary 2.10. *If* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ *and* $\bar{\mathbf{d}}$ *are sequences of positive and real numbers, then:*

$$\frac{1}{2}\left[\sum_{i=1}^{n}\frac{a_i^3}{c_i}\sum_{i=1}^{n}\frac{b_i^3}{d_i}+\sum_{i=1}^{n}a_ic_i\sum_{i=1}^{n}b_id_i\right]\geq\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2, \tag{2.14}$$

$$\frac{1}{2}\left[\sum_{i=1}^{n}\frac{a_i^2b_i}{c_i}\sum_{i=1}^{n}\frac{b_i^2a_i}{d_i}+\sum_{i=1}^{n}b_ic_i\sum_{i=1}^{n}a_id_i\right]\geq\left(\sum_{i=1}^{n}a_ib_i\right)^2. \tag{2.15}$$

Finally, we also have [5, p. 6].

Corollary 2.11. *If* $\bar{\mathbf{a}}$, *and* $\bar{\mathbf{b}}$ *are positive, then*

$$\frac{1}{2}\left[\sum_{i=1}^{n}\frac{a_i^3}{b_i}\sum_{i=1}^{n}\frac{b_i^3}{a_i}-\left(\sum_{i=1}^{n}a_ib_i\right)^2\right]\geq\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2\geq 0.$$

The following version for complex numbers also holds.

Theorem 2.12. *Let* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{c}}=(c_1,\dots,c_n)$ *and* $\bar{\mathbf{d}}=(d_1,\dots,d_n)$ *be sequences of complex numbers and* $\bar{\mathbf{p}}=(p_1,\dots,p_n)$, $\bar{\mathbf{q}}=(q_1,\dots,q_n)$ *are nonnegative. Then one has the inequality*

$$\sum_{i=1}^{n}p_i|a_i|^2\sum_{i=1}^{n}q_i|b_i|^2+\sum_{i=1}^{n}p_i|c_i|^2\sum_{i=1}^{n}q_i|d_i|^2\geq 2\operatorname{Re}\left[\sum_{i=1}^{n}p_ia_i\bar{c}_i\sum_{i=1}^{n}q_ib_i\bar{d}_i\right]. \tag{2.16}$$

The case of equality for $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ *positive holds iff* $a_ib_j=c_id_j$ *for any* $i,j\in\{1,\dots,n\}$.

Proof. From the elementary inequality for complex numbers

$$|a|^2+|b|^2\geq 2\operatorname{Re}\left[a\bar{b}\right],\quad a,b\in\mathbb{C},$$

with equality iff $a=b$, we have

$$|a_i|^2|b_j|^2+|c_i|^2|d_j|^2\geq 2\operatorname{Re}\left[a_i\bar{c}_ib_j\bar{d}j\right] \tag{2.17}$$

for any $i,j\in\{1,\dots,n\}$. Multiplying (2.17) by $p_iq_j\geq 0$ and summing over i and j from 1 to n, we deduce (2.16).

The case of equality is obvious and we omit the details. □

Remark 2.13. Similar particular cases may be stated but we omit the details.

2.4. **A Related Additive Inequality.** The following inequality was obtained in [5, Theorem 1.1].

Theorem 2.14. *If* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ *are sequences of real numbers and* $\bar{\mathbf{c}}=(c_1,\dots,c_n)$, $\bar{\mathbf{d}}=(d_1,\dots,d_n)$ *are nonnegative, then*

$$\sum_{i=1}^{n}d_i\sum_{i=1}^{n}c_ia_i^2+\sum_{i=1}^{n}c_i\sum_{i=1}^{n}d_ib_i^2\geq 2\sum_{i=1}^{n}c_ia_i\sum_{i=1}^{n}d_ib_i. \tag{2.18}$$

If c_i *and* d_i $(i=1,\dots,n)$ *are positive, then equality holds in (2.18) iff* $\bar{\mathbf{a}}=\bar{\mathbf{b}}=\bar{\mathbf{k}}$ *where* $\bar{\mathbf{k}}=(k,k,\dots,k)$ *is a constant sequence.*

Proof. We will follow the proof from [5].

From the elementary inequality

$$a^2+b^2\geq 2ab\ \text{ for any }\ a,b\in\mathbb{R} \tag{2.19}$$

with equality iff $a = b$; we have

$$a_i^2 + b_j^2 \geq 2a_i b_j \text{ for any } i, j \in \{1, \dots, n\}. \tag{2.20}$$

Multiplying (2.20) by $c_i d_j \geq 0$, $i, j \in \{1, \dots, n\}$ and summing over i from 1 to n and over j from 1 to n, we deduce (2.18).

If $c_i, d_j > 0$ $(i = 1, \dots, n)$, then the equality holds in (2.18) iff $a_i = b_j$ for any $i, j \in \{1, \dots, n\}$ which is equivalent with the fact that $a_i = b_i = k$ for any $i \in \{1, \dots, n\}$. □

The following corollary holds [5, p. 4].

Corollary 2.15. *If* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *are nonnegative sequences, then*

$$\frac{1}{2}\left[\sum_{i=1}^n a_i^3 \sum_{i=1}^n b_i + \sum_{i=1}^n a_i \sum_{i=1}^n b_i^3\right] \geq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2; \tag{2.21}$$

$$\frac{1}{2}\left[\sum_{i=1}^n a_i \sum_{i=1}^n a_i^2 b_i + \sum_{i=1}^n b_i \sum_{i=1}^n b_i^2 a_i\right] \geq \left(\sum_{i=1}^n a_i b_i\right)^2. \tag{2.22}$$

Another corollary that may be obtained is [5, p. 4 – 5].

Corollary 2.16. *If* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *are sequences of positive real numbers, then*

$$\sum_{i=1}^n \frac{a_i^2 + b_i^2}{2a_i b_i} \geq \frac{\sum_{i=1}^n \frac{1}{a_i} \sum_{i=1}^n \frac{1}{b_i}}{\sum_{i=1}^n \frac{1}{a_i b_i}}, \tag{2.23}$$

$$\sum_{i=1}^n a_i \sum_{i=1}^n \frac{1}{b_i} + \sum_{i=1}^n \frac{1}{a_i} \sum_{i=1}^n b_i \geq 2n^2, \tag{2.24}$$

and

$$n \sum_{i=1}^n \frac{a_i^2 + b_i^2}{2a_i^2 b_i^2} \geq \sum_{i=1}^n \frac{1}{a_i} \sum_{i=1}^n \frac{1}{b_i}. \tag{2.25}$$

The following version for complex numbers also holds.

Theorem 2.17. *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are sequences of complex numbers, then for* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *and* $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *two sequences of nonnegative real numbers, one has the inequality*

$$\sum_{i=1}^n q_i \sum_{i=1}^n p_i |a_i|^2 + \sum_{i=1}^n p_i \sum_{i=1}^n q_i |b_i|^2 \geq 2 \operatorname{Re}\left[\sum_{i=1}^n p_i a_i \sum_{i=1}^n q_i \bar{b}_i\right]. \tag{2.26}$$

For $\bar{\mathbf{p}}, \bar{\mathbf{q}}$ *positive sequences, the equality holds in (2.26) iff* $\bar{\mathbf{a}} = \bar{\mathbf{b}} = \bar{\mathbf{k}} = (k, \dots, k)$.

The proof goes in a similar way with the one in Theorem 2.14 on making use of the following elementary inequality holding for complex numbers

$$|a|^2 + |b|^2 \geq 2 \operatorname{Re}\left[a\bar{b}\right], \quad a, b \in \mathbb{C}; \tag{2.27}$$

with equality iff $a = b$.

2.5. **A Parameter Additive Inequality.** The following inequality was obtained in [5, Theorem 4.1].

Theorem 2.18. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers and* $\bar{\mathbf{c}} = (c_1, \dots, c_n)$, $\bar{\mathbf{d}} = (d_1, \dots, d_n)$ *be nonnegative. If* $\alpha, \beta > 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma^2 \le \alpha\beta$, *then*

$$\alpha \sum_{i=1}^n d_i \sum_{i=1}^n a_i^2 c_i + \beta \sum_{i=1}^n c_i \sum_{i=1}^n b_i^2 d_i \ge 2\gamma \sum_{i=1}^n c_i a_i \sum_{i=1}^n d_i b_i. \tag{2.28}$$

Proof. We will follow the proof from [5].

Since $\alpha, \beta > 0$ and $\gamma^2 \le \alpha\beta$, it follows that for any $x, y \in \mathbb{R}$ one has

$$\alpha x^2 + \beta y^2 \ge 2\gamma xy. \tag{2.29}$$

Choosing in (2.29) $x = a_i$, $y = b_j$ $(i, j = 1, \dots, n)$, we get

$$\alpha a_i^2 + \beta b_j^2 \ge 2\gamma a_i b_j \text{ for any } i, j \in \{1, \dots, n\}. \tag{2.30}$$

If we multiply (2.30) by $c_i d_j \ge 0$ and sum over i and j from 1 to n, we deduce the desired inequality (2.28). □

The following corollary holds.

Corollary 2.19. *If* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are nonnegative sequences and* α, β, γ *are as in Theorem 2.18, then*

$$\alpha \sum_{i=1}^n b_i \sum_{i=1}^n a_i^3 + \beta \sum_{i=1}^n a_i \sum_{i=1}^n b_i^3 \ge 2\gamma \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2, \tag{2.31}$$

$$\alpha \sum_{i=1}^n a_i \sum_{i=1}^n a_i^2 b_i + \beta \sum_{i=1}^n b_i \sum_{i=1}^n b_i^2 a_i \ge 2\gamma \left(\sum_{i=1}^n a_i b_i\right)^2. \tag{2.32}$$

The following particular case is important [5, p. 8].

Theorem 2.20. *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ *be sequences of real numbers. If* $\bar{\mathbf{p}}$ *is a sequence of nonnegative real numbers with* $\sum_{i=1}^n p_i > 0$, *then:*

$$\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 \ge \frac{\sum_{i=1}^n p_i a_i b_i \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i}{\sum_{i=1}^n p_i}. \tag{2.33}$$

In particular,

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \ge \frac{1}{n} \sum_{i=1}^n a_i b_i \sum_{i=1}^n a_i \sum_{i=1}^n b_i. \tag{2.34}$$

Proof. We will follow the proof from [5, p. 8].

If we choose in Theorem 2.18, $c_i = d_i = p_i$ $(i = 1, \dots, n)$ and $\alpha = \sum_{i=1}^n p_i b_i^2$, $\beta = \sum_{i=1}^n p_i a_i^2$, $\gamma = \sum_{i=1}^n p_i a_i b_i$, we observe, by the $(CBS)-$inequality with the weights p_i $(i = 1, \dots, n)$ one has $\gamma^2 \le \alpha\beta$, and then by (2.28) we deduce (2.33). □

Remark 2.21. If we assume that $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are *asynchronous*, i.e.,

$$(a_i - a_j)(b_i - b_j) \le 0 \text{ for any } i, j \in \{1, \dots, n\},$$

then by Čebyšev's inequality

$$\sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \ge \sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i \tag{2.35}$$

respectively

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geq n \sum_{i=1}^{n} a_i b_i, \tag{2.36}$$

we have the following refinements of the (CBS) $-$inequality

$$\begin{aligned} \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 &\geq \frac{\sum_{i=1}^{n} p_i a_i b_i \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i}{\sum_{i=1}^{n} p_i} \\ &\geq \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \end{aligned} \tag{2.37}$$

provided $\sum_{i=1}^{n} p_i a_i b_i \geq 0$, respectively

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \frac{1}{n} \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2 \tag{2.38}$$

provided $\sum_{i=1}^{n} a_i b_i \geq 0$.

2.6. **A Generalisation Provided by Young's Inequality.** The following result was obtained in [5, Theorem 5.1].

Theorem 2.22. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *and* $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *be sequences of nonnegative real numbers and* $\alpha, \beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *Then one has the inequality*

$$\alpha \sum_{i=1}^{n} q_i \sum_{i=1}^{n} p_i b_i^{\beta} + \beta \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i a_i^{\alpha} \geq \alpha\beta \sum_{i=1}^{n} p_i b_i \sum_{i=1}^{n} q_i a_i. \tag{2.39}$$

If $\bar{\mathbf{p}}$ *and* $\bar{\mathbf{q}}$ *are sequences of positive real numbers, then the equality holds in (2.39) iff there exists a constant* $k \geq 0$ *such that* $a_i^{\alpha} = b_i^{\beta} = k$ *for each* $i \in \{1, \dots, n\}$.

Proof. It is, by the Arithmetic-Geometric inequality [6, p. 15], well known that

$$\frac{1}{\alpha} x + \frac{1}{\beta} y \geq x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}} \text{ for } x, y \geq 0, \ \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \ \alpha, \beta > 1 \tag{2.40}$$

with equality iff $x = y$.

Applying (2.40) for $x = a_i^{\alpha}$, $y = b_j^{\beta}$ $(i, j = 1, \dots, n)$ we have

$$\alpha b_j^{\beta} + \beta a_i^{\alpha} \geq \alpha\beta a_i b_j \text{ for any } i, j \in \{1, \dots, n\} \tag{2.41}$$

with equality iff $a_i^{\alpha} = b_j^{\beta}$ for any $i, j \in \{1, \dots, n\}$.

If we multiply (2.41) by $q_i p_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and sum over i and j from 1 to n we deduce (2.39).

The case of equality is obvious by the above considerations. □

The following corollary is a natural consequence of the above theorem.

Corollary 2.23. *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, α *and* β *be as in Theorem 2.22. Then*

$$\frac{1}{\alpha} \sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i^{\alpha+1} + \frac{1}{\beta} \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i^{\beta+1} \geq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2; \tag{2.42}$$

$$(2.43)\qquad \frac{1}{\alpha}\sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_ia_i^{\alpha}+\frac{1}{\beta}\sum_{i=1}^{n}b_i\sum_{i=1}^{n}a_ib_i^{\beta}\geq\left(\sum_{i=1}^{n}a_ib_i\right)^2.$$

The following result which provides a generalisation of the (CBS) −inequality may be obtained by Theorem 2.22 as well [5, Theorem 5.2].

Theorem 2.24. *Let $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ be sequences of positive real numbers. If α,β are as above, then*

$$(2.44)\qquad \left(\frac{1}{\alpha}\sum_{i=1}^{n}x_i^{\alpha}y_i^{2-\alpha}+\frac{1}{\beta}\sum_{i=1}^{n}x_i^{\beta}y_i^{2-\beta}\right)\cdot\sum_{i=1}^{n}y_i^2\geq\left(\sum_{i=1}^{n}x_iy_i\right)^2.$$

The equality holds iff $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are proportional.

Proof. Follows by Theorem 2.22 on choosing $p_i=q_i=y_i^2$, $a_i=\frac{x_i}{y_i}$, $b_i=\frac{x_i}{y_i}$, $i\in\{1,\dots,n\}$. □

Remark 2.25. For $\alpha=\beta=2$, we recapture the (CBS) −inequality.

Remark 2.26. For $a_i=|z_i|$, $b_i=|w_i|$, with $z_i,w_i\in\mathbb{C}$; $i=1,\dots,n$, we may obtain similar inequalities for complex numbers. We omit the details.

2.7. **Further Generalisations via Young's Inequality.** The following inequality is known in the literature as Young's inequality

$$(2.45)\qquad px^q+qy^p\geq pqxy,\quad x,y\geq 0\ \text{ and }\ \frac{1}{p}+\frac{1}{q}=1,\ p>1$$

with equality iff $x^q=y^p$.

The following result generalising the (CBS) −inequality was obtained in [7, Theorem 2.1] (see also [8, Theorem 1]).

Theorem 2.27. *Let $\bar{\mathbf{x}}=(x_1,\dots,x_n)$, $\bar{\mathbf{y}}=(y_1,\dots,y_n)$ be sequences of complex numbers and $\bar{\mathbf{p}}=(p_1,\dots,p_n)$, $\bar{\mathbf{q}}=(q_1,\dots,q_n)$ be two sequences of nonnegative real numbers. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$(2.46)\qquad \frac{1}{p}\sum_{k=1}^{n}p_k|x_k|^p\sum_{k=1}^{n}q_k|y_k|^p+\frac{1}{q}\sum_{k=1}^{n}q_k|x_k|^q\sum_{k=1}^{n}p_k|y_k|^q\geq\sum_{k=1}^{n}p_k|x_ky_k|\sum_{k=1}^{n}q_k|x_ky_k|.$$

Proof. We shall follow the proof in [7].

Choosing $x=|x_j||y_i|$, $y=|x_i||y_j|$, $i,j\in\{1,\dots,n\}$, we get from (2.45)

$$(2.47)\qquad q|x_i|^p|y_j|^p+p|x_j|^q|y_i|^q\geq pq|x_iy_i||x_jy_j|$$

for any $i,j\in\{1,\dots,n\}$.

Multiplying with $p_iq_j\geq 0$ and summing over i and j from 1 to n, we deduce the desired result (2.46). □

The following corollary is a natural consequence of the above theorem [7, Corollary 2.2] (see also [8, p. 105]).

Corollary 2.28. *If $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are as in Theorem 2.27 and $\bar{\mathbf{m}}=(m_1,\dots,m_n)$ is a sequence of nonnegative real numbers, then*

$$(2.48)\qquad \frac{1}{p}\sum_{k=1}^{n}m_k|x_k|^p\sum_{k=1}^{n}m_k|y_k|^p+\frac{1}{q}\sum_{k=1}^{n}m_k|x_k|^q\sum_{k=1}^{n}m_k|y_k|^q\geq\left(\sum_{k=1}^{n}m_k|x_ky_k|\right)^2,$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.29. If in (2.48) we assume that $m_k = 1$, $k \in \{1, \dots, n\}$, then we obtain [7, p. 7] (see also [8, p. 105])

$$\frac{1}{p}\sum_{k=1}^{n}|x_k|^p\sum_{k=1}^{n}|y_k|^p + \frac{1}{q}\sum_{k=1}^{n}|x_k|^q\sum_{k=1}^{n}|y_k|^q \geq \left(\sum_{k=1}^{n}|x_k y_k|\right)^2, \tag{2.49}$$

which, in the particular case $p = q = 2$ will provide the (CBS) −inequality.

The second generalisation of the (CBS) −inequality via Young's inequality is incorporated in the following theorem [7, Theorem 2.4] (see also [8, Theorem 2]).

Theorem 2.30. *Let* $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{p}}, \bar{\mathbf{q}}$ *and* p, q *be as in Theorem 2.27. Then one has the inequality*

$$\frac{1}{p}\sum_{k=1}^{n}p_k|x_k|^p\sum_{k=1}^{n}q_k|y_k|^q + \frac{1}{q}\sum_{k=1}^{n}q_k|x_k|^q\sum_{k=1}^{n}p_k|y_k|^p \geq \sum_{k=1}^{n}p_k|x_k||y_k|^{p-1}\sum_{k=1}^{n}q_k|x_k||y_k|^{q-1}. \tag{2.50}$$

Proof. We shall follow the proof in [7].

Choosing in (2.45), $x = \frac{|x_j|}{|y_j|}$, $y = \frac{|x_i|}{|y_i|}$, we get

$$p\left(\frac{|x_j|}{|y_j|}\right)^q + q\left(\frac{|x_i|}{|y_i|}\right)^p \geq pq\frac{|x_i||x_j|}{|y_i||y_j|} \tag{2.51}$$

for any $y_i \neq 0$, $i, j \in \{1, \dots, n\}$.

It is easy to see that (2.51) is equivalent to

$$q|x_i|^p|y_j|^q + p|y_i|^p|x_j|^q \geq pq|x_i||y_i|^{p-1}|x_j||y_j|^{q-1} \tag{2.52}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying (2.52) by $p_i q_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequlality (2.50). □

The following corollary holds [7, Corollary 2.5] (see also [8, p. 106]).

Corollary 2.31. *Let* $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{m}}$ *and* $\bar{\mathbf{p}}, \bar{\mathbf{q}}$ *be as in Corollary 2.28. Then*

$$\frac{1}{p}\sum_{k=1}^{n}m_k|x_k|^p\sum_{k=1}^{n}m_k|y_k|^q + \frac{1}{q}\sum_{k=1}^{n}m_k|x_k|^q\sum_{k=1}^{n}m_k|y_k|^p \geq \sum_{k=1}^{n}m_k|x_k||y_k|^{p-1}\sum_{k=1}^{n}m_k|x_k||y_k|^{q-1}. \tag{2.53}$$

Remark 2.32. If in (2.53) we assume that $m_k = 1$, $k \in \{1, \dots, n\}$, then we obtain [7, p. 8] (see also [8, p. 106])

$$\frac{1}{p}\sum_{k=1}^{n}|x_k|^p\sum_{k=1}^{n}|y_k|^q + \frac{1}{q}\sum_{k=1}^{n}|x_k|^q\sum_{k=1}^{n}|y_k|^p \geq \sum_{k=1}^{n}|x_k||y_k|^{p-1}\sum_{k=1}^{n}|x_k||y_k|^{q-1}, \tag{2.54}$$

which, in the particular case $p = q = 2$ will provide the (CBS) −inequality.

The third result is embodied in the following theorem [7, Theorem 2.7] (see also [8, Theorem 3]).

Theorem 2.33. *Let* $\bar{\mathrm{x}}$, $\bar{\mathrm{y}}$, $\bar{\mathrm{p}}$, $\bar{\mathrm{q}}$ *and* p, q *be as in Theorem 2.27. Then one has the inequality*

$$\frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^p \sum_{k=1}^{n} q_k |y_k|^q + \frac{1}{q}\sum_{k=1}^{n} q_k |x_k|^p \sum_{k=1}^{n} p_k |y_k|^q \geq \sum_{k=1}^{n} p_k |x_k y_k| \sum_{k=1}^{n} p_k |x_k|^{p-1} |y_k|^{q-1}. \tag{2.55}$$

Proof. We shall follow the proof in [7].

If we choose $x = \frac{|y_i|}{|y_j|}$ and $y = \frac{|x_i|}{|x_j|}$ in (2.45) we get

$$p\left(\frac{|y_i|}{|y_j|}\right)^q + q\left(\frac{|x_i|}{|x_j|}\right)^p \geq pq\frac{|x_i|\,|y_i|}{|x_j|\,|y_j|},$$

for any $x_i, y_j \neq 0$, $i, j \in \{1, \dots, n\}$, giving

$$q|x_i|^p |y_j|^q + p|y_i|^q |x_j|^p \geq pq|x_i y_i| |x_j|^{p-1} |y_j|^{q-1} \tag{2.56}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying (2.56) by $p_i q_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequality (2.55). □

The following corollary is a natural consequence of the above theorem [8, p. 106].

Corollary 2.34. *Let* $\bar{\mathrm{x}}$, $\bar{\mathrm{y}}$, $\bar{\mathrm{m}}$ *and* $\bar{\mathrm{p}}$, $\bar{\mathrm{q}}$ *be as in Corollary 2.28. Then one has the inequality:*

$$\sum_{k=1}^{n} m_k |x_k|^p \sum_{k=1}^{n} m_k |y_k|^q \geq \sum_{k=1}^{n} m_k |x_k y_k| \sum_{k=1}^{n} m_k |x_k|^{p-1} |y_k|^{q-1}. \tag{2.57}$$

Remark 2.35. If in (2.57) we assume that $m_k = 1$, $k = \{1, \dots, n\}$, then we obtain [7, p. 8] (see also [8, p. 10])

$$\sum_{k=1}^{n} |x_k|^p \sum_{k=1}^{n} |y_k|^q \geq \sum_{k=1}^{n} |x_k y_k| \sum_{k=1}^{n} |x_k|^{p-1} |y_k|^{q-1}, \tag{2.58}$$

which, in the particular case $p = q = 2$ will provide the (CBS) $-$inequality.

The fourth generalisation of the (CBS) $-$inequality is embodied in the following theorem [7, Theorem 2.9] (see also [8, Theorem 4]).

Theorem 2.36. *Let* $\bar{\mathrm{x}}$, $\bar{\mathrm{y}}$, $\bar{\mathrm{p}}$, $\bar{\mathrm{q}}$ *and* p, q *be as in Theorem 2.27. Then one has the inequality*

$$\frac{1}{q}\sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} q_k |y_k|^q + \frac{1}{p}\sum_{k=1}^{n} p_k |y_k|^2 \sum_{k=1}^{n} q_k |x_k|^p \geq \sum_{k=1}^{n} q_k |x_k y_k| \sum_{k=1}^{n} p_k |x_k|^{\frac{2}{q}} |y_k|^{\frac{2}{p}}. \tag{2.59}$$

Proof. We shall follow the proof in [7].

Choosing in (2.45), $x = |x_i|^{\frac{2}{q}} |y_j|$, $y = |x_j| |y_i|^{\frac{2}{p}}$, we get

$$p|x_i|^2 |y_j|^q + q|x_j|^p |y_i|^2 \geq pq|x_i|^{\frac{2}{q}} |y_i|^{\frac{2}{p}} |x_j y_j| \tag{2.60}$$

for any $i, j \in \{1, \dots, n\}$.

Multiply (2.60) by $p_i q_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequality (2.60). □

The following corollary holds [7, Corollary 2.10] (see also [8, p. 107]).

Corollary 2.37. *Let* $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{m}}$ *and* p, q *be as in Corollary 2.28. Then one has the inequality:*

$$\frac{1}{q}\sum_{k=1}^{n} m_k |x_k|^2 \sum_{k=1}^{n} m_k |y_k|^q + \frac{1}{p}\sum_{k=1}^{n} m_k |y_k|^2 \sum_{k=1}^{n} m_k |x_k|^p \geq \sum_{k=1}^{n} m_k |x_k y_k| \sum_{k=1}^{n} m_k |x_k|^{\frac{2}{q}} |y_k|^{\frac{2}{p}}. \tag{2.61}$$

Remark 2.38. If in (2.61) we take $m_k = 1, k \in \{1, \dots, n\}$, then we get

$$\frac{1}{q}\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^q + \frac{1}{p}\sum_{k=1}^{n} |y_k|^2 \sum_{k=1}^{n} |x_k|^p \geq \sum_{k=1}^{n} |x_k y_k| \sum_{k=1}^{n} |x_k|^{\frac{2}{q}} |y_k|^{\frac{2}{p}}, \tag{2.62}$$

which, in the particular case $p = q = 2$ will provide the (CBS) −inequality.

The fifth result generalising the (CBS) −inequality is embodied in the following theorem [7, Theorem 2.12] (see also [8, Theorem 5]).

Theorem 2.39. *Let* $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{p}}, \bar{\mathbf{q}}$ *and* p, q *be as in Theorem 2.27. Then one has the inequality*

$$\frac{1}{p}\sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} q_k |y_k|^q + \frac{1}{q}\sum_{k=1}^{n} p_k |y_k|^2 \sum_{k=1}^{n} q_k |x_k|^p \geq \sum_{k=1}^{n} p_k |x_k|^{\frac{2}{p}} |y_k|^{\frac{2}{q}} \sum_{k=1}^{n} q_k |x_k|^{p-1} |y_k|^{q-1}. \tag{2.63}$$

Proof. We will follow the proof in [7].

Choosing in (2.45), $x = \frac{|y_i|^{\frac{2}{q}}}{|y_j|}, y = \frac{|x_i|^{\frac{2}{p}}}{|x_j|}, y_i, x_j \neq 0, i, j \in \{1, \dots, n\}$, we may write

$$p\left(\frac{|y_i|^{\frac{2}{q}}}{|y_j|}\right)^q + q\left(\frac{|x_i|^{\frac{2}{p}}}{|x_j|}\right)^p \geq pq\frac{|y_i|^{\frac{2}{q}} |x_i|^{\frac{2}{p}}}{|x_j| |y_j|},$$

from where results

$$p |y_i|^2 |x_j|^p + q |x_i|^2 |y_j|^q \geq pq |x_i|^{\frac{2}{p}} |y_i|^{\frac{2}{q}} |x_j|^{p-1} |y_j|^{q-1} \tag{2.64}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying (2.64) by $p_i q_j \geq 0 \ (i, j \in \{1, \dots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequality (2.63). □

The following corollary holds [7, Corollary 2.13] (see also [8, p. 108]).

Corollary 2.40. *Let* $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{m}}$ *and* p, q *be as in Corollary 2.28. Then one has the inequality:*

$$\frac{1}{p}\sum_{k=1}^{n} m_k |x_k|^2 \sum_{k=1}^{n} m_k |y_k|^q + \frac{1}{q}\sum_{k=1}^{n} m_k |y_k|^2 \sum_{k=1}^{n} m_k |x_k|^p \geq \sum_{k=1}^{n} m_k |x_k|^{\frac{2}{p}} |y_k|^{\frac{2}{q}} \sum_{k=1}^{n} m_k |x_k|^{p-1} |y_k|^{q-1}. \tag{2.65}$$

Remark 2.41. If in (2.46) we choose $m_k = 1$, $k \in \{1, \dots, n\}$, then we get [7, p. 10] (see also [8, p. 108])

$$\frac{1}{p}\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^q+\frac{1}{q}\sum_{k=1}^{n}|y_k|^2\sum_{k=1}^{n}|x_k|^p\geq\sum_{k=1}^{n}|x_k|^{\frac{2}{p}}|y_k|^{\frac{2}{q}}\sum_{k=1}^{n}|x_k|^{p-1}|y_k|^{q-1}, \tag{2.66}$$

which in the particular case $p = q = 2$ will provide the (CBS) $-$inequality.

Finally, the following result generalising the (CBS) $-$inequality holds [7, Theorem 2.15] (see also [8, Theorem 6]).

Theorem 2.42. *Let* $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{q}}$ *and* p, q *be as in Theorem 2.27. Then one has the inequality:*

$$\frac{1}{p}\sum_{k=1}^{n}p_k|x_k|^2\sum_{k=1}^{n}q_k|y_k|^p+\frac{1}{q}\sum_{k=1}^{n}q_k|y_k|^2\sum_{k=1}^{n}p_k|x_k|^q \geq\sum_{k=1}^{n}p_k|x_k|^{\frac{2}{p}}|y_k|\sum_{k=1}^{n}q_k|x_k|^{\frac{2}{q}}|y_k|. \tag{2.67}$$

Proof. We shall follow the proof in [7].

From (2.45) one has the inequality

$$q\left(|x_i|^{\frac{2}{p}}|y_j|\right)^p+p\left(|x_j|^{\frac{2}{q}}|y_i|\right)^q\geq pq\,|x_i|^{\frac{2}{p}}|y_i|\,|x_j|^{\frac{2}{q}}|y_j| \tag{2.68}$$

for any $i, j \in \{1, \dots, n\}$.

Multiplying (2.68) by $p_i q_j \geq 0$ $(i, j \in \{1, \dots, n\})$ and summing over i and j from 1 to n, we deduce the desired inequality (2.67). □

The following corollary also holds [7, Corollary 2.16] (see also [8, p. 108]).

Corollary 2.43. *With the assumptions in Corollary 2.28, one has the inequality*

$$\sum_{k=1}^{n}m_k|x_k|^2\sum_{k=1}^{n}m_k\left(\frac{1}{p}|y_k|^p+\frac{1}{q}|y_k|^q\right)\geq\sum_{k=1}^{n}m_k|x_k|^{\frac{2}{p}}|y_k|\sum_{k=1}^{n}m_k|x_k|^{\frac{2}{q}}|y_k|. \tag{2.69}$$

Remark 2.44. If in (2.69) we choose $m_k = 1$ $(k \in \{1, \dots, n\})$, then we get

$$\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}\left(\frac{1}{p}|y_k|^p+\frac{1}{q}|y_k|^q\right)\geq\sum_{k=1}^{n}|x_k|^{\frac{2}{p}}|y_k|\sum_{k=1}^{n}|x_k|^{\frac{2}{q}}|y_k|, \tag{2.70}$$

which, in the particular case $p = q = 2$, provides the (CBS) $-$inequality.

2.8. **A Generalisation Involving** $J-$**Convex Functions.** For $a > 1$, we denote by $\exp_a$ the function

$$\exp_a : \mathbb{R} \to (0, \infty), \quad \exp_a(x) = a^x. \tag{2.71}$$

Definition 2.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $J-$convex on an interval I if

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2} \quad \text{for any} \quad x, y \in I. \tag{2.72}$$

It is obvious that any convex function on I is a J convex function on I, but the converse does not generally hold.

The following lemma holds (see [7, Lemma 4.3]).

Lemma 2.45. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $J-$convex function on I, $a > 1$ and $x, y \in \mathbb{R}\setminus\{0\}$ with $\log_a x^2, \log_a y^2 \in I$. Then $\log_a |xy| \in I$ and*

$$\{\exp_b [f(\log_a |xy|)]\}^2 \le \exp_b \left[f\left(\log_a x^2\right)\right] \exp_b \left[f\left(\log_a y^2\right)\right] \tag{2.73}$$

for any $b > 1$.

Proof. I, being an interval, is a convex set in $\mathbb{R}$ and thus

$$\log_a |xy| = \frac{1}{2}\left[\log_a x^2 + \log_a y^2\right] \in I.$$

Since f is $J-$convex, one has

$$\begin{aligned} f(\log_a |xy|) &= f\left[\frac{1}{2}\left(\log_a x^2 + \log_a y^2\right)\right] \\ &\le \frac{f(\log_a x^2) + f(\log_a y^2)}{2}. \end{aligned} \tag{2.74}$$

Taking the $\exp_b$ in both parts, we deduce

$$\begin{aligned} \exp_b [f(\log_a |xy|)] &\le \exp_b \left[\frac{f(\log_a x^2) + f(\log_a y^2)}{2}\right] \\ &= \left\{\exp_b \left[f\left(\log_a x^2\right)\right] \exp_b \left[f\left(\log_a y^2\right)\right]\right\}^{\frac{1}{2}}, \end{aligned}$$

which is equivalent to (2.73). □

The following generalisation of the $(CBS)-$inequality in terms of a $J-$ convex function holds [7, Theorem 4.4].

Theorem 2.46. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $J-$convex function on I, $a, b > 1$ and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ sequences of nonzero real numbers. If $\log_a a_k^2, \log_a b_k^2 \in I$ for all $k \in \{1, \dots, n\}$, then one has the inequality:*

$$\left\{\sum_{k=1}^n \exp_b [f(\log_a |a_k b_k|)]\right\}^2 \le \sum_{k=1}^n \exp_b \left[f\left(\log_a a_k^2\right)\right] \sum_{k=1}^n \exp_b \left[f\left(\log_a b_k^2\right)\right]. \tag{2.75}$$

Proof. Using Lemma 2.45 and the $(CBS)-$inequality one has

$$\begin{aligned} &\sum_{k=1}^n \exp_b [f(\log_a |a_k b_k|)] \\ &\quad \le \sum_{k=1}^n \left[\exp_b \left[f\left(\log_a a_k^2\right)\right] \exp_b \left[f\left(\log_a b_k^2\right)\right]\right]^{\frac{1}{2}} \\ &\quad \le \left(\sum_{k=1}^n \left\{\left[\exp_b \left[f\left(\log_a a_k^2\right)\right]\right]^{\frac{1}{2}}\right\}^2 \sum_{k=1}^n \left\{\left[\exp_b \left[f\left(\log_a b_k^2\right)\right]\right]^{\frac{1}{2}}\right\}^2\right)^{\frac{1}{2}} \end{aligned}$$

which is clearly equivalent to (2.75). □

Remark 2.47. If in (2.75) we choose $a = b > 1$ and $f(x) = x$, $x \in \mathbb{R}$, then we recapture the $(CBS)-$inequality.

2.9. **A Functional Generalisation.** The following result was proved in [10, Theorem 2].

Theorem 2.48. *Let A be a subset of real numbers $\mathbb{R}$, $f : A \to \mathbb{R}$ and $\mathbf{\bar{a}} = (a_1, \dots, a_n)$, $\mathbf{\bar{b}} = (b_1, \dots, b_n)$ sequences of real numbers with the properties that*

(i) $a_i b_i, a_i^2, b_i^2 \in A$ *for any* $i \in \{1, \dots, n\}$,

(ii) $f(a_i^2), f(b_i^2) \geq 0$ *for any* $i \in \{1, \dots, n\}$,

(iii) $f^2(a_i b_i) \leq f(a_i^2) f(b_i^2)$ *for any* $i \in \{1, \dots, n\}$.

Then one has the inequality:

$$\left[\sum_{i=1}^{n} f(a_i b_i)\right]^2 \leq \sum_{i=1}^{n} f\left(a_i^2\right) \sum_{i=1}^{n} f\left(b_i^2\right). \tag{2.76}$$

Proof. We give here a simpler proof than that found in [10].

We have

$$\begin{aligned}\left|\sum_{i=1}^{n} f(a_i b_i)\right| &\leq \sum_{i=1}^{n} |f(a_i b_i)| \\ &\leq \sum_{i=1}^{n} \left[f\left(a_i^2\right)\right]^{\frac{1}{2}} \left[f\left(b_i^2\right)\right]^{\frac{1}{2}} \\ &\leq \left[\sum_{i=1}^{n} \left(\left[f\left(a_i^2\right)\right]^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} \left(\left[f\left(b_i^2\right)\right]^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \text{ (by the } (CBS)\text{-inequality)} \\ &= \left[\sum_{i=1}^{n} f\left(a_i^2\right) \sum_{i=1}^{n} f\left(b_i^2\right)\right]^{\frac{1}{2}}\end{aligned}$$

and the inequality (2.76) is proved. □

Remark 2.49. It is obvious that for $A = \mathbb{R}$ and $f(x) = x$, we recapture the (CBS) −inequality.

Assume that $\varphi : \mathbb{N} \to \mathbb{N}$ is Euler's indicator. In 1940, T. Popoviciu [11] proved the following inequality for φ

$$[\varphi(ab)]^2 \leq \varphi\left(a^2\right) \varphi\left(b^2\right) \quad \text{for any natural number } a, b; \tag{2.77}$$

with equality iff a and b have the same prime factors.

A simple proof of this fact may be done by using the representation

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ [9, p. 109].

The following generalisation of Popoviciu's result holds [10, Theorem 1].

Theorem 2.50. *Let $a_i, b_i \in \mathbb{N}$ $(i = 1, \dots, n)$. Then one has the inequality*

$$\left[\sum_{i=1}^{n} \varphi(a_i b_i)\right]^2 \leq \sum_{i=1}^{n} \varphi\left(a_i^2\right) \sum_{i=1}^{n} \varphi\left(b_i^2\right). \tag{2.78}$$

Proof. Follows by Theorem 2.48 on taking into account that, by (2.77),

$$[\varphi(a_i b_i)]^2 \leq \varphi\left(a_i^2\right) \varphi\left(b_i^2\right) \quad \text{for any } i \in \{1, \dots, n\}.$$

□

Further, let us denote by $s(n)$ the sum of all relatively prime numbers with n and less than n. Then the following result also holds [10, Theorem 1].

Theorem 2.51. *Let* $a_i, b_i \in \mathbb{N}\ (i = 1, \dots, n)$. *Then one has the inequality*

$$\left[\sum_{i=1}^{n} s(a_i b_i)\right]^2 \leq \sum_{i=1}^{n} s\left(a_i^2\right) \sum_{i=1}^{n} s\left(b_i^2\right). \tag{2.79}$$

Proof. It is known (see for example [9, p. 109]) that for any $n \in \mathbb{N}$ one has

$$s(n) = \frac{1}{2} n \varphi(n). \tag{2.80}$$

Thus

$$[s(a_i b_i)]^2 = \frac{1}{4} a_i^2 b_i^2 \varphi^2(a_i b_i) \leq \frac{1}{4} a_i^2 b_i^2 \varphi\left(a_i^2\right) \varphi\left(b_i^2\right) = s\left(a_i^2\right) s\left(b_i^2\right) \tag{2.81}$$

for each $i \in \{1, \dots, n\}$.

Using Theorem 2.48 we then deduce the desired inequality (2.79). □

The following corollaries of Theorem 2.48 are also natural to be considered [10, p. 126].

Corollary 2.52. *Let* $a_i, b_i \in \mathbb{R}\ (i = 1, \dots, n)$ *and* $a > 1$. *Denote* $\exp_a x = a^x$, $x \in \mathbb{R}$. *Then one has the inequality*

$$\left[\sum_{i=1}^{n} \exp_a(a_i b_i)\right]^2 \leq \sum_{i=1}^{n} \exp_a\left(a_i^2\right) \sum_{i=1}^{n} \exp_a\left(b_i^2\right). \tag{2.82}$$

Corollary 2.53. *Let* $a_i, b_i \in (-1, 1) \quad (i = 1, \dots, n)$ *and* $m > 0$. *Then one has the inequality:*

$$\left[\sum_{i=1}^{n} \frac{1}{(1 - a_i b_i)^m}\right]^2 \leq \sum_{i=1}^{n} \frac{1}{\left(1 - a_i^2\right)^m} \sum_{i=1}^{n} \frac{1}{\left(1 - b_i^2\right)^m}. \tag{2.83}$$

2.10. **A Generalisation for Power Series.** The following result holds [12, Remark 2].

Theorem 2.54. *Let* $F : (-r, r) \to \mathbb{R}$, $F(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ *with* $\alpha_k \geq 0$, $k \in \mathbb{N}$. *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are sequences of real numbers such that*

$$a_i b_i,\ a_i^2,\ b_i^2 \in (-r, r) \quad \text{for any} \quad i \in \{1, \dots, n\}, \tag{2.84}$$

then one has the inequality:

$$\sum_{i=1}^{n} F\left(a_i^2\right) \sum_{i=1}^{n} F\left(b_i^2\right) \geq \left[\sum_{i=1}^{n} F(a_i b_i)\right]^2. \tag{2.85}$$

Proof. Firstly, let us observe that if $x, y \in \mathbb{R}$ such that $xy, x^2, y^2 \in (-r, r)$, then one has the inequality

$$[F(xy)]^2 \leq F\left(x^2\right) F\left(y^2\right). \tag{2.86}$$

Indeed, by the $(CBS)-$inequality, we have

$$\left[\sum_{k=0}^{n} \alpha_k x^k y^k\right]^2 \leq \sum_{k=0}^{n} \alpha_k x^{2k} \sum_{k=0}^{n} \alpha_k y^{2k}, \quad n \geq 0. \tag{2.87}$$

Taking the limit as $n \to \infty$ in (2.87), we deduce (2.86).

Using the (CBS) −inequality and (2.86) we have

$$\begin{aligned}\left|\sum_{i=1}^{n} F(a_i b_i)\right| &\le \sum_{i=1}^{n} |F(a_i b_i)| \\ &\le \sum_{i=1}^{n} \left[F\left(a_i^2\right)\right]^{\frac{1}{2}} \left[F\left(b_i^2\right)\right]^{\frac{1}{2}} \\ &\le \left\{\sum_{i=1}^{n} \left(\left[F\left(a_i^2\right)\right]^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} \left(\left[F\left(b_i^2\right)\right]^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^{n} F\left(a_i^2\right) \sum_{i=1}^{n} F\left(b_i^2\right)\right]^{\frac{1}{2}},\end{aligned}$$

which is clearly equivalent to (2.85). □

The following particular inequalities of (CBS) −type hold [12, p. 164].

(1) If $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$ are sequences of real numbers, then one has the inequality

$$\sum_{k=1}^{n} \exp\left(a_k^2\right) \sum_{k=1}^{n} \exp\left(b_k^2\right) \ge \left[\sum_{k=1}^{n} \exp(a_k b_k)\right]^2; \tag{2.88}$$

$$\sum_{k=1}^{n} \sinh\left(a_k^2\right) \sum_{k=1}^{n} \sinh\left(b_k^2\right) \ge \left[\sum_{k=1}^{n} \sinh(a_k b_k)\right]^2; \tag{2.89}$$

$$\sum_{k=1}^{n} \cosh\left(a_k^2\right) \sum_{k=1}^{n} \cosh\left(b_k^2\right) \ge \left[\sum_{k=1}^{n} \cosh(a_k b_k)\right]^2. \tag{2.90}$$

(2) If $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$ are such that $a_i, b_i \in (-1, 1)$, $i \in \{1, \dots, n\}$, then one has the inequalities

$$\sum_{k=1}^{n} \tan\left(a_k^2\right) \sum_{k=1}^{n} \tan\left(b_k^2\right) \ge \left[\sum_{k=1}^{n} \tan(a_k b_k)\right]^2; \tag{2.91}$$

$$\sum_{k=1}^{n} \arcsin\left(a_k^2\right) \sum_{k=1}^{n} \arcsin\left(b_k^2\right) \ge \left[\sum_{k=1}^{n} \arcsin(a_k b_k)\right]^2; \tag{2.92}$$

$$\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k^2}{1-a_k^2}\right)\right] \ln\left[\prod_{k=1}^{n}\left(\frac{1+b_k^2}{1-b_k^2}\right)\right] \ge \left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k b_k}{1-a_k b_k}\right)\right]\right\}^2; \tag{2.93}$$

$$\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k^2}\right)\right] \ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-b_k^2}\right)\right] \ge \left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k b_k}\right)\right]\right\}^2; \tag{2.94}$$

$$\sum_{k=1}^{n} \frac{1}{\left(1-a_k^2\right)^m} \sum_{k=1}^{n} \frac{1}{\left(1-b_k^2\right)^m} \ge \left[\sum_{k=1}^{n} \frac{1}{\left(1-a_k b_k\right)^m}\right]^2, \quad m > 0. \tag{2.95}$$

2.11. **A Generalisation of Callebaut's Inequality.** The following result holds (see also [12, Theorem 2] for a generalisation for positive linear functionals).

Theorem 2.55. *Let* $F : (-r, r) \to \mathbb{R}$, $F(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ *with* $\alpha_k \geq 0,\ k \in \mathbb{N}$. *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are sequences of nonnegative real numbers such that*

$$a_i b_i,\ a_i^{\alpha} b_i^{2-\alpha},\ a_i^{2-\alpha} b_i^{\alpha} \in (0, r) \text{ for any } i \in \{1, \dots, n\};\ \alpha \in [0, 2], \tag{2.96}$$

then one has the inequality

$$\left[\sum_{i=1}^{n} F(a_i b_i)\right]^2 \leq \sum_{i=1}^{n} F\left(a_i^{\alpha} b_i^{2-\alpha}\right) \sum_{i=1}^{n} F\left(a_i^{2-\alpha} b_i^{\alpha}\right). \tag{2.97}$$

Proof. Firstly, we note that for any $x, y > 0$ such that xy, $x^{\alpha} y^{2-\alpha}$, $x^{2-\alpha} y^{\alpha} \in (0, r)$ one has

$$[F(xy)]^2 \leq F\left(x^{\alpha} y^{2-\alpha}\right) F\left(x^{2-\alpha} y^{\alpha}\right). \tag{2.98}$$

Indeed, using Callebaut's inequality, i.e., we recall it [4]

$$\left(\sum_{i=1}^{m} \alpha_i x_i y_i\right)^2 \leq \sum_{i=1}^{m} \alpha_i x_i^{\alpha} y_i^{2-\alpha} \sum_{i=1}^{m} \alpha_i x_i^{2-\alpha} y_i^{\alpha}, \tag{2.99}$$

we may write, for $m \geq 0$, that

$$\left(\sum_{i=0}^{m} \alpha_i x^i y^i\right)^2 \leq \sum_{i=0}^{m} \alpha_i \left(x^{\alpha} y^{2-\alpha}\right)^i \sum_{i=0}^{m} \alpha_i \left(x^{2-\alpha} y^{\alpha}\right)^i. \tag{2.100}$$

Taking the limit as $m \to \infty$, we deduce (2.98).

Using the (CBS) $-$inequality and (2.98) we may write:

$$\begin{aligned}
\left|\sum_{i=1}^{n} F(a_i b_i)\right| &\leq \sum_{i=1}^{n} |F(a_i b_i)| \\
&\leq \sum_{i=1}^{n} \left[F\left(a_i^{\alpha} b_i^{2-\alpha}\right)\right]^{\frac{1}{2}} \left[F\left(a_i^{2-\alpha} b_i^{\alpha}\right)\right]^{\frac{1}{2}} \\
&\leq \left\{\sum_{i=1}^{n} \left(\left[F\left(a_i^{\alpha} b_i^{2-\alpha}\right)\right]^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} \left(\left[F\left(a_i^{2-\alpha} b_i^{\alpha}\right)\right]^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}} \\
&= \left[\sum_{i=1}^{n} F\left(a_i^{\alpha} b_i^{2-\alpha}\right) \sum_{i=1}^{n} F\left(a_i^{2-\alpha} b_i^{\alpha}\right)\right]^{\frac{1}{2}}
\end{aligned}$$

which is clearly equivalent to (2.97). □

The following particular inequalities also hold [12, pp. 165-166].

(1) Let $\bar{\mathrm{a}}$ and $\bar{\mathrm{b}}$ be sequences of nonnegative real numbers. Then one has the inequalities

$$\left[\sum_{k=1}^{n}\exp\left(a_kb_k\right)\right]^2\le\sum_{k=1}^{n}\exp\left(a_k^{\alpha}b_k^{2-\alpha}\right)\sum_{k=1}^{n}\exp\left(a_k^{2-\alpha}b_k^{\alpha}\right); \tag{2.101}$$

$$\left[\sum_{k=1}^{n}\sinh\left(a_kb_k\right)\right]^2\le\sum_{k=1}^{n}\sinh\left(a_k^{\alpha}b_k^{2-\alpha}\right)\sum_{k=1}^{n}\sinh\left(a_k^{2-\alpha}b_k^{\alpha}\right); \tag{2.102}$$

$$\left[\sum_{k=1}^{n}\cosh\left(a_kb_k\right)\right]^2\le\sum_{k=1}^{n}\cosh\left(a_k^{\alpha}b_k^{2-\alpha}\right)\sum_{k=1}^{n}\cosh\left(a_k^{2-\alpha}b_k^{\alpha}\right). \tag{2.103}$$

(2) Let $\bar{\mathrm{a}}$ and $\bar{\mathrm{b}}$ be such that $a_k, b_k \in (0,1)$ for any $k \in \{1,\dots,n\}$. Then one has the inequalities:

$$\left[\sum_{k=1}^{n}\tan\left(a_kb_k\right)\right]^2\le\sum_{k=1}^{n}\tan\left(a_k^{\alpha}b_k^{2-\alpha}\right)\sum_{k=1}^{n}\tan\left(a_k^{2-\alpha}b_k^{\alpha}\right); \tag{2.104}$$

$$\left[\sum_{k=1}^{n}\arcsin\left(a_kb_k\right)\right]^2\le\sum_{k=1}^{n}\arcsin\left(a_k^{\alpha}b_k^{2-\alpha}\right)\sum_{k=1}^{n}\arcsin\left(a_k^{2-\alpha}b_k^{\alpha}\right); \tag{2.105}$$

$$\left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_kb_k}{1-a_kb_k}\right)\right]\right\}^2\le\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k^{\alpha}b_k^{2-\alpha}}{1-a_k^{\alpha}b_k^{2-\alpha}}\right)\right]\ln\left[\prod_{k=1}^{n}\left(\frac{1+a_k^{2-\alpha}b_k^{\alpha}}{1-a_k^{2-\alpha}b_k^{\alpha}}\right)\right]; \tag{2.106}$$

$$\left\{\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_kb_k}\right)\right]\right\}^2\le\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k^{\alpha}b_k^{2-\alpha}}\right)\right]\ln\left[\prod_{k=1}^{n}\left(\frac{1}{1-a_k^{2-\alpha}b_k^{\alpha}}\right)\right]. \tag{2.107}$$

2.12. Wagner's Inequality for Real Numbers.

The following generalisation of the $(CBS)-$inequality for sequences of real numbers is known in the literature as Wagner's inequality [15], or [14] (see also [4, p. 85]).

Theorem 2.56. *Let* $\bar{\mathrm{a}} = (a_1,\dots,a_n)$ *and* $\bar{\mathrm{b}} = (b_1,\dots,b_n)$ *be sequences of real numbers. If* $0 \le x \le 1$, *then one has the inequality*

$$\left(\sum_{k=1}^{n}a_kb_k+x\sum_{1\le i\ne j\le n}a_ib_j\right)^2\le\left[\sum_{k=1}^{n}a_k^2+2x\sum_{1\le i<j\le n}a_ia_j\right]\left[\sum_{k=1}^{n}b_k^2+2x\sum_{1\le i<j\le n}b_ib_j\right]. \tag{2.108}$$

Proof. We shall follow the proof in [13] (see also [4, p. 85]).

For any $x \in [0,1]$, consider the quadratic polynomial in y

$$\begin{aligned}
P(y) &:= (1-x)\sum_{k=1}^{n}(a_k y - b_k)^2 + x\left[\sum_{k=1}^{n}(a_k y - b_k)\right]^2 \\
&= (1-x)\left[y^2\sum_{k=1}^{n}a_k^2 - 2y\sum_{k=1}^{n}a_k b_k + \sum_{k=1}^{n}b_k^2\right] \\
&\qquad + x\left[y^2\left(\sum_{k=1}^{n}a_k\right)^2 - 2y\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) + \left(\sum_{k=1}^{n}b_k\right)^2\right] \\
&= \left[(1-x)\sum_{k=1}^{n}a_k^2 + x\left(\sum_{k=1}^{n}a_k\right)^2\right]y^2 - 2y\left[(1-x)\sum_{k=1}^{n}a_k b_k + x\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k\right] \\
&\qquad + (1-x)\sum_{k=1}^{n}b_k^2 + x\left(\sum_{k=1}^{n}b_k\right)^2 \\
&= \left\{\sum_{k=1}^{n}a_k^2 + x\left[\left(\sum_{k=1}^{n}a_k\right)^2 - \sum_{k=1}^{n}a_k^2\right]\right\}y^2 \\
&\qquad - 2y\left[\sum_{k=1}^{n}a_k b_k + x\left(\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k - \sum_{k=1}^{n}a_k b_k\right)\right] \\
&\qquad + \sum_{k=1}^{n}b_k^2 + x\left[\left(\sum_{k=1}^{n}b_k\right)^2 - \sum_{k=1}^{n}b_k^2\right].
\end{aligned}$$

Since, it is obvious that:

$$\left(\sum_{k=1}^{n}a_k\right)^2 - \sum_{k=1}^{n}a_k^2 = 2\sum_{1\le i<j\le n}a_i a_j,$$
$$\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k - \sum_{k=1}^{n}a_k b_k = \sum_{1\le i\ne j\le n}a_i b_j$$

and

$$\left(\sum_{k=1}^{n}b_k\right)^2 - \sum_{k=1}^{n}b_k^2 = 2\sum_{1\le i<j\le n}b_i b_j,$$

we get

$$\begin{aligned}
P(y) &= \left(\sum_{k=1}^{n}a_k^2 + 2x\sum_{1\le i<j\le n}a_i a_j\right)y^2 \\
&\qquad - 2y\left(\sum_{k=1}^{n}a_k b_k + x\sum_{1\le i\ne j\le n}a_i b_j\right) + \sum_{k=1}^{n}b_k^2 + 2x\sum_{1\le i<j\le n}b_i b_j.
\end{aligned}$$

Taking into consideration, by the definition of P, that $P(y) \geq 0$ for any $y \in \mathbb{R}$, it follows that the discriminant $\Delta \leq 0$, i.e.,

$$0 \geq \frac{1}{4}\Delta = \left(\sum_{k=1}^{n} a_k b_k + x \sum_{1\leq i\neq j\leq n} a_i b_j\right)^2 - \left(\sum_{k=1}^{n} a_k^2 + 2x \sum_{1\leq i<j\leq n} a_i a_j\right)\left(\sum_{k=1}^{n} b_k^2 + 2x \sum_{1\leq i<j\leq n} b_i b_j\right)$$

and the inequality (2.108) is proved. □

Remark 2.57. If $x = 0$, then from (2.108) we recapture the (CBS) −inequality for real numbers.

2.13. **Wagner's inequality for Complex Numbers.** The following inequality which provides a version for complex numbers of Wagner's result holds [16].

Theorem 2.58. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of complex numbers. Then for any* $x \in [0,1]$ *one has the inequality*

$$\left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \sum_{1\leq i\neq j\leq n} \operatorname{Re}\left(a_i \bar{b}_j\right)\right]^2 \leq \left[\sum_{k=1}^{n} |a_k|^2 + 2x \sum_{1\leq i<j\leq n} \operatorname{Re}\left(a_i \bar{a}_j\right)\right]\left[\sum_{k=1}^{n} |b_k|^2 + 2x \sum_{1\leq i<j\leq n} \operatorname{Re}\left(b_i \bar{b}_j\right)\right]. \tag{2.109}$$

Proof. Start with the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = (1-x)\sum_{k=1}^{n} |ta_k - b_k|^2 + x\left|\sum_{k=1}^{n} (ta_k - b_k)\right|^2. \tag{2.110}$$

We have

$$\begin{aligned} f(t) &= (1-x)\sum_{k=1}^{n} (ta_k - b_k)\left(t\bar{a}_k - \bar{b}_k\right) \\ &\quad + x\left(t\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k\right)\left(t\sum_{k=1}^{n} \bar{a}_k - \sum_{k=1}^{n} \bar{b}_k\right) \\ &= (1-x)\left[t^2\sum_{k=1}^{n} |a_k|^2 - t\sum_{k=1}^{n} b_k\bar{a}_k - t\sum_{k=1}^{n} a_k\bar{b}_k + \sum_{k=1}^{n} |b_k|^2\right] \\ &\quad + x\left[t^2\sum_{k=1}^{n} |a_k|^2 - t\sum_{k=1}^{n} b_k\sum_{k=1}^{n} \bar{a}_k - t\sum_{k=1}^{n} a_k\sum_{k=1}^{n} \bar{b}_k + \sum_{k=1}^{n} |b_k|^2\right] \end{aligned} \tag{2.111}$$

$$= \left[(1-x) \sum_{k=1}^{n} |a_k|^2 + x \left| \sum_{k=1}^{n} a_k \right|^2 \right] t^2$$
$$+ 2 \left[(1-x) \sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \operatorname{Re}\left[\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k \right] \right] t$$
$$+ (1-x) \sum_{k=1}^{n} |b_k|^2 + x \left| \sum_{k=1}^{n} b_k \right|^2 .$$

Observe that

$$\left| \sum_{k=1}^{n} a_k \right|^2 = \sum_{i,j=1}^{n} a_i \bar{a}_j = \sum_{i=1}^{n} |a_i|^2 + \sum_{1 \le i \ne j \le n} a_i \bar{a}_j = \sum_{i=1}^{n} |a_i|^2 + \sum_{1 \le i < j \le n} a_i \bar{a}_j + \sum_{1 \le j < i \le n} a_i \bar{a}_j = \sum_{i=1}^{n} |a_i|^2 + 2 \sum_{1 \le i < j \le n} \operatorname{Re}\left(a_i \bar{a}_j\right) \tag{2.112}$$

and, similarly,

$$\left| \sum_{k=1}^{n} b_k \right|^2 = \sum_{i=1}^{n} |b_i|^2 + 2 \sum_{1 \le i < j \le n} \operatorname{Re}\left(b_i \bar{b}_j\right) . \tag{2.113}$$

Also

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k = \sum_{i=1}^{n} a_i \bar{b}_i + \sum_{1 \le i \ne j \le n} a_i \bar{b}_j$$

and thus

$$\operatorname{Re}\left(\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \bar{b}_k \right) = \sum_{i=1}^{n} \operatorname{Re}\left(a_i \bar{b}_i\right) + \sum_{1 \le i \ne j \le n} \operatorname{Re}\left(a_i \bar{b}_j\right) . \tag{2.114}$$

Utilising (2.112) – (2.114), by (2.111), we deduce

$$f(t) = \left[\sum_{k=1}^{n} |a_k|^2 + 2x \sum_{1 \le i < j \le n} \operatorname{Re}\left(a_i \bar{a}_j\right) \right] t^2 + 2 \left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \sum_{1 \le i \ne j \le n} \operatorname{Re}\left(a_i \bar{b}_j\right) \right] t + \sum_{k=1}^{n} |b_k|^2 + 2x \sum_{1 \le i < j \le n} \operatorname{Re}\left(b_i \bar{b}_j\right) . \tag{2.115}$$

Since, by (2.110), $f(t) \geq 0$ for any $t \in \mathbb{R}$, it follows that the discriminant of the quadratic function given by (2.115) is negative, i.e.,

$$\begin{aligned} 0 &\geq \frac{1}{4}\Delta \\ &= \left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right) + x \sum_{1\leq i\neq j\leq n} \operatorname{Re}\left(a_i \bar{b}_j\right)\right]^2 \\ &\quad - \left[\sum_{k=1}^{n} |a_k|^2 + 2x \sum_{1\leq i<j\leq n} \operatorname{Re}\left(a_i \bar{a}_j\right)\right]\left[\sum_{k=1}^{n} |b_k|^2 + 2x \sum_{1\leq i<j\leq n} \operatorname{Re}\left(b_i \bar{b}_j\right)\right] \end{aligned}$$

and the inequality (2.109) is proved. □

Remark 2.59. If $x = 0$, then we get the (CBS) −inequality

$$\left[\sum_{k=1}^{n} \operatorname{Re}\left(a_k \bar{b}_k\right)\right]^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 . \tag{2.116}$$

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3. Refinements of the (CBS) −Inequality

3.1. A Refinement in Terms of Moduli.

The following result was proved in [1].

Theorem 3.1. *Let* $\bar{\mathrm{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathrm{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers. Then one has the inequality*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \geq \left|\sum_{k=1}^{n} a_k |a_k| \sum_{k=1}^{n} b_k |b_k| - \sum_{k=1}^{n} a_k |b_k| \sum_{k=1}^{n} |a_k| b_k\right| \geq 0. \tag{3.1}$$

Proof. We will follow the proof from [1].

For any $i, j \in \{1, \dots, n\}$ the next elementary inequality is true:

$$|a_i b_j - a_j b_i| \geq ||a_i b_j| - |a_j b_i|| . \tag{3.2}$$

By multiplying this inequality with $|a_i b_j - a_j b_i| \geq 0$ we get

$$\begin{aligned} (a_i b_j - a_j b_i)^2 &\geq |(a_i b_j - a_j b_i)(|a_i| |b_j| - |a_j| |b_i|)| \\ &= |a_i |a_i| b_j |b_j| + b_i |b_i| a_j |a_j| - |a_i| b_i a_j |b_j| - a_i b_j |a_j| |b_i|| . \end{aligned} \tag{3.3}$$

Summing (3.3) over i and j from 1 to n, we deduce

$$\begin{aligned} &\sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2 \\ &\qquad \geq \sum_{i,j=1}^{n} \left| a_i |a_i| b_j |b_j| + b_i |b_i| a_j |a_j| - |a_i| b_i a_j |b_j| - a_i b_j |a_j| |b_i| \right| \\ &\qquad \geq \left| \sum_{i,j=1}^{n} (a_i |a_i| b_j |b_j| + b_i |b_i| a_j |a_j| - |a_i| b_i a_j |b_j| - a_i b_j |a_j| |b_i|) \right| , \end{aligned}$$

giving the desired inequality (3.1). □

The following corollary is a natural consequence of (3.1) [1, Corollary 4].

Corollary 3.2. *Let* $\bar{\mathrm{a}}$ *be a sequence of real numbers. Then*

$$\frac{1}{n} \sum_{k=1}^{n} a_k^2 - \left(\frac{1}{n} \sum_{k=1}^{n} a_k\right)^2 \geq \left| \frac{1}{n} \sum_{k=1}^{n} a_k |a_k| - \frac{1}{n} \sum_{k=1}^{n} a_k \cdot \frac{1}{n} \sum_{k=1}^{n} |a_k| \right| \geq 0. \tag{3.4}$$

There are some particular inequalities that may also be deduced from the above Theorem 3.1 (see [1, p. 80]).

(1) Suppose that for $\bar{\mathrm{a}}$ and $\bar{\mathrm{b}}$ sequences of real numbers, one has $\operatorname{sgn}(a_k) = \operatorname{sgn}(b_k) = e_k \in \{-1, 1\}$. Then one has the inequality

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \geq \left| \sum_{k=1}^{n} e_k a_k^2 \sum_{k=1}^{n} e_k b_k^2 - \left(\sum_{k=1}^{n} e_k a_k b_k\right)^2 \right| \geq 0. \tag{3.5}$$

(2) If $\bar{\mathrm{a}} = (a_1, \dots, a_{2n})$, then we have the inequality

$$2n \sum_{k=1}^{2n} a_k^2 - \left[\sum_{k=1}^{2n} (-1)^k a_k\right]^2 \geq \left| \sum_{k=1}^{2n} a_k \sum_{k=1}^{2n} (-1)^k |a_k| \right| \geq 0. \tag{3.6}$$

(3) If $\bar{\mathbf{a}} = (a_1, \dots, a_{2n+1})$, then we have the inequality

$$(2n+1)\sum_{k=1}^{2n+1} a_k^2 - \left(\sum_{k=1}^{2n+1} (-1)^k a_k\right)^2 \geq \left|\sum_{k=1}^{2n+1} a_k \sum_{k=1}^{2n+1} (-1)^k |a_k|\right| \geq 0. \tag{3.7}$$

The following version for complex numbers is valid as well.

Theorem 3.3. *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be sequences of complex numbers. Then one has the inequality*

$$\sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \left|\sum_{i=1}^n a_i b_i\right|^2 \geq \left|\sum_{i=1}^n |a_i| \bar{a}_i \sum_{i=1}^n |b_i| b_i - \sum_{i=1}^n |a_i| b_i \sum_{i=1}^n |b_i| \bar{a}_i\right| \geq 0. \tag{3.8}$$

Proof. We have for any $i, j \in \{1, \dots, n\}$ that

$$|\bar{a}_i b_j - \bar{a}_j b_i| \geq ||a_i| |b_j| - |a_j| |b_i||.$$

Multiplying by $|\bar{a}_i b_j - \bar{a}_j b_i| \geq 0$, we get

$$|\bar{a}_i b_j - \bar{a}_j b_i|^2 \geq ||a_i| \bar{a}_i |b_j| b_j + |a_j| \bar{a}_j |b_i| b_i - |a_i| b_i |b_j| \bar{a}_j - |b_i| \bar{a}_i |a_j| b_j|.$$

Summing over i and j from 1 to n and using the Lagrange's identity for complex numbers:

$$\sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \left|\sum_{i=1}^n a_i b_i\right|^2 = \frac{1}{2}\sum_{i,j=1}^n |\bar{a}_i b_j - \bar{a}_j b_i|^2$$

we deduce the desired inequality (3.8). □

Remark 3.4. Similar particular inequalities may be stated, but we omit the details.

3.2. **A Refinement for a Sequence Whose Norm is One.** The following result holds [1, Theorem 6].

Theorem 3.5. *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be sequences of real numbers and $\bar{\mathbf{e}} = (e_1, \dots, e_n)$ be such that $\sum_{i=1}^n e_i^2 = 1$. Then the following inequality holds*

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \left[\left|\sum_{k=1}^n a_k b_k - \sum_{k=1}^n e_k a_k \sum_{k=1}^n e_k b_k\right| + \left|\sum_{k=1}^n e_k a_k \sum_{k=1}^n e_k b_k\right|\right]^2 \geq \left(\sum_{k=1}^n a_k b_k\right)^2. \tag{3.9}$$

Proof. We will follow the proof from [1].

From the (CBS) $-$inequality, one has

$$\sum_{k=1}^n \left[a_k - \left(\sum_{i=1}^n e_i a_i\right) e_k\right]^2 \sum_{k=1}^n \left[b_k - \left(\sum_{i=1}^n e_i b_i\right) e_k\right]^2 \geq \left\{\sum_{k=1}^n \left[a_k - \left(\sum_{i=1}^n e_i a_i\right) e_k\right]\left[b_k - \left(\sum_{i=1}^n e_i b_i\right) e_k\right]\right\}^2. \tag{3.10}$$

Since $\sum_{k=1}^{n} e_k^2 = 1$, a simple calculation shows that

$$\sum_{k=1}^{n}\left[a_k-\left(\sum_{i=1}^{n}e_ia_i\right)e_k\right]^2=\sum_{k=1}^{n}a_k^2-\left(\sum_{k=1}^{n}e_ka_k\right)^2,$$
$$\sum_{k=1}^{n}\left[b_k-\left(\sum_{i=1}^{n}e_ib_i\right)e_k\right]^2=\sum_{k=1}^{n}b_k^2-\left(\sum_{k=1}^{n}e_kb_k\right)^2,$$

and

$$\sum_{k=1}^{n}\left[a_k-\left(\sum_{i=1}^{n}e_ia_i\right)e_k\right]\left[b_k-\left(\sum_{i=1}^{n}e_ib_i\right)e_k\right]=\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k$$

and then the inequality (3.10) becomes

$$\left[\sum_{k=1}^{n}a_k^2-\left(\sum_{k=1}^{n}e_ka_k\right)^2\right]\left[\sum_{k=1}^{n}b_k^2-\left(\sum_{k=1}^{n}e_kb_k\right)^2\right] \geq \left(\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right)^2 \geq 0. \tag{3.11}$$

Using the elementary inequality

$$\left(m^2-l^2\right)\left(p^2-q^2\right)\leq (mp-lq)^2, \quad m,l,p,q\in\mathbb{R}$$

for the choices

$$m=\left(\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}, \quad l=\left|\sum_{k=1}^{n}e_ka_k\right|, \quad p=\left(\sum_{k=1}^{n}b_k^2\right)^{\frac{1}{2}}$$
$$\text{and} \quad q=\left|\sum_{k=1}^{n}e_kb_k\right|$$

the above inequality (3.11) provides the following result

$$\left[\left(\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}b_k^2\right)^{\frac{1}{2}}-\left|\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right|\right]^2 \geq \left|\sum_{k=1}^{n}a_kb_k-\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right|^2. \tag{3.12}$$

Since

$$\left(\sum_{k=1}^{n}a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}b_k^2\right)^{\frac{1}{2}}\geq\left|\sum_{k=1}^{n}e_ka_k\sum_{k=1}^{n}e_kb_k\right|$$

then, by taking the square root in (3.12) we deduce the first part of (3.9).

The second part is obvious, and the theorem is proved. □

The following corollary is a natural consequence of the above theorem [1, Corollary 7].

Corollary 3.6. *Let $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{e}}$ be as in Theorem 3.5. If $\sum_{k=1}^{n} a_k b_k = 0$, then one has the inequality:*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq 4 \left(\sum_{k=1}^{n} e_k a_k \right)^2 \left(\sum_{k=1}^{n} e_k b_k \right)^2 . \tag{3.13}$$

The following inequalities are interesting as well [1, p. 81].

(1) For any $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ one has the inequality

$$\begin{aligned} \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 &\geq \left[\left| \sum_{k=1}^{n} a_k b_k - \frac{1}{n} \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \right| + \frac{1}{n} \left| \sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \right| \right]^2 \\ &\geq \left(\sum_{k=1}^{n} a_k b_k \right)^2 . \end{aligned} \tag{3.14}$$

(2) If $\sum_{k=1}^{n} a_k b_k = 0$, then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \frac{4}{n^2} \left(\sum_{k=1}^{n} a_k \right)^2 \left(\sum_{k=1}^{n} b_k \right)^2 . \tag{3.15}$$

In a similar manner, we may state and prove the following result for complex numbers.

Theorem 3.7. *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be sequences of complex numbers and $\bar{\mathbf{e}} = (e_1, \dots, e_n)$ a sequence of complex numbers satisfying the condition $\sum_{i=1}^{n} |e_i|^2 = 1$. Then the following refinement of the (CBS) $-$inequality holds*

$$\begin{aligned} \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 &\geq \left[\left| \sum_{k=1}^{n} a_k \bar{b}_k - \sum_{k=1}^{n} a_k \bar{e}_k \cdot \sum_{k=1}^{n} e_k \bar{b}_k \right| + \left| \sum_{k=1}^{n} a_k \bar{e}_k \cdot \sum_{k=1}^{n} e_k \bar{b}_k \right| \right]^2 \\ &\geq \left| \sum_{k=1}^{n} a_k \bar{b}_k \right|^2 . \end{aligned} \tag{3.16}$$

The proof is similar to the one in Theorem 3.5 on using the corresponding (CBS) $-$inequality for complex numbers.

Remark 3.8. Similar particular inequalities may be stated, but we omit the details.

3.3. **A Second Refinement in Terms of Moduli.** The following lemma holds.

Lemma 3.9. *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ be a sequence of real numbers and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ a sequence of positive real numbers with $\sum_{i=1}^{n} p_i = 1$. Then one has the inequality:*

$$\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \geq \left| \sum_{i=1}^{n} p_i |a_i| a_i - \sum_{i=1}^{n} p_i |a_i| \sum_{i=1}^{n} p_i a_i \right| . \tag{3.17}$$

Proof. By the properties of moduli we have

$$(a_i - a_j)^2 = |(a_i - a_j)(a_i - a_j)| \geq |(|a_i| - |a_j|)(a_i - a_j)|$$

for any $i, j \in \{1, \dots, n\}$. This is equivalent to

$$a_i^2 - 2a_i a_j + a_j^2 \geq ||a_i| a_i + |a_j| a_j - |a_i| a_j - |a_j| a_i| \tag{3.18}$$

for any $i, j \in \{1, \dots, n\}$.

If we multiply (3.18) by $p_ip_j \geq 0$ and sum over i and j from 1 to n we deduce

$$\begin{aligned}\sum_{j=1}^n p_j \sum_{i=1}^n p_i a_i^2 - 2\sum_{i=1}^n p_i a_i \sum_{j=1}^n p_j a_j + \sum_{i=1}^n p_i \sum_{j=1}^n p_j a_j^2 \\ \geq \sum_{i,j=1}^n p_ip_j \left| |a_i| a_i + |a_j| a_j - |a_i| a_j - |a_j| a_i \right| \\ \geq \left| \sum_{i,j=1}^n p_ip_j \left(|a_i| a_i + |a_j| a_j - |a_i| a_j - |a_j| a_i \right) \right|,\end{aligned}$$

which is clearly equivalent to (3.17). □

Using the above lemma, we may prove the following refinement of the (CBS) -inequality.

Theorem 3.10. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers. Then one has the inequality*

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_ib_i\right)^2 \geq \left| \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n sgn\,(a_i)\,|b_i|\, b_i - \sum_{i=1}^n |a_ib_i| \sum_{i=1}^n a_ib_i \right| \geq 0. \tag{3.19}$$

Proof. If we choose (for $a_i \neq 0$, $i \in \{1, \dots, n\}$) in (3.17), that

$$p_i := \frac{a_i^2}{\sum_{k=1}^n a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i \in \{1, \dots, n\},$$

we get

$$\begin{aligned}\sum_{i=1}^n \frac{a_i^2}{\sum_{k=1}^n a_k^2} \cdot \left(\frac{b_i}{a_i}\right)^2 - \left(\sum_{i=1}^n \frac{a_i^2}{\sum_{k=1}^n a_k^2} \cdot \frac{b_i}{a_i}\right)^2 \\ \geq \left| \sum_{i=1}^n \frac{a_i^2}{\sum_{k=1}^n a_k^2} \left|\frac{b_i}{a_i}\right| \cdot \frac{b_i}{a_i} - \sum_{i=1}^n \frac{a_i^2}{\sum_{k=1}^n a_k^2} \left|\frac{b_i}{a_i}\right| \sum_{i=1}^n \frac{a_i^2}{\sum_{k=1}^n a_k^2} \cdot \frac{b_i}{a_i} \right|\end{aligned}$$

from where we get

$$\sum_{i=1}^n \frac{b_i^2}{\sum_{k=1}^n a_k^2} - \frac{\left(\sum_{i=1}^n a_ib_i\right)^2}{\left(\sum_{k=1}^n a_k^2\right)^2} \geq \left| \frac{\sum_{i=1}^n \frac{|a_i|}{a_i} |b_i|\, b_i}{\sum_{k=1}^n a_k^2} - \frac{\sum_{i=1}^n |a_ib_i| \sum_{i=1}^n a_ib_i}{\left(\sum_{k=1}^n a_k^2\right)^2} \right|$$

which is clearly equivalent to (3.19). □

The case for complex numbers is as follows.

Lemma 3.11. *Let* $\bar{\mathbf{z}} = (z_1, \dots, z_n)$ *be a sequence of complex numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *a sequence of positive real numbers with* $\sum_{i=1}^n p_i = 1$. *Then one has the inequality:*

$$\sum_{i=1}^n p_i |z_i|^2 - \left|\sum_{i=1}^n p_iz_i\right|^2 \geq \left| \sum_{i=1}^n p_i |z_i|\, z_i - \sum_{i=1}^n p_i |z_i| \sum_{i=1}^n p_iz_i \right|. \tag{3.20}$$

Proof. By the properties of moduli for complex numbers we have

$$|z_i - z_j|^2 \geq |(|z_i| - |z_j|)(z_i - z_j)|$$

for any $i, j \in \{1, \dots, n\}$, which is clearly equivalent to

$$|z_i|^2 - 2\,\mathrm{Re}\,(z_i\bar{z}_j) + |z_j|^2 \geq ||z_i|\, z_i + |z_j|\, z_j - z_i |z_j| - |z_i|\, z_j|$$

for any $i, j \in \{1, \dots, n\}$.

If we multiply with $p_i p_j \geq 0$ and sum over i and j from 1 to $n,$ we deduce the desired inequality (3.20). □

Now, in a similar manner to the one in Theorem 3.10, we may state the following result for complex numbers.

Theorem 3.12. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ $(a_i \neq 0,\ i = 1, \dots, n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of complex numbers. Then one has the inequality:*

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} \bar{a}_i b_i \right|^2 \geq \left| \sum_{i=1}^{n} \frac{|a_i|}{a_i} |b_i| \, b_i - \sum_{i=1}^{n} |a_i| \, b_i \sum_{i=1}^{n} \bar{a}_i b_i \right| \geq 0. \tag{3.21}$$

3.4. A Refinement for a Sequence Less than the Weights.

The following result was obtained in [1, Theorem 9] (see also [2, Theorem 3.10]).

Theorem 3.13. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *be sequences of nonnegative real numbers such that* $p_k \geq q_k$ *for any* $k \in \{1, \dots, n\}$ *. Then we have the inequality*

$$\begin{aligned} \sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 &\geq \left[\left| \sum_{k=1}^{n} (p_k - q_k) \, a_k b_k \right| + \left(\sum_{k=1}^{n} q_k a_k^2 \sum_{k=1}^{n} q_k b_k^2 \right)^{\frac{1}{2}} \right]^2 \\ &\geq \left[\left| \sum_{k=1}^{n} (p_k - q_k) \, a_k b_k \right| + \left| \sum_{k=1}^{n} q_k a_k b_k \right| \right]^2 \\ &\geq \left(\sum_{k=1}^{n} p_k a_k b_k \right)^2 . \end{aligned} \tag{3.22}$$

Proof. We shall follow the proof in [1].

Since $p_k - q_k \geq 0$, then the (CBS) −inequality for the weights $r_k := p_k - q_k$ will give

$$\left(\sum_{k=1}^{n} p_k a_k^2 - \sum_{k=1}^{n} q_k a_k^2 \right) \left(\sum_{k=1}^{n} p_k b_k^2 - \sum_{k=1}^{n} q_k b_k^2 \right) \geq \left[\sum_{k=1}^{n} (p_k - q_k) \, a_k b_k \right]^2 . \tag{3.23}$$

Using the elementary inequality

$$(ac - bd)^2 \geq \left(a^2 - b^2\right) \left(c^2 - d^2\right), \quad a, b, c, d \in \mathbb{R}$$

for the choices

$$a = \left(\sum_{k=1}^{n} p_k a_k^2 \right)^{\frac{1}{2}}, \quad b = \left(\sum_{k=1}^{n} q_k a_k^2 \right)^{\frac{1}{2}}, \quad c = \left(\sum_{k=1}^{n} p_k b_k^2 \right)^{\frac{1}{2}}$$

and

$$d = \left(\sum_{k=1}^{n} q_k b_k^2 \right)^{\frac{1}{2}}$$

we deduce by (3.23) that

$$\left[\left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}}-\left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}}\right]^2 \geq \left[\sum_{k=1}^{n}\left(p_k-q_k\right) a_k b_k\right]^2. \tag{3.24}$$

Since, obviously,

$$\left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} \geq \left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}}$$

then, by (3.24), on taking the square root, we would get

$$\left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} \geq \left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}}+\left|\sum_{k=1}^{n}\left(p_k-q_k\right) a_k b_k\right|,$$

which provides the first inequality in (3.22).

The other inequalities are obvious and we omit the details. □

The following corollary is a natural consequence of the above theorem [2, Corollary 3.11].

Corollary 3.14. *Let* $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$ *be sequences of real numbers and* $\bar{\mathrm{s}}=(s_1,\dots,s_n)$ *be such that* $0\leq s_k\leq 1$ *for any* $k\in\{1,\dots,n\}$ *. Then one has the inequalities*

$$\begin{aligned}\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 &\geq \left[\left|\sum_{k=1}^{n}\left(1-s_k\right) a_k b_k\right|+\left(\sum_{k=1}^{n} s_k a_k^2 \sum_{k=1}^{n} s_k b_k^2\right)^{\frac{1}{2}}\right]^2 \\ &\geq \left[\left|\sum_{k=1}^{n}\left(1-s_k\right) a_k b_k\right|+\left|\sum_{k=1}^{n} s_k a_k b_k\right|\right]^2 \\ &\geq \left(\sum_{k=1}^{n} a_k b_k\right)^2.\end{aligned} \tag{3.25}$$

Remark 3.15. Assume that $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$ and $\bar{\mathrm{s}}$ are as in Corollary 3.14. The following inequalities hold (see [2, p. 15]).

a) If $\sum_{k=1}^{n} a_k b_k=0$, then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq 4\left(\sum_{k=1}^{n} s_k a_k b_k\right)^2. \tag{3.26}$$

b) If $\sum_{k=1}^{n} s_k a_k b_k=0$, then

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \left[\left|\sum_{k=1}^{n} a_k b_k\right|+\left(\sum_{k=1}^{n} \alpha_k a_k^2 \sum_{k=1}^{n} \alpha_k b_k^2\right)^{\frac{1}{2}}\right]^2. \tag{3.27}$$

In particular, we may obtain the following particular inequalities involving trigonometric functions (see [2, p. 15])

$$\begin{aligned}\text{(3.28)} \qquad \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 &\geq \left[\left|\sum_{k=1}^{n} a_k b_k \cos^2 \alpha_k\right| + \left(\sum_{k=1}^{n} a_k^2 \sin^2 \alpha_k \sum_{k=1}^{n} b_k^2 \sin^2 \alpha_k\right)^{\frac{1}{2}}\right]^2 \\ &\geq \left[\left|\sum_{k=1}^{n} a_k b_k \cos^2 \alpha_k\right| + \left|\sum_{k=1}^{n} a_k b_k \sin^2 \alpha_k\right|\right]^2 \\ &\geq \left(\sum_{k=1}^{n} a_k b_k\right)^2,\end{aligned}$$

where $a_k, b_k, \alpha_k \in \mathbb{R}$, $k = 1, \dots, n$.

If one would assume that $\sum_{k=1}^{n} a_k b_k = 0$, then

$$\text{(3.29)} \qquad \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq 4 \left(\sum_{k=1}^{n} a_k b_k \sin^2 \alpha_k\right)^2.$$

If $\sum_{k=1}^{n} a_k b_k \sin^2 \alpha_k = 0$, then

$$\text{(3.30)} \qquad \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \geq \left[\left|\sum_{k=1}^{n} a_k b_k\right| + \left(\sum_{k=1}^{n} a_k^2 \sin^2 \alpha_k \sum_{k=1}^{n} b_k^2 \sin^2 \alpha_k\right)^{\frac{1}{2}}\right]^2.$$

3.5. A Conditional Inequality Providing a Refinement.

The following lemma holds [2, Lemma 4.1].

Lemma 3.16. *Consider the sequences of real numbers* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n)$ *and* $\bar{\mathbf{z}} = (z_1, \dots, z_n)$. *If*

$$\text{(3.31)} \qquad y_k^2 \leq |x_k z_k| \quad \textit{for any } k \in \{1, \dots, n\},$$

then one has the inequality:

$$\text{(3.32)} \qquad \left(\sum_{k=1}^{n} |y_k|\right)^2 \leq \sum_{k=1}^{n} |x_k| \sum_{k=1}^{n} |z_k|.$$

Proof. We will follow the proof in [2]. Using the condition (3.31) and the (CBS) $-$inequality, we have

$$\begin{aligned}\sum_{k=1}^{n} |y_k| &\leq \sum_{k=1}^{n} |x_k|^{\frac{1}{2}} |z_k|^{\frac{1}{2}} \\ &\leq \left[\sum_{k=1}^{n} \left(|x_k|^{\frac{1}{2}}\right)^2 \sum_{k=1}^{n} \left(|z_k|^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^{n} |x_k| \sum_{k=1}^{n} |z_k|\right)^{\frac{1}{2}}\end{aligned}$$

which is clearly equivalent to (3.32). □

The following result holds [2, Theorem 4.6].

Theorem 3.17. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *and* $\bar{\mathbf{c}} = (c_1, \dots, c_n)$ *be sequences of real numbers such that*

(i) $|b_k| + |c_k| \neq 0 \ \ (k \in \{1, \dots, n\})$*;*

(ii) $|a_k| \leq \frac{2|b_k c_k|}{|b_k|+|c_k|}$ *for any* $k \in \{1, \dots, n\}$.

Then one has the inequality

$$\sum_{k=1}^{n} |a_k| \leq \frac{2 \sum_{k=1}^{n} |b_k| \sum_{k=1}^{n} |c_k|}{\sum_{k=1}^{n} (|b_k| + |c_k|)}. \tag{3.33}$$

Proof. We will follow the proof in [2]. By (ii) we observe that

$$|a_k| \leq \frac{2|b_k c_k|}{|b_k| + |c_k|} \leq \begin{cases} 2|b_k| \\ \\ 2|c_k| \end{cases} \text{ for any } k \in \{1, \dots, n\}$$

and thus

$$\begin{aligned} x_k &:= 2|b_k| - |a_k| \geq 0 \qquad \text{and} \\ z_k &:= 2|c_k| - |a_k| \geq 0 \qquad \text{for any } k \in \{1, \dots, n\}. \end{aligned} \tag{3.34}$$

A simple calculation also shows that the relation (ii) is equivalent to

$$a_k^2 \leq (2|b_k| - |a_k|)(2|c_k| - |a_k|) \quad \text{for any } k \in \{1, \dots, n\}. \tag{3.35}$$

If we consider $y_k := a_k$ and take $x_k,\ z_k\ (k = 1, \dots, n)$ as defined by (3.34), then we get $y_k^2 \leq x_k z_k$ (with $x_k,\ z_k \geq 0$) for any $k \in \{1, \dots, n\}$. Applying Lemma 3.16 we deduce

$$\left(\sum_{k=1}^{n} |a_k|\right)^2 \leq \left(2\sum_{k=1}^{n} |b_k| - \sum_{k=1}^{n} |a_k|\right)\left(2\sum_{k=1}^{n} |c_k| - \sum_{k=1}^{n} |a_k|\right) \tag{3.36}$$

which is clearly equivalent to (3.33). □

The following corollary is a natural consequence of the above theorem [2, Corollary 4.7].

Corollary 3.18. *For any sequence* $\bar{\mathbf{x}}$ *and* $\bar{\mathbf{y}}$ *of real numbers, with* $|x_k| + |y_k| \neq 0\ (k = 1, \dots, n)$, *one has:*

$$\sum_{k=1}^{n} \frac{|x_k y_k|}{|x_k| + |y_k|} \leq \frac{2 \sum_{k=1}^{n} |x_k| \sum_{k=1}^{n} |y_k|}{\sum_{k=1}^{n} (|x_k| + |y_k|)}. \tag{3.37}$$

For two positive real numbers, let us recall the following means

$$A(a, b) := \frac{a+b}{2} \qquad (\textit{the arithmetic mean})$$

$$G(a, b) := \sqrt{ab} \qquad (\textit{the geometric mean})$$

and

$$H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}} \qquad (\textit{the harmonic mean}).$$

We remark that if $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are sequences of real numbers, then obviously

$$\sum_{i=1}^{n} A(a_i, b_i) = A\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right), \tag{3.38}$$

and, by the (CBS) −inequality,

$$\sum_{i=1}^{n} G(a_i, b_i) \le G\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right). \tag{3.39}$$

The following similar result for harmonic means also holds [2, p. 19].

Theorem 3.19. *For any two sequences of positive real numbers* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *we have the property:*

$$\sum_{i=1}^{n} H(a_i, b_i) \le H\left(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right). \tag{3.40}$$

Proof. Follows by Corollary 3.18 on choosing $x_k = a_k$, $y_k = b_k$ and multiplying the inequality (3.37) with 2. ☐

The following refinement of the (CBS) −inequality holds [2, Corollary 4.9]. This result is known in the literature as **Milne's inequality** [8].

Theorem 3.20. *For any two sequences of real numbers* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *with* $|p_k| + |q_k| \neq 0$ $(k = 1, \dots, n)$, *one has the inequality:*

$$\left(\sum_{k=1}^{n} p_k q_k\right)^2 \le \sum_{k=1}^{n} \left(p_k^2 + q_k^2\right) \sum_{k=1}^{n} \frac{p_k^2 q_k^2}{p_k^2 + q_k^2} \le \sum_{k=1}^{n} p_k^2 \sum_{k=1}^{n} q_k^2. \tag{3.41}$$

Proof. We shall follow the proof in [2]. The first inequality is obvious by Lemma 3.16 on choosing $y_k = p_k q_k$, $x_k = p_k^2 + q_k^2$ and $z_k = \frac{p_k^2 q_k^2}{p_k^2 + q_k^2}$ $(k = 1, \dots, n)$.

The second inequality follows by Corollary 3.18 on choosing $x_k = p_k^2$ and $y = q_k^2$ $(k = 1, \dots, n)$. ☐

Remark 3.21. The following particular inequality is obvious by (3.41)

$$\left(\sum_{i=1}^{n} \sin\alpha_k \cos\alpha_k\right)^2 \le n \sum_{i=1}^{n} \sin^2\alpha_k \cos^2\alpha_k \le \sum_{i=1}^{n} \sin^2\alpha_k \sum_{i=1}^{n} \cos^2\alpha_k; \tag{3.42}$$

for any $\alpha_k \in \mathbb{R}$, $k \in \{1, \dots, n\}$.

3.6. **A Refinement for Non-Constant Sequences.** The following result was proved in [3, Theorem 1].

Theorem 3.22. *Let* $\bar{\mathbf{a}} = (a_i)_{i\in\mathbb{N}}$, $\bar{\mathbf{b}} = (b_i)_{i\in\mathbb{N}}$, $\bar{\mathbf{p}} = (p_i)_{i\in\mathbb{N}}$ *be sequences of real numbers such that*

(i) $a_i \neq a_j$ *and* $b_i \neq b_j$ *for* $i \neq j$, $i, j \in \mathbb{N}$;
(ii) $p_i > 0$ *for all* $i \in \mathbb{N}$.

Then for any H *a finite part of* $\mathbb{N}$ *one has the inequality:*

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \ge \max\{A, B\} \ge 0, \tag{3.43}$$

where

$$A := \max_{\substack{J\subseteq H \\ J\neq\emptyset}} \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{j\in J} p_j a_j - \sum_{i\in H} p_i a_i^2 \sum_{j\in J} p_j b_j\right]^2}{P_J \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in J} p_i a_i\right)^2} \tag{3.44}$$

and

$$B:=\max_{\substack{J\subseteq H\\ J\neq\emptyset}}\frac{\left[\sum_{i\in H}p_ia_ib_i\sum_{j\in J}p_jb_j-\sum_{i\in H}p_ib_i^2\sum_{j\in J}p_ja_j\right]^2}{P_J\sum_{i\in H}p_ib_i^2-\left(\sum_{i\in J}p_ib_i\right)^2} \tag{3.45}$$

and $P_J:=\sum_{j\in J}p_j$.

Proof. We shall follow the proof in [3].

Let J be a part of H. Define the mapping $f_J:\mathbb{R}\to\mathbb{R}$ given by

$$f_J(t)=\sum_{i\in H}p_ia_i^2\left[\sum_{i\in H\setminus J}p_ib_i^2+\sum_{i\in J}p_i(b_i+t)^2\right]-\left[\sum_{i\in H\setminus J}p_ia_ib_i+\sum_{i\in J}p_ia_i(b_i+t)\right]^2.$$

Then by the $(CBS)-$inequality we have that $f_J(t)\geq 0$ for all $t\in\mathbb{R}$.

On the other hand we have

$$\begin{aligned}f_J(t)&=\sum_{i\in H}p_ia_i^2\left[\sum_{i\in H}p_ib_i^2+2t\sum_{i\in H}p_ib_i+t^2P_J\right]-\left[\sum_{i\in H}p_ia_ib_i+t\sum_{i\in J}p_ia_i\right]^2\\&=t^2\left[P_J\sum_{i\in H}p_ia_i^2-\left(\sum_{i\in J}p_ia_i\right)^2\right]\\&\quad+2t\left[\sum_{i\in H}p_ia_i^2\sum_{i\in J}p_ib_i-\sum_{i\in H}p_ia_ib_i\sum_{i\in J}p_ia_i\right]\\&\quad+\left[\sum_{i\in H}p_ia_i^2\sum_{i\in H}p_ib_i^2-\left(\sum_{i\in H}p_ia_ib_i\right)^2\right]\end{aligned}$$

for all $t\in\mathbb{R}$.

Since

$$P_J\sum_{i\in H}p_ia_i^2-\left(\sum_{i\in J}p_ia_i\right)^2\geq P_J\sum_{i\in J}p_ia_i^2-\left(\sum_{i\in J}p_ia_i\right)^2>0$$

as $a_i\neq a_j$ for all $i,j\in\{1,\dots,n\}$ with $i\neq j$, then, by the inequality $f_J(t)\geq 0$ for any $t\in\mathbb{R}$ we get that

$$\begin{aligned}0\geq\frac{1}{4}\Delta=&\left[\sum_{i\in H}p_ia_ib_i\sum_{j\in J}p_ja_j-\sum_{i\in H}p_ia_i^2\sum_{j\in J}p_jb_j\right]^2\\&-\left[P_J\sum_{i\in H}p_ia_i^2-\left(\sum_{i\in J}p_ia_i\right)^2\right]\left[\sum_{i\in H}p_ia_i^2\sum_{i\in H}p_ib_i^2-\left(\sum_{i\in H}p_ia_ib_i\right)^2\right]\end{aligned}$$

from where results the inequality

$$\sum_{i\in H}p_ia_i^2\sum_{i\in H}p_ib_i^2-\left(\sum_{i\in H}p_ia_ib_i\right)^2\geq A.$$

The second part of the proof goes likewise for the mapping $g_J : \mathbb{R} \to \mathbb{R}$ given by

$$g_J(t) = \left[\sum_{i\in H\setminus J} p_i a_i^2 + \sum_{i\in J} p_i (a_i + t)^2\right] \sum_{i\in H} p_i b_i^2 - \left[\sum_{i\in H\setminus J} p_i a_i b_i + \sum_{i\in J} p_i b_i (a_i + t)\right]^2$$

and we omit the details. □

The following corollary also holds [3, Corollary 1].

Corollary 3.23. *With the assumptions of Theorem 3.22 and if*

$$C := \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{i\in H} p_i a_i - \sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i\right]^2}{P_H \sum_{i\in H} p_i a_i^2 - \left(\sum_{i\in H} p_i a_i\right)^2}, \tag{3.46}$$

$$D := \frac{\left[\sum_{i\in H} p_i a_i b_i \sum_{i\in H} p_i b_i - \sum_{i\in H} p_i b_i^2 \sum_{i\in H} p_i a_i\right]^2}{P_H \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i b_i\right)^2}, \tag{3.47}$$

then one has the inequality

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \max\{C, D\} \geq 0. \tag{3.48}$$

The following corollary also holds [3, Corollary 2].

Corollary 3.24. *If $a_i, b_i \neq 0$ for $i \in \mathbb{N}$ and H is a finite part of $\mathbb{N}$, then one has the inequality*

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \frac{1}{card(H) - 1} \max\left\{\frac{\sum_{j\in H} p_j c_j^2}{\sum_{i\in H} p_i a_i^2}, \frac{\sum_{j\in H} p_j d_j^2}{\sum_{i\in H} p_i b_i^2}\right\} \geq 0, \tag{3.49}$$

where

$$c_j := a_j \sum_{i\in H} p_i a_i b_i - b_j \sum_{i\in H} p_i a_i^2, \quad j \in H \tag{3.50}$$

and

$$d_j := a_j \sum_{i\in H} p_i b_i^2 - b_j \sum_{i\in H} p_i a_i b_i, \quad j \in H. \tag{3.51}$$

Proof. Choosing in Theorem 3.22, $J = \{j\}$, we get the inequality

$$\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2 \geq \frac{p_j^2 c_j^2}{p_j \sum_{i\in H} p_i a_i^2 - p_j^2 a_j^2}, \quad j \in H$$

from where we obtain

$$\left(\sum_{i\in H} p_i a_i^2 - p_j a_j^2\right) \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2\right] \geq p_j c_j^2 \text{ for any } j \in H.$$

Summing these inequalities over $j \in H$, we get

$$[card(H) - 1] \sum_{i\in H} p_i a_i^2 \left[\sum_{i\in H} p_i a_i^2 \sum_{i\in H} p_i b_i^2 - \left(\sum_{i\in H} p_i a_i b_i\right)^2\right] \geq \sum_{j\in H} p_j c_j^2$$

from where we get the first part of (3.49).

The second part goes likewise and we omit the details. □

Remark 3.25. The following particular inequalities provide refinement for the (CBS) −inequality [3, p. 60 – p. 61].

(1) Assume that $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ are nonconstant sequences of real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \max\left\{ \frac{\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} a_i b_i\right]^2}{n\sum\limits_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2}, \frac{\left[\sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i^2\right]^2}{n\sum\limits_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} b_i\right)^2} \right\}. \tag{3.52}$$

(2) Assume that $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are sequences of real numbers with not all elements equal to zero, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \frac{1}{n-1} \max\left\{ \frac{\sum\limits_{j=1}^{n} \left(a_j \sum\limits_{i=1}^{n} a_i b_i - b_j \sum\limits_{i=1}^{n} a_i^2\right)^2}{\sum\limits_{i=1}^{n} a_i^2}, \frac{\sum\limits_{j=1}^{n} \left(a_j \sum\limits_{i=1}^{n} a_i^2 - b_j \sum\limits_{i=1}^{n} a_i b_i\right)^2}{\sum\limits_{i=1}^{n} b_i^2} \right\}. \tag{3.53}$$

3.7. **De Bruijn's Inequality.** The following refinement of the (CBS) −inequality was proved by N.G. de Bruijn in 1960, [4] (see also [5, p. 89]).

Theorem 3.26. *If $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ is a sequence of real numbers and $\bar{\mathbf{z}} = (z_1, \ldots, z_n)$ is a sequence of complex numbers, then*

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k^2\right|\right]. \tag{3.54}$$

Equality holds in (3.54) if and only if for $k \in \{1, \ldots, n\}$, $a_k = \mathrm{Re}(\lambda z_k)$, where λ is a complex number such that $\sum_{k=1}^{n} \lambda^2 z_k^2$ is a nonnegative real number.

Proof. We shall follow the proof in [5, p. 89 – p. 90].

By a simultaneous rotation of all the z_k's about the origin, we get

$$\sum_{k=1}^{n} a_k z_k \geq 0.$$

This rotation does not affect the moduli

$$\left|\sum_{k=1}^{n} a_k z_k\right|, \quad \left|\sum_{k=1}^{n} z_k^2\right| \quad \text{and} \quad |z_k| \quad \text{for} \quad k \in \{1, \dots, n\}.$$

Hence, it is sufficient to prove inequality (3.54) for the case where $\sum_{k=1}^{n} a_k z_k \geq 0$.

If we put $z_k = x_k + i y_k$ $(k \in \{1, \dots, n\})$, then, by the (CBS) −inequality for real numbers, we have

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 = \left(\sum_{k=1}^{n} a_k z_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} x_k^2. \tag{3.55}$$

Since

$$2x_k^2 = |z_k|^2 + \operatorname{Re} z_k^2 \quad \text{for any} \quad k \in \{1, \dots, n\}$$

we obtain, by (3.55), that

$$\left|\sum_{k=1}^{n} a_k z_k\right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} a_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \sum_{k=1}^{n} \operatorname{Re} z_k^2\right]. \tag{3.56}$$

As

$$\sum_{k=1}^{n} \operatorname{Re} z_k^2 = \operatorname{Re}\left(\sum_{k=1}^{n} z_k^2\right) \leq \left|\sum_{k=1}^{n} z_k^2\right|,$$

then by (3.56) we deduce the desired inequality (3.54). □

3.8. McLaughlin's Inequality.

The following refinement of the (CBS) −inequality for sequences of real numbers was obtained in 1966 by H.W. McLaughlin [7, p. 66].

Theorem 3.27. *If* $\bar{\mathbf{a}} = (a_1, \dots, a_{2n})$, $\bar{\mathbf{b}} = (b_1, \dots, b_{2n})$ *are sequences of real numbers, then*

$$\left(\sum_{i=1}^{2n} a_i b_i\right)^2 + \left[\sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i)\right]^2 \leq \sum_{i=1}^{2n} a_i^2 \sum_{i=1}^{2n} b_i^2 \tag{3.57}$$

with equality if and only if for any $i, j \in \{1, \dots, n\}$

$$a_i b_j - a_j b_i - a_{n+i} b_{n+j} + a_{n+j} b_{n+i} = 0 \tag{3.58}$$

and

$$a_i b_{n+j} - a_j b_{n+i} + a_{n+i} b_j - a_{n+j} b_i = 0. \tag{3.59}$$

Proof. We shall follow the proof in [6] by M.O. Drîmbe.

The following identity may be obtained by direct computation

$$\begin{aligned} \sum_{i=1}^{2n} a_i^2 \sum_{i=1}^{2n} b_i^2 - \left(\sum_{i=1}^{2n} a_i b_i\right)^2 - \left[\sum_{i=1}^{n} (a_i b_{n+i} - a_{n+i} b_i)\right]^2 \\ = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i - a_{n+i} b_{n+j} + a_{n+j} b_{n+i})^2 \\ + \sum_{1 \leq i < j \leq n} (a_i b_{n+j} - a_j b_{n+i} + a_{n+i} b_j - a_{n+j} b_i)^2. \end{aligned} \tag{3.60}$$

It is obvious that (3.57) is a simple consequence of the identity (3.60). The case of equality is also obvious. □

Remark 3.28. For other similar (CBS) −type inequalties see the survey paper [7]. An analogous inequality to (3.57) for sequences $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ having $4n$ terms each may be found in [7, p. 70].

3.9. **A Refinement due to Daykin-Eliezer-Carlitz.** We will present now the version due to Mitrinović, Pečarić and Fink [5, p. 87] of Daykin-Eliezer-Carlitz's refinement of the discrete (CBS) −inequality [8].

Theorem 3.29. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of positive numbers. The ineuality*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} f(a_i b_i) \sum_{i=1}^{n} g(a_i b_i) \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \tag{3.61}$$

holds if and only if

$$f(a,b)\, g(a,b) = a^2 b^2, \tag{3.62}$$

$$f(ka, kb) = k^2 f(a,b), \tag{3.63}$$

$$\frac{bf(a,1)}{af(b,1)} + \frac{af(b,1)}{bf(a,1)} \le \frac{a}{b} + \frac{b}{a} \tag{3.64}$$

for any $a, b, k > 0.$

Proof. We shall follow the proof in [5, p. 88 – p. 89].
Necessity. Indeed, for $n = 1,$ the inequality (3.61) becomes

$$(ab)^2 \le f(a,b)\, g(a,b) \le a^2 b^2, \qquad a, b > 0$$

which gives the condition (3.62).
For $n = 2$ in (3.61), using (3.62), we get

$$2a_1b_1a_2b_2 \le f(a_1,b_1)\, g(a_2,b_2) + f(a_2,b_2)\, g(a_1,b_1) \le a_1^2 b_2^2 + a_2^2 b_1^2.$$

By eliminating $g,$ we get

$$2 \le \frac{f(a_1,b_1)}{f(a_2,b_2)} \cdot \frac{a_2b_2}{a_1b_1} + \frac{f(a_2,b_2)}{f(a_1,b_1)} \cdot \frac{a_1b_1}{a_2b_2} \le \frac{a_1b_2}{a_2b_1} + \frac{a_2b_1}{a_1b_2}. \tag{3.65}$$

By substituting in (3.65) a, b for a_1, b_1 and ka, kb for a_2, b_2 ($k > 0$), we get

$$2 \le \frac{f(a,b)}{f(ka,kb)} k^2 + \frac{f(ka,kb)}{f(a,b)} k^{-2} \le 2$$

and this is valid only if $k^2 f(a,b)\,(f(ka,kb)) = 1,$ i.e., the condition (3.63) holds.
Using (3.65), for $a_1 = a$, $b_1 = b$, $a_2 = b$, $b_2 = 1,$ we have

$$2 \le \frac{\frac{f(a,1)}{a}}{\frac{f(b,1)}{b}} + \frac{\frac{f(b,1)}{b}}{\frac{f(a,1)}{a}} \le \frac{a}{b} + \frac{b}{a}. \tag{3.66}$$

The first inequality in (3.66) is always satisfied while the second inequality is equivalent to (3.64).

Sufficiency. Suppose that (3.62) holds. Then inequality (3.61) can be written in the form

$$2\sum_{1\le i<j\le n} a_i b_i a_j b_j \le \sum_{1\le i<j\le n} \left[f(a_i,b_i)\, g(a_j,b_j) + f(a_j,b_j)\, g(a_i,b_i) \right] \le \sum_{1\le i<j\le n} \left(a_i^2 b_j^2 + a_j^2 b_i^2 \right).$$

Therefore, it is enough to prove

$$2a_i b_i a_j b_j \le f(a_i,b_i)\, g(a_j,b_j) + f(a_j,b_j)\, g(a_i,b_i) \le a_i^2 b_j^2 + a_j^2 b_i^2. \tag{3.67}$$

Suppose that (3.64) holds. Then (3.66) holds and putting $a = \frac{a_i}{b_i}$, $b = \frac{a_j}{b_j}$ in (3.66) and using (3.63), we get

$$2 \le \frac{f(a_i,b_i)}{f(a_j,b_j)} \cdot \frac{a_j b_j}{a_i b_i} + \frac{f(a_j,b_j)}{f(a_i,b_i)} \cdot \frac{a_i b_i}{a_j b_j} \le \frac{a_i b_j}{a_j b_i} + \frac{a_j b_i}{a_i b_j}.$$

Multiplying the last inequality by $a_i b_i a_j b_j$ and using (3.62), we obtain (3.67). □

Remark 3.30. In [8] (see [5, p. 89]) the condition (3.64) is given as

$$f(b,1) \le f(a,1), \quad \frac{f(a,1)}{a^2} \le \frac{f(b,1)}{b^2} \quad \text{for } a \ge b > 0. \tag{3.68}$$

Remark 3.31. O.E. Daykin, C.J. Eliezer and C. Carlitz [8] stated that examples for f, g satisfying (3.62) – (3.64) were obtained in the literature. The choice $f(x,y) = x^2 + y^2$, $g(x,y) = \frac{x^2y^2}{x^2+y^2}$ will give the **Milne's inequality**

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \sum_{i=1}^n \left(a_i^2 + b_i^2\right) \cdot \sum_{i=1}^n \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \le \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2. \tag{3.69}$$

For a different proof of this fact, see Section 3.5.

The choice $f(x,y) = x^{1+\alpha} y^{1-\alpha}$, $g(x,y) = x^{1-\alpha} y^{1+\alpha}$ $(\alpha \in [0,1])$ will give the ***Callebaut inequality***

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \sum_{i=1}^n a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^n a_i^{1-\alpha} b_i^{1+\alpha} \le \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2. \tag{3.70}$$

3.10. **A Refinement via Dunkl-Williams' Inequality.** We will use the following version of Dunkl-Williams' inequality established in 1964 in inner product spaces [9].

Lemma 3.32. *Let a, b be two non-null complex numbers. Then*

$$|a-b| \ge \frac{1}{2}\left(|a| + |b|\right) \left| \frac{a}{|a|} - \frac{b}{|b|} \right|. \tag{3.71}$$

Proof. We start with the identity (see also [5, pp. 515 – 516])

$$\begin{aligned}\left|\frac{a}{|a|}-\frac{b}{|b|}\right|^2 &= \left(\frac{a}{|a|}-\frac{b}{|b|}\right)\left(\frac{\bar{a}}{|a|}-\frac{\bar{b}}{|b|}\right) \\ &= 2-2\operatorname{Re}\left(\frac{a}{|a|}\cdot\frac{\bar{b}}{|b|}\right) \\ &= \frac{1}{|a|\,|b|}\left(2\,|a|\,|b|-2\operatorname{Re}\left(a\cdot\bar{b}\right)\right) \\ &= \frac{1}{|a|\,|b|}\left[2\,|a|\,|b|-\left(|a|^2+|b|^2-|a-b|^2\right)\right] \\ &= \frac{1}{|a|\,|b|}\left[|a-b|^2-(|a|-|b|)^2\right].\end{aligned}$$

Hence

$$|a-b|^2-\left[\frac{1}{2}(|a|+|b|)\right]^2\left|\frac{a}{|a|}-\frac{b}{|b|}\right|^2=\frac{(|a|-|b|)^2}{4\,|a|\,|b|}\left[(|a|+|b|)^2-|a-b|^2\right]$$

and (3.71) is proved. □

Using the above result, we may prove the following refinement of the (CBS) −inequality for complex numbers.

Theorem 3.33. *If* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ *are two sequences of nonzero complex numbers, then*

$$\begin{aligned}(3.72)\quad \sum_{k=1}^n|a_k|^2\sum_{k=1}^n|b_k|^2-\left|\sum_{k=1}^n a_kb_k\right|^2& \\ \geq\frac{1}{8}\sum_{i,j=1}^n&\left|\bar{a}_ib_j-\bar{a}_jb_i+\frac{|b_i|}{|a_i|}\bar{a}_i\cdot\frac{|a_j|}{|b_j|}b_j-\frac{|a_i|}{|b_i|}b_i\cdot\frac{|b_j|}{|a_j|}\bar{a}_j\right|^2\geq 0.\end{aligned}$$

Proof. The inequality (3.71) is clearly equivalent to

$$|a-b|^2\geq\frac{1}{4}\left|a-b+\frac{|b|}{|a|}\cdot a-\frac{|a|}{|b|}\cdot b\right|^2 \tag{3.73}$$

for any $a,b\in\mathbb{C}$, $a,b\neq 0$.

We know the Lagrange's identity for sequences of complex numbers

$$\sum_{k=1}^n|a_k|^2\sum_{k=1}^n|b_k|^2-\left|\sum_{k=1}^n a_kb_k\right|^2=\frac{1}{2}\sum_{i,j=1}^n|\bar{a}_ib_j-\bar{a}_jb_i|^2. \tag{3.74}$$

By (3.73), we have

$$|\bar{a}_ib_j-\bar{a}_jb_i|^2\geq\frac{1}{4}\left|\bar{a}_ib_j-\bar{a}_jb_i+\frac{|a_j|\,|b_i|}{|a_i|\,|b_j|}\bar{a}_ib_j-\frac{|a_i|\,|b_j|}{|a_j|\,|b_i|}\bar{a}_jb_i\right|^2.$$

Summing over i,j from 1 to n and using the (CBS) −inequality for double sums, we deduce (3.72). □

3.11. **Some Refinements due to Alzer and Zheng.** In 1992, H. Alzer [10] presented the following refinement of the Cauchy-Schwarz inequality written in the form

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n} x_k^2 y_k. \tag{3.75}$$

Theorem 3.34. *Let* x_k *and* y_k $(k = 1, \dots, n)$ *be real numbers satisfying* $0 = x_0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}$ *and* $0 < y_n \leq y_{n-1} \leq \cdots \leq y_1$. *Then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \left[x_k^2 - \frac{1}{4} x_{k-1} x_k\right] y_k, \tag{3.76}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, \dots, n)$ *and* $y_1 = \cdots = y_n$.

In 1998, Liu Zheng [11] pointed out an error in the proof given in [10], which can be corrected as shown in [11]. Moreover, Liu Zheng established the following result which sharpens (3.76).

Theorem 3.35. *Let* x_k *and* y_k $(k = 1, \dots, n)$ *be real numbers satisfying* $0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}$ *and* $0 < y_n \leq y_{n-1} \leq \cdots \leq y_1$. *Then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \delta_k y_k, \tag{3.77}$$

with

$$\delta_1 = x_1^2 \text{ and } \delta_k = \frac{7k+1}{8k} x_k^2 - \frac{k}{8(k-1)} x_{k-1}^2 \quad (k \geq 2). \tag{3.78}$$

Equality holds in (3.77) if and only if $x_k = kx_1$ $(k = 1, \dots, n)$ *and* $y_1 = \cdots = y_n$.

In 1999, H. Alzer improved the above results as follows.

To present his results, we will follow [12].

In order to prove the main result, we need some technical lemmas.

Lemma 3.36. *Let* x_k $(k = 1, \dots, n)$ *be real numbers such that*

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}.$$

Then

$$2\sum_{k=1}^{n} x_k \leq (n+1) x_n, \tag{3.79}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, \dots, n)$.

A proof of Lemma 3.36 is given in [10].

Lemma 3.37. *Let* x_k $(k = 1, \dots, n)$ *be real numbers such that*

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n}.$$

Then

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2, \tag{3.80}$$

with equality holding if and only if $x_k = kx_1$ $(k = 1, \dots, n)$.

Proof. Let

$$S_n = S_n(x_1,\dots,x_n) = n\sum_{k=1}^{n}\frac{3k+1}{4k}x_k^2 - \left(\sum_{k=1}^{n}x_k\right)^2.$$

Then we have for $n \geq 2$:

$$S_n - S_{n-1} = \sum_{k=1}^{n-1}\frac{3k+1}{4k}x_k^2 - 2x_n\sum_{k=1}^{n-1}x_k + \frac{3(n-1)}{4}x_n^2 = f(x_n), \quad \text{say.} \tag{3.81}$$

We differentiate with respect to x_n and use (3.79) and $x_n \geq \frac{n}{n-1}x_{n-1}$. This yields

$$f'(x_n) = \frac{3(n-1)}{2}x_n - 2\sum_{k=1}^{n-1}x_k \geq \frac{n}{2}x_{n-1} > 0$$

and

$$\begin{aligned} f(x_n) &\geq f\left(\frac{n}{n-1}x_{n-1}\right) \\ &= \sum_{k=1}^{n}\frac{3k+1}{4k}x_k^2 - \frac{2n}{n-1}x_{n-1}\sum_{k=1}^{n-1}x_k + \frac{3n^2}{4(n-1)}x_{n-1}^2 \\ &= T_{n-1}(x_1,\dots,x_{n-1}), \quad \text{say.} \end{aligned} \tag{3.82}$$

We use induction on n to establish that $T_{n-1}(x_1,\dots,x_{n-1}) \geq 0$ for $n \geq 2$. We have $T_1(x_1) = 0$. Let $n \geq 3$; applying (3.79) we obtain

$$\begin{aligned} \frac{\partial}{\partial x_{n-1}}T_{n-1}(x_1,\dots,x_{n-1}) &= \frac{3n+2}{2}x_{n-1} - \frac{2n}{n-1}\sum_{k=1}^{n-1}x_k \\ &\geq \frac{(n-2)(n+1)}{2(n-1)}x_{n-1} > 0 \end{aligned}$$

and

$$T_{n-1}(x_1,\dots,x_{n-1}) \geq T_{n-1}\left(x_1,\dots,x_{n-2},\frac{n-1}{n-2}x_{n-2}\right). \tag{3.83}$$

Using the induction hypothesis $T_{n-2}(x_1,\dots,x_{n-2}) \geq 0$ and (3.79), we get

$$T_{n-1}\left(x_1,\dots,x_{n-2},\frac{n-1}{n-2}x_{n-2}\right) \geq \frac{x_{n-2}}{n-2}\left[(n-1)x_{n-2} - 2\sum_{k=1}^{n-2}x_k\right]. \tag{3.84}$$

From (3.83) and (3.84) we conclude $T_{n-1}(x_1,\dots,x_{n-1}) \geq 0$ for $n \geq 2$, so that (3.81) and (3.82) imply

$$S_n \geq S_{n-1} \geq \cdots \geq S_2 \geq S_1 = 0. \tag{3.85}$$

This proves inequality (3.80). We discuss the cases of equality. A simple calculation reveals that $S_n(x_1, 2x_1,\dots,nx_1) = 0$. We use induction on n to prove the implication

$$S_n(x_1,\dots,x_n) = 0 \Longrightarrow x_k = kx_1 \text{ for } k = 1,\dots,n. \tag{3.86}$$

If $n = 1$, then (3.86) is obviously true. Next, we assume that (3.86) holds with $n-1$ instead of n. Let $n \geq 2$ and $S_n(x_1,\dots,x_n) = 0$. Then (3.85) leads to $S_{n-1}(x_1,\dots,x_{n-1}) = 0$ which implies $x_k = kx_1$ for $k = 1,\dots,n-1$. Thus, we have $S_n(x_1, 2x_1,\dots,(n-1)x_1, x_n) = 0$ which is equivalent to $(x_n - nx_1)(3x_n - nx_1) = 0$. Since $3x_n > nx_1$, we get $x_n = nx_1$. □

Lemma 3.38. *Let x_k $(k = 1, \dots, n)$ be real numbers such that*

$$0 < x_1 \le \frac{x_2}{2} \le \cdots \le \frac{x_n}{n}.$$

If the natural numbers n and q satisfy $n \ge q + 1$, then

$$0 < \left(\sum_{k=1}^{q} x_k \right)^2 - 2q x_n \sum_{k=1}^{q} x_k + \frac{(3n+1)\,q^2}{4n} x_n^2. \tag{3.87}$$

Proof. We denote the expression on the right-hand side of (3.87) by $u(x_n)$. Then we differentiate with respect to x_n and apply (3.79), $x_n \ge \left(\frac{n}{q}\right) x_q$ and $n \ge q+1$. This yields

$$\begin{aligned} \frac{1}{q} u'(x_n) &= \frac{(3n+1)\,q}{2n} x_n - 2q \sum_{k=1}^{q} x_k \\ &\ge \frac{(3n+1)\,q}{2n} x_n - (q+1)\, x_q \\ &\ge \frac{3n - 2q - 1}{2} x_q > 0. \end{aligned}$$

Hence, we get

$$u(x_n) \ge u\left(\frac{n}{q} x_q \right) = \frac{(3n+1)\,n}{4} x_q^2 - 2n x_q \sum_{k=1}^{q} x_k + \left(\sum_{k=1}^{q} x_k \right)^2. \tag{3.88}$$

Let

$$v(t) = \frac{(3t+1)\,t}{4} x_q - 2t \sum_{k=1}^{q} x_k \quad \text{and} \quad t \ge q+1;$$

from (3.79) we conclude that

$$v'(t) = \frac{6t+1}{4} x_q - 2 \sum_{k=1}^{q} x_k \ge \frac{2q+3}{4} x_q > 0.$$

This implies that the expression on the right-hand side of (3.88) is increasing on $[q+1, \infty)$ with respect to n. Since $n \ge q+1$, we get from (3.88):

$$\begin{aligned} u(x_n) &\ge \frac{(3q+4)\,(q+1)}{4} x_q^2 - 2\,(q+1)\, x_q \sum_{k=1}^{q} x_k + \left(\sum_{k=1}^{q} x_k \right)^2 \\ &= P_q(x_1, \dots, x_q)\,, \quad \text{say.} \end{aligned} \tag{3.89}$$

We use induction on q to show that $P_q(x_1, \dots, x_q) > 0$ for $q \ge 1$. We have $P_1(x_1) = \frac{1}{2} x_1^2$. If $P_{q-1}(x_1, \dots, x_{q-1}) > 0$, then we obtain for $q \ge 2$:

$$\begin{aligned} P_q(x_1, \dots, x_q) > 2q\,(x_{q-1} - x_q) \sum_{k=1}^{q-1} x_k - \frac{(3q+1)\,q}{4} x_{q-1}^2 \\ + \frac{q\,(3q-1)}{4} x_q^2 = w(x_q)\,, \quad \text{say.} \end{aligned} \tag{3.90}$$

We differentiate with respect to x_q and use (3.79) and $x_q \geq \left(\frac{q}{q-1}\right) x_{q-1}$. Then we get

$$\begin{aligned} w'(x_q) &= q\left[\frac{3q-1}{2}x_q - 2\sum_{k=1}^{q-1} x_k\right] \\ &\geq \frac{q^2(q+1)}{2(q-1)}x_{q-1} \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} w(x_q) &\geq w\left(\frac{q}{q-1}x_{q-1}\right) \\ &= \frac{q}{q-1}x_{q-1}\left[\frac{4q^2-q-1}{4(q-1)}x_{q-1} - 2\sum_{k=1}^{q-1} x_k\right] \\ &\geq \frac{(3q-1)q}{4(q-1)}x_{q-1}^2 \\ &> 0. \end{aligned} \tag{3.91}$$

From (3.89), (3.90) and (3.91), we obtain $u(x_n) > 0$. □

We are now in a position to prove the following companion of inequalities (3.76) and (3.77) (see [12]).

Theorem 3.39. *The inequality*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \sum_{k=1}^{n} y_k \sum_{k=1}^{n}\left(\alpha + \frac{\beta}{k}\right) y_k, \tag{3.92}$$

holds for all natural numbers n and for all real numbers x_k and y_k $(k = 1, \dots, n)$ with

$$0 < x_1 \leq \frac{x_2}{2} \leq \cdots \leq \frac{x_n}{n} \text{ and } 0 < y_n \leq y_{n-1} \leq \cdots \leq y_1, \tag{3.93}$$

if and only if

$$\alpha \geq \frac{3}{4} \text{ and } \beta \geq 1 - \alpha.$$

Proof. First, we assume that (3.92) is valid for all $n \geq 1$ and for all real numbers x_k and y_k $(k = 1, \dots, n)$ which satisfy (3.93). We set $x_k = k$ and $y_k = 1$ $(k = 1, \dots, n)$. Then (3.92) leads to

$$0 \leq \left(\alpha - \frac{3}{4}\right) 2n + \alpha + 3\beta - \frac{3}{2} \qquad (n \geq 1). \tag{3.94}$$

This implies $\alpha \geq \frac{3}{4}$. And, (3.94) with $n = 1$ yields $\alpha + \beta \geq 1$.

Now, we suppose that $\alpha \geq \frac{3}{4}$ and $\beta \geq 1 - \alpha$. Then we obtain for $k \geq 1$:

$$\alpha + \frac{\beta}{k} \geq \alpha + \frac{1-\alpha}{k} \geq \frac{3}{4} + \frac{1}{4k},$$

so that is suffices to show that (3.92) holds with $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$. Let

$$F(y_1, \dots, y_n) = \sum_{k=1}^{n} y_k \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2 y_k - \left(\sum_{k=1}^{n} x_k y_k\right)^2$$

and

$$F_q(y) = F(y, \dots, y, y_{q+1}, \dots, y_n) \qquad (1 \le q \le n-1).$$

We shall prove that F_q is strictly increasing on $[y_{q+1}, \infty)$. Since $y_{q+1} \le y_q$, we obtain

$$F_q(y_q) \ge F_q(y_{q+1}) = F_{q+1}(y_{q+1}) \qquad (1 \le q \le n-1), \tag{3.95}$$

and Lemma 3.37 imply

$$\begin{aligned} F(y_1, \dots, y_n) &= F_1(y_1) \ge F_1(y_2) = F_2(y_2) \ge F_2(y_3) \\ &\ge \dots \ge F_{n-1}(y_{n-1}) \ge F_{n-1}(y) \\ &= y_n^2 \left[n \sum_{k=1}^{n} \frac{3k+1}{4k} x_k^2 - \left(\sum_{k=1}^{n} x_k \right)^2 \right] \ge 0. \end{aligned}$$

If $F(y_1, \dots, y_n) = 0$, then we conclude from the strict monotonicity of F_q and from Lemma 3.37 that $y_1 = \dots = y_n$ and $x_k = kx_1$ $(k = 1, \dots, n)$.

It remains to show that F_q is strictly increasing on $[y_{q+1}, \infty)$. Let $y \ge y_{q+1}$; we differentiate F_q and apply Lemma 3.37. This yields

$$\begin{aligned} F_q'(y) = 2y \left[q \sum_{k=1}^{q} \frac{3k+1}{4k} x_k^2 - \left(\sum_{k=1}^{q} x_k \right)^2 \right] + q \sum_{k=q+1}^{n} \frac{3k+1}{4k} x_k^2 y_k \\ + \sum_{k=q+1}^{n} y_k \sum_{k=1}^{q} \frac{3k+1}{4k} x_k^2 - 2 \sum_{k=q+1}^{n} x_k y_k \sum_{k=1}^{q} x_k \end{aligned}$$

and

$$\frac{1}{2} F_q''(y) = q \sum_{k=1}^{q} \frac{3k+1}{4k} x_k^2 - \left(\sum_{k=1}^{q} x_k \right)^2 \ge 0.$$

Hence, we have

$$\begin{aligned} F_q'(y) &\ge F_q'(y_{q+1}) \\ &= \left(2qy_{q+1} + \sum_{k=q+1}^{n} y_k \right) \left[\sum_{k=1}^{q} \frac{3k+1}{4k} x_k^2 - \frac{1}{q} \left(\sum_{k=1}^{q} x_k \right)^2 \right] \\ &\quad + \frac{1}{q} \sum_{k=q+1}^{n} y_k \left\{ \frac{(3k+1) q^2}{4k} x_k^2 - 2qx_k \sum_{i=1}^{q} x_i + \left(\sum_{i=1}^{q} x_i \right)^2 \right\}. \end{aligned} \tag{3.96}$$

From Lemma 3.37 and Lemma 3.38 we obtain $F_q'(y_{q+1}) > 0$, so that (3.96) implies $F_q'(y) > 0$ for $y \ge y_{q+1}$. This completes the proof of the theorem. □

Remark 3.40. The proof of the theorem reveals that the sign of equality holds in (3.92) (with $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$) if and only if $x_k = kx_1$ $(k = 1, \dots, n)$ and $y_1 = \dots = y_n$.

Remark 3.41. If δ_k is defined by (3.78), then we have for $k \ge 2$:

$$\delta_k - \left(\frac{3}{4} + \frac{1}{4k} \right) x_k^2 = \frac{k(k-1)}{8} \left[\left(\frac{x_k}{k} \right)^2 - \left(\frac{x_{k-1}}{k-1} \right)^2 \right],$$

which implies that inequality (3.92) (with $\alpha = \frac{3}{4}$ and $\beta = \frac{1}{4}$) sharpens (3.77).

Remark 3.42. It is shown in [10] that if a sequence (x_k) satisfies $x_0 = 0$ and $2x_k \leq x_{k-1} + x_{k+1}$ $(k \geq 1)$, then $\left(\frac{x_k}{k}\right)$ is increasing. Hence, inequality (3.92) is valid for all sequences (x_k) which are positive and convex.

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4. FUNCTIONAL PROPERTIES

4.1. A Monotonicity Property.

The following result was obtained in [1, Theorem].

Theorem 4.1. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of real numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *be sequences of nonnegative real numbers such that* $p_k \geq q_k$ *for any* $k \in \{1, \dots, n\}$ *. Then one has the inequality*

$$\left(\sum_{i=1}^{n} p_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i a_i b_i\right| \geq \left(\sum_{i=1}^{n} q_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} q_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} q_i a_i b_i\right| \geq 0. \tag{4.1}$$

Proof. We shall follow the proof in [1].

Since $p_k - q_k \geq 0$, then the (CBS) −inequality for the weights $r_k = p_k - q_k$ $(k \in \{1, \dots, n\})$ will produce

$$\left(\sum_{k=1}^{n} p_k a_k^2 - \sum_{k=1}^{n} q_k a_k^2\right)\left(\sum_{k=1}^{n} p_k b_k^2 - \sum_{k=1}^{n} q_k b_k^2\right) \geq \left[\sum_{k=1}^{n} (p_k - q_k) a_k b_k\right]^2 . \tag{4.2}$$

Using the elementary inequality

$$(ac - bd)^2 \geq \left(a^2 - b^2\right)\left(c^2 - d^2\right), \quad a, b, c, d \in \mathbb{R}$$

for the choices

$$a = \left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}}, \quad b = \left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}}, \quad c = \left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} \quad \text{and} \quad d = \left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}}$$

we deduce by (4.2), that

$$\left(\sum_{k=1}^{n} p_k a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} p_k b_k^2\right)^{\frac{1}{2}} - \left(\sum_{k=1}^{n} q_k a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} q_k b_k^2\right)^{\frac{1}{2}} \geq \left|\sum_{k=1}^{n} p_k a_k b_k - \sum_{k=1}^{n} q_k a_k b_k\right| \geq \left|\sum_{k=1}^{n} p_k a_k b_k\right| - \left|\sum_{k=1}^{n} q_k a_k b_k\right|$$

proving the desired inequality (4.1). □

The following corollary holds [1, Corollary 1].

Corollary 4.2. *Let* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *be as in Theorem 4.1. Denote*

$$S_n(\mathbf{1}) := \{\bar{\mathbf{x}} = (x_1, \dots, x_n) \,|\, 0 \leq x_i \leq 1,\ i \in \{1, \dots, n\}\}.$$

Then

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i\right| = \sup_{\bar{\mathbf{x}} \in S_n(1)} \left[\left(\sum_{i=1}^{n} x_i a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} x_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} x_i a_i b_i\right|\right] \geq 0. \tag{4.3}$$

Remark 4.3. The following inequality is a natural particular case that may be obtained from (4.1) [1, p. 79]

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i\right| \geq \left[\sum_{i=1}^{n} a_i^2 \operatorname{trig}^2(\alpha_i)\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n} b_i^2 \operatorname{trig}^2(\alpha_i)\right]^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i \operatorname{trig}^2(\alpha_i)\right| \geq 0, \tag{4.4}$$

where $\operatorname{trig}(x) = \sin x$ or $\cos x$, $x \in \mathbb{R}$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a sequence of real numbers.

4.2. **A Superadditivity Property in Terms of Weights.** Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $S(\mathbb{K})$ the linear space of real or complex numbers, i.e.,

$$S(\mathbb{K}) := \left\{\overline{\mathbf{x}} | \overline{\mathbf{x}} = (x_i)_{i\in\mathbb{N}},\ \ x_i \in \mathbb{K},\ i \in \mathbb{N}\right\}$$

and $S_+(\mathbb{R})$ the family of nonnegative real sequences. Define the mapping

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) := \left(\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i x_i \bar{y}_i\right|, \tag{4.5}$$

where $\overline{\mathbf{p}} \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in S(\mathbb{K})$.

The following superadditivity property in terms of weights holds [2, p. 16].

Theorem 4.4. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ *and* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in S(\mathbb{K})$ *we have*

$$S(\overline{\mathbf{p}} + \overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + S(\overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq 0. \tag{4.6}$$

Proof. Using the (CBS) $-$inequality for real numbers

$$\left(a^2 + b^2\right)^{\frac{1}{2}}\left(c^2 + d^2\right)^{\frac{1}{2}} \geq ac + bd; \quad a, b, c, d \geq 0, \tag{4.7}$$

we have

$$\begin{aligned} S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) &= \left(\sum_{i\in I} p_i |x_i|^2 + \sum_{i\in I} q_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2 + \sum_{i\in I} q_i |y_i|^2\right)^{\frac{1}{2}} \\ &\qquad - \left|\sum_{i\in I} p_i x_i \bar{y}_i + \sum_{i\in I} q_i x_i \bar{y}_i\right| \\ &\geq \left(\sum_{i\in I} p_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in I} q_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} q_i |y_i|^2\right)^{\frac{1}{2}} \\ &\qquad - \left|\sum_{i\in I} p_i x_i \bar{y}_i\right| - \left|\sum_{i\in I} q_i x_i \bar{y}_i\right| \\ &= S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + S(\overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}), \end{aligned}$$

and the inequality (4.6) is proved. □

The following corollary concerning the monotonicity of $S(\cdot, I, \overline{\mathbf{x}}, \overline{\mathbf{y}})$ also holds [2, p. 16].

Corollary 4.5. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$ *with* $\overline{\mathbf{p}} \geq \overline{\mathbf{q}}$ *and* $I \in \mathcal{P}_f(\mathbb{N}), \overline{\mathbf{x}}, \overline{\mathbf{y}} \in S(\mathbb{K})$ *one has the inequality:*

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq 0. \tag{4.8}$$

Proof. Using Theorem 4.4, we have

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) = S((\overline{\mathbf{p}} - \overline{\mathbf{q}}) + \overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}} - \overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + S(\overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}})$$

giving

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) - S(\overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}} - \overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq 0$$

and the inequality (4.8) is proved. □

Remark 4.6. The following inequalities follow by the above results [2, p. 17].

(1) Let $\alpha_i \in \mathbb{R}$ $(i \in \{1, \dots, n\})$ and $x_i, y_i \in \mathbb{K}$ $(i \in \{1, \dots, n\})$. Then one has the inequality:

$$\begin{aligned} &\left(\sum_{i=1}^n |x_i|^2 \sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n x_i \bar{y}_i\right| \\ &\quad\geq \left(\sum_{i=1}^n |x_i|^2 \sin^2 \alpha_i \sum_{i=1}^n |y_i|^2 \sin^2 \alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n x_i \bar{y}_i \sin^2 \alpha_i\right| \\ &\qquad + \left(\sum_{i=1}^n |x_i|^2 \cos^2 \alpha_i \sum_{i=1}^n |y_i|^2 \cos^2 \alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n x_i \bar{y}_i \cos^2 \alpha_i\right| \geq 0. \end{aligned} \tag{4.9}$$

(2) Denote $S_n(\mathbf{1}) := \{\overline{\mathbf{p}} \in S_+(\mathbb{R}) \,| p_i \leq 1 \text{ for all } i \in \{1,\dots,n\}\}$. Then for all $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in S(\mathbb{K})$ one has the bound (see also Corollary 4.2):

$$0 \leq \left(\sum_{i=1}^n |x_i|^2 \sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n x_i \bar{y}_i\right| = \sup_{\overline{\mathbf{p}} \in S_n(\mathbf{1})} \left\{ \left(\sum_{i=1}^n p_i |x_i|^2 \sum_{i=1}^n p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^n p_i x_i \bar{y}_i\right| \right\}. \tag{4.10}$$

4.3. **The Superadditivity as an Index Set Mapping.** We assume that we are under the hypothesis and notations in Section 4.2. Reconsider the functional $S(\cdot,\cdot,\cdot,\cdot) : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{K}) \times S(\mathbb{K}) \to \mathbb{R}$,

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) := \left(\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i x_i \bar{y}_i\right|. \tag{4.11}$$

The following superadditivity property as an index set mapping holds [2].

Theorem 4.7. *For any $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap J = \emptyset$, one has the inequality*

$$S(\overline{\mathbf{p}}, I \cup J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + S(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq 0. \tag{4.12}$$

Proof. Using the elementary inequality for real numbers

$$\left(a^2+b^2\right)^{\frac{1}{2}} \left(c^2+d^2\right)^{\frac{1}{2}} \geq ac+bd; \quad a,b,c,d \geq 0, \tag{4.13}$$

we have

$$\begin{aligned} S(\overline{\mathbf{p}}, I \cup J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) &= \left(\sum_{i\in I} p_i |x_i|^2 + \sum_{j\in J} p_j |x_j|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2 + \sum_{j\in J} p_j |y_j|^2\right)^{\frac{1}{2}} \\ &\quad - \left|\sum_{i\in I} p_i x_i \bar{y}_i + \sum_{j\in J} p_j x_j \bar{y}_j\right| \\ &\geq \left(\sum_{i\in I} p_i |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in J} p_j |x_j|^2\right)^{\frac{1}{2}} \left(\sum_{j\in J} p_j |y_j|^2\right)^{\frac{1}{2}} \\ &\quad - \left|\sum_{i\in I} p_i x_i \bar{y}_i\right| - \left|\sum_{j\in J} p_j x_j \bar{y}_j\right| \\ &= S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + S(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \end{aligned}$$

and the inequality (4.12) is proved. □

The following corollary concerning the monotonicity of $S(\overline{\mathbf{p}}, \cdot, \overline{\mathbf{x}}, \overline{\mathbf{y}})$ as an index set mapping also holds [2, p. 16].

Corollary 4.8. *For any $I, J \in \mathcal{P}_f(\mathbb{N})$ with $I \supseteq J \neq \emptyset$, one has*

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq 0. \tag{4.14}$$

Proof. Using Theorem 4.7, we may write

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) = S(\overline{\mathbf{p}}, (I \backslash J) \cup J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}}, I \backslash J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + S(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}})$$

giving

$$S(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) - S(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq S(\overline{\mathbf{p}}, I \backslash J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq 0$$

which proves the desired inequality (4.14). □

Remark 4.9. The following inequalities follow by the above results [2, p. 17].

(1) Let $p_i \geq 0$ $(i \in \{1, \ldots, 2n\})$ and $x_i, y_i \in \mathbb{K}$ $(i \in \{1, \ldots, 2n\})$. Then we have the inequality

$$\begin{aligned} &\left(\sum_{i=1}^{2n} p_i |x_i|^2 \sum_{i=1}^{2n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{2n} p_i x_i \bar{y}_i\right| \\ &\geq \left(\sum_{i=1}^{n} p_{2i} |x_{2i}|^2 \sum_{i=1}^{n} p_{2i} |y_{2i}|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_{2i} x_i \bar{y}_i\right| \\ &\quad + \left(\sum_{i=1}^{n} p_{2i-1} |x_{2i-1}|^2 \sum_{i=1}^{n} p_{2i-1} |y_{2i-1}|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_{2i-1} x_{2i-1} \bar{y}_{2i-1}\right| \\ &\geq 0. \end{aligned} \tag{4.15}$$

(2) We have the bound

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i x_i \bar{y}_i\right| \\ &= \sup_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} \left[\left(\sum_{i \in I} p_i |x_i|^2 \sum_{i \in I} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i \in I} p_i x_i \bar{y}_i\right|\right] \geq 0. \end{aligned} \tag{4.16}$$

(3) Define the sequence

$$S_n := \left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} p_i x_i \bar{y}_i\right| \geq 0 \tag{4.17}$$

where $\overline{\mathbf{p}} = (p_i)_{i \in \mathbb{N}} \in S_+(\mathbb{R})$, $\overline{\mathbf{x}} = (x_i)_{i \in \mathbb{N}}$, $\overline{\mathbf{y}} = (y_i)_{i \in \mathbb{N}} \in S(\mathbb{K})$. Then S_n is *monotontic nondecreasing* and we have the following *lower bound*

$$\begin{aligned} S_n \geq &\max_{1 \leq i,j \leq n} \Big\{ \left(p_i |x_i|^2 + p_j |x_j|^2\right)^{\frac{1}{2}} \left(p_i |y_i|^2 + p_j |y_j|^2\right)^{\frac{1}{2}} \\ &\qquad - |p_i x_i \bar{y}_i + p_j x_j \bar{y}_j| \Big\} \\ \geq &\ 0. \end{aligned} \tag{4.18}$$

4.4. **Strong Superadditivity in Terms of Weights.** With the notations in Section 4.2, define the mapping

$$\bar{S}(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) := \sum_{i \in I} p_i |x_i|^2 \sum_{i \in I} p_i |y_i|^2 - \left|\sum_{i \in I} p_i x_i \bar{y}_i\right|^2, \tag{4.19}$$

where $\overline{\mathbf{p}} \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in S(\mathbb{K})$.

Denote also by $\|\cdot\|_{\ell,H}$ the weighted Euclidean norm

$$\|\mathbf{\overline{x}}\|_{\ell,H} := \left(\sum_{i\in H}\ell_i |x_i|^2\right)^{\frac{1}{2}}, \quad \ell \in S_+(\mathbb{R}),\ H\in\mathcal{P}_f(\mathbb{N}). \tag{4.20}$$

The following strong superadditivity property in terms of weights holds [2, p. 18].

Theorem 4.10. *For any* $\overline{\mathbf{p}},\overline{\mathbf{q}}\in S_+(\mathbb{R}),\ I\in\mathcal{P}_f(\mathbb{N})$ *and* $\overline{\mathbf{x}},\overline{\mathbf{y}}\in S(\mathbb{K})$ *we have*

$$\begin{aligned}\bar{S}(\overline{\mathbf{p}}+\overline{\mathbf{q}},I,\overline{\mathbf{x}},\overline{\mathbf{y}})-\bar{S}(\overline{\mathbf{p}},I,\overline{\mathbf{x}},\overline{\mathbf{y}})-\bar{S}(\overline{\mathbf{q}},I,\overline{\mathbf{x}},\overline{\mathbf{y}})\\ \geq\left(\det\begin{bmatrix}\|\overline{\mathbf{x}}\|_{\overline{\mathbf{p}},I} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{p}},I}\\ \|\overline{\mathbf{x}}\|_{\overline{\mathbf{q}},I} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{q}},I}\end{bmatrix}\right)^2\geq 0.\end{aligned}\tag{4.21}$$

Proof. We have

$$\begin{aligned}\bar{S}(\overline{\mathbf{p}}+\overline{\mathbf{q}},I,\overline{\mathbf{x}},\overline{\mathbf{y}}) &= \left(\sum_{i\in I}p_i|x_i|^2+\sum_{i\in I}q_i|x_i|^2\right)\left(\sum_{i\in I}p_i|y_i|^2+\sum_{i\in I}q_i|y_i|^2\right)\\ &\quad-\left|\sum_{i\in I}p_ix_i\bar{y}_i+\sum_{i\in I}q_ix_i\bar{y}_i\right|^2\\ &\geq \sum_{i\in I}p_i|x_i|^2\sum_{i\in I}p_i|y_i|^2+\sum_{i\in I}q_i|x_i|^2\sum_{i\in I}q_i|y_i|^2\\ &\quad-\left(\left|\sum_{i\in I}p_ix_i\bar{y}_i\right|+\left|\sum_{i\in I}q_ix_i\bar{y}_i\right|\right)^2\\ &= \bar{S}(\overline{\mathbf{p}},I,\overline{\mathbf{x}},\overline{\mathbf{y}})+\bar{S}(\overline{\mathbf{q}},I,\overline{\mathbf{x}},\overline{\mathbf{y}})+\sum_{i\in I}p_i|x_i|^2\sum_{i\in I}q_i|y_i|^2\\ &\quad+\sum_{i\in I}q_i|x_i|^2\sum_{i\in I}p_i|y_i|^2-2\left|\sum_{i\in I}p_ix_i\bar{y}_i\right|\left|\sum_{i\in I}q_ix_i\bar{y}_i\right|.\end{aligned}\tag{4.22}$$

By (CBS) $-$inequality, we have

$$\left|\sum_{i\in I}p_ix_i\bar{y}_i\right|\left|\sum_{i\in I}q_ix_i\bar{y}_i\right|\leq\left[\sum_{i\in I}p_i|x_i|^2\sum_{i\in I}p_i|y_i|^2\sum_{i\in I}q_i|x_i|^2\sum_{i\in I}q_i|y_i|^2\right]^{\frac{1}{2}}$$

and thus

$$\begin{aligned}&\sum_{i\in I}p_i|x_i|^2\sum_{i\in I}q_i|y_i|^2+\sum_{i\in I}q_i|x_i|^2\sum_{i\in I}p_i|y_i|^2\\ &\quad-2\left|\sum_{i\in I}p_ix_i\bar{y}_i\right|\left|\sum_{i\in I}q_ix_i\bar{y}_i\right|\geq\left[\left(\sum_{i\in I}p_i|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I}q_i|y_i|^2\right)^{\frac{1}{2}}\right.\\ &\qquad\left.-\left(\sum_{i\in I}q_i|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in I}p_i|y_i|^2\right)^{\frac{1}{2}}\right]^2.\end{aligned}\tag{4.23}$$

Utilising (4.22) and (4.23) we deduce the desired inequality (4.21). □

The following corollary concerning a strong monotonicity result also holds [2, p. 18].

Corollary 4.11. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$ *with* $\overline{\mathbf{p}} \geq \overline{\mathbf{q}}$ *one has the inequality:*

$$\bar{S}(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) - \bar{S}(\overline{\mathbf{q}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq \left(\det \begin{bmatrix} \|\overline{\mathbf{x}}\|_{\overline{\mathbf{q}},I} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{q}},I} \\ \|\overline{\mathbf{x}}\|_{\overline{\mathbf{p}}-\overline{\mathbf{q}},I} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{p}}-\overline{\mathbf{q}},I} \end{bmatrix} \right)^2 \geq 0. \tag{4.24}$$

Remark 4.12. The following refinement of the (CBS) −inequality is a natural consequence of (4.21) [2, p. 19]

$$\begin{aligned} \sum_{i\in I}|x_i|^2\sum_{i\in I}|y_i|^2 - \left|\sum_{i\in I}x_i\bar{y}_i\right|^2 &\geq \sum_{i\in I}|x_i|^2\sin^2\alpha_i\sum_{i\in I}|y_i|^2\sin^2\alpha_i - \left|\sum_{i\in I}x_i\bar{y}_i\sin^2\alpha_i\right|^2 \\ &\quad + \sum_{i\in I}|x_i|^2\cos^2\alpha_i\sum_{i\in I}|y_i|^2\cos^2\alpha_i - \left|\sum_{i\in I}x_i\bar{y}_i\cos^2\alpha_i\right|^2 \\ &\quad + \left(\det \begin{bmatrix} \left(\sum_{i\in I}|x_i|^2\sin^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I}|y_i|^2\sin^2\alpha_i\right)^{\frac{1}{2}} \\ \left(\sum_{i\in I}|x_i|^2\cos^2\alpha_i\right)^{\frac{1}{2}} & \left(\sum_{i\in I}|y_i|^2\cos^2\alpha_i\right)^{\frac{1}{2}} \end{bmatrix} \right)^2 \geq 0. \end{aligned} \tag{4.25}$$

where $\alpha_i \in \mathbb{R}$, $i \in I$.

4.5. **Strong Superadditivity as an Index Set Mapping.**

We assume that we are under the hypothesis and notations in Section 4.2. Reconsider the functional $\bar{S}(\cdot,\cdot,\cdot,\cdot) : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{K}) \times S(\mathbb{K}) \to \mathbb{R}$,

$$\bar{S}(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) := \sum_{i\in I}p_i|x_i|^2\sum_{i\in I}p_i|y_i|^2 - \left|\sum_{i\in I}p_ix_i\bar{y}_i\right|^2. \tag{4.26}$$

The following strong supperadditivity property as an index set mapping holds [2, p. 18].

Theorem 4.13. *For any* $\overline{\mathbf{p}} \in S_+(\mathbb{R})$, $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *with* $I \cap J = \emptyset$ *and* $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in S(\mathbb{K})$, *we have*

$$\begin{aligned} \bar{S}(\overline{\mathbf{p}}, I\cup J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) - \bar{S}(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) - \bar{S}(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \\ \geq \left(\det \begin{bmatrix} \|\overline{\mathbf{x}}\|_{\overline{\mathbf{p}},I} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{p}},I} \\ \|\overline{\mathbf{x}}\|_{\overline{\mathbf{p}},J} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{p}},J} \end{bmatrix} \right)^2 \geq 0. \end{aligned} \tag{4.27}$$

Proof. We have

$$\begin{aligned}
(4.28)\qquad & \bar{S}(\overline{\mathbf{p}}, I \cup J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \\
&= \left(\sum_{i\in I} p_i |x_i|^2 + \sum_{j\in J} p_j |x_j|^2\right)\left(\sum_{i\in I} p_i |y_i|^2 + \sum_{j\in J} p_j |y_j|^2\right) \\
&\quad - \left|\sum_{i\in I} p_i x_i \bar{y}_i + \sum_{j\in I} p_j x_j \bar{y}_j\right|^2 \\
&\geq \sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 + \sum_{j\in J} p_j |x_j|^2 \sum_{j\in J} p_j |y_j|^2 \\
&\quad + \sum_{i\in I} p_i |x_i|^2 \sum_{j\in J} p_j |y_j|^2 + \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 \\
&\quad - \left(\left|\sum_{i\in I} p_i x_i \bar{y}_i\right| + \left|\sum_{j\in I} p_j x_j \bar{y}_j\right|\right)^2 \\
&= \bar{S}(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + \bar{S}(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) + \sum_{i\in I} p_i |x_i|^2 \sum_{j\in J} p_j |y_j|^2 \\
&\quad + \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 - 2\left|\sum_{i\in I} p_i x_i \bar{y}_i\right|\left|\sum_{j\in I} p_j x_j \bar{y}_j\right|.
\end{aligned}$$

By the (CBS) $-$inequality, we have

$$\left|\sum_{i\in I} p_i x_i \bar{y}_i\right|\left|\sum_{j\in I} p_j x_j \bar{y}_j\right| \leq \left[\sum_{i\in I} p_i |x_i|^2 \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 \sum_{j\in J} p_j |y_j|^2\right]^{\frac{1}{2}}$$

and thus

$$\begin{aligned}
(4.29)\qquad & \sum_{i\in I} p_i |x_i|^2 \sum_{j\in J} p_j |y_j|^2 + \sum_{i\in I} p_i |y_i|^2 \sum_{j\in J} p_j |x_j|^2 \\
&\quad - 2\left|\sum_{i\in I} p_i x_i \bar{y}_i\right|\left|\sum_{j\in I} p_j x_j \bar{y}_j\right| \geq \Bigg[\left(\sum_{i\in I} p_i |x_i|^2\right)^{\frac{1}{2}}\left(\sum_{j\in J} p_j |y_j|^2\right)^{\frac{1}{2}} \\
&\qquad\qquad - \left(\sum_{i\in I} p_i |y_i|^2\right)^{\frac{1}{2}}\left(\sum_{j\in J} p_j |x_j|^2\right)^{\frac{1}{2}}\Bigg]^2.
\end{aligned}$$

If we use now (4.28) and (4.29), we may deduce the desired inequality (4.27). □

The following corollary concerning strong monotonicity also holds [2, p. 18].

Corollary 4.14. *For any* $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *with* $I \supseteq J$ *one has the inequality*

$$(4.30)\qquad \bar{S}(\overline{\mathbf{p}}, I, \overline{\mathbf{x}}, \overline{\mathbf{y}}) - \bar{S}(\overline{\mathbf{p}}, J, \overline{\mathbf{x}}, \overline{\mathbf{y}}) \geq \left(\det\begin{bmatrix} \|\overline{\mathbf{x}}\|_{\overline{\mathbf{p}},J} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{p}},J} \\ \|\overline{\mathbf{x}}\|_{\overline{\mathbf{p}},I\setminus J} & \|\overline{\mathbf{y}}\|_{\overline{\mathbf{p}},I\setminus J}\end{bmatrix}\right)^2 \geq 0.$$

Remark 4.15. The following refinement of the (CBS) $-$inequality is a natural consequence of (4.27) [2, p. 19].

Suppose $p_i \geq 0,\ i \in \{1, \dots, 2n\}$ and $x_i, y_i \in \mathbb{K}$, $i \in \{1, \dots, 2n\}$. Then we have the inequality

$$\sum_{i=1}^{2n} p_i |x_i|^2 \sum_{i=1}^{2n} p_i |y_i|^2 - \left| \sum_{i=1}^{2n} p_i x_i \bar{y}_i \right|^2 \geq \sum_{i=1}^{n} p_{2i} |x_{2i}|^2 \sum_{i=1}^{n} p_{2i} |y_{2i}|^2 - \left| \sum_{i=1}^{n} p_{2i} x_{2i} \bar{y}_{2i} \right|^2 + \sum_{i=1}^{n} p_{2i-1} |x_{2i-1}|^2 \sum_{i=1}^{n} p_{2i-1} |y_{2i-1}|^2 - \left| \sum_{i=1}^{n} p_{2i-1} x_{2i-1} \bar{y}_{2i-1} \right|^2 + \left(\det \begin{bmatrix} \left(\sum_{i=1}^{n} p_{2i} |x_{2i}|^2 \right)^{\frac{1}{2}} & \left(\sum_{i=1}^{n} p_{2i} |y_{2i}|^2 \right)^{\frac{1}{2}} \\ \left(\sum_{i=1}^{n} p_{2i-1} |x_{2i-1}|^2 \right)^{\frac{1}{2}} & \left(\sum_{i=1}^{n} p_{2i-1} |y_{2i-1}|^2 \right)^{\frac{1}{2}} \end{bmatrix} \right)^2 \geq 0. \tag{4.31}$$

4.6. **Another Superadditivity Property.** Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers, $S(\mathbb{R})$ the linear space of real sequences and $S_+(\mathbb{R})$ the family of nonnegative real sequences.

Consider the mapping $C : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{R}) \times S(\mathbb{R}) \to \mathbb{R}$

$$C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) := \sum_{i \in I} p_i a_i^2 \sum_{i \in I} p_i b_i^2 - \left(\sum_{i \in I} p_i a_i b_i \right)^2. \tag{4.32}$$

The following identity holds [3, p. 115].

Lemma 4.16. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$ *one has*

$$C\left(\overline{\mathbf{p}} + \overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) = C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + C\left(\overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + \sum_{(i,j) \in I \times I} p_i q_j (a_i b_j - a_j b_i)^2. \tag{4.33}$$

Proof. Using the well-known Lagrange's identity, we have

$$C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) = \frac{1}{2} \sum_{(i,j) \in I \times I} p_i p_j (a_i b_j - a_j b_i)^2. \tag{4.34}$$

Thus

$$\begin{aligned} & C\left(\overline{\mathbf{p}} + \overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \\ &= \frac{1}{2} \sum_{(i,j) \in I \times I} (p_i + q_i)(p_j + q_j)(a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{(i,j) \in I \times I} p_i p_j (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{(i,j) \in I \times I} q_i q_j (a_i b_j - a_j b_i)^2 \\ &\quad + \frac{1}{2} \sum_{(i,j) \in I \times I} p_i q_j (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{(i,j) \in I \times I} p_j q_i (a_i b_j - a_j b_i)^2 \\ &= C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + C\left(\overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + \sum_{(i,j) \in I \times I} p_i q_j (a_i b_j - a_j b_i)^2 \end{aligned}$$

since, by symmetry,

$$\sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 = \sum_{(i,j)\in I\times I} p_j q_i (a_i b_j - a_j b_i)^2 .$$

□

Consider the following mapping:

$$D\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) := \left[C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\right]^{\frac{1}{2}} = \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2\right]^{\frac{1}{2}} .$$

The following result has been obtained in [4, p. 88] as a particular case of a more general result holding in inner product spaces.

Theorem 4.17. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$, $I \in \mathcal{P}_f(\mathbb{N})$ *and* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$, *we have the superadditive property*

$$D\left(\overline{\mathbf{p}} + \overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \geq D\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + D\left(\overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \geq 0. \tag{4.35}$$

Proof. We will give here an elementary proof following the one in [3, p. 116 – p. 117].

By Lemma 4.16, we obviously have

$$D^2\left(\overline{\mathbf{p}} + \overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) = D^2\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + D^2\left(\overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) + \sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 . \tag{4.36}$$

We claim that

$$\sum_{(i,j)\in I\times I} p_i q_j (a_i b_j - a_j b_i)^2 \geq 2D\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) D\left(\overline{\mathbf{q}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right) . \tag{4.37}$$

Taking the square in both sides of (4.37), we must prove that

$$\begin{aligned} &\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} q_i b_i^2 + \sum_{i\in I} q_i a_i^2 \sum_{i\in I} p_i b_i^2 - 2\sum_{i\in I} p_i a_i b_i \sum_{i\in I} q_i a_i b_i\right]^2 \\ &\qquad \geq 4\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i\right)^2\right] \\ &\qquad\qquad \times \left[\sum_{i\in I} q_i a_i^2 \sum_{i\in I} q_i b_i^2 - \left(\sum_{i\in I} q_i a_i b_i\right)^2\right] . \end{aligned} \tag{4.38}$$

Let us denote

$$a := \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}}, \quad x := \left(\sum_{i\in I} q_i a_i^2\right)^{\frac{1}{2}}, \quad b := \left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}},$$

$$y := \left(\sum_{i\in I} q_i b_i^2\right)^{\frac{1}{2}}, \quad c := \sum_{i\in I} p_i a_i b_i, \quad z := \sum_{i\in I} q_i a_i b_i.$$

With these notations (4.38) may be written in the following form

$$\left(a^2 y^2 + b^2 x^2 - 2cz\right)^2 \geq 4\left(a^2 b^2 - c^2\right)\left(x^2 y^2 - z^2\right) . \tag{4.39}$$

Using the elementary inequality

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \leq (mp - nq)^2 , \quad m, n, p, q \in \mathbb{R}$$

we may state that

$$4\left(abxy-cz\right)^2 \geq 4\left(a^2b^2-c^2\right)\left(x^2y^2-z^2\right) \geq 0. \tag{4.40}$$

Since, by the (CBS) $-$inequality, we observe that $abxy \geq |cz| \geq |cz|$, we can state that

$$a^2y^2+b^2x^2-2cz \geq 2\left(abxy-cz\right) \geq 0$$

giving

$$\left(a^2y^2+b^2x^2-2cz\right)^2 \geq 4\left(abxy-cz\right)^2. \tag{4.41}$$

Utilizing (4.40) and (4.41) we deduce the inequality (4.39), and (4.37) is proved.
Finally, by (4.36) and (4.37) we have

$$D^2\left(\overline{\mathbf{p}}+\overline{\mathbf{q}},I,\overline{\mathbf{a}},\overline{\mathbf{b}}\right) \geq \left[D\left(\overline{\mathbf{p}},I,\overline{\mathbf{a}},\overline{\mathbf{b}}\right)+D\left(\overline{\mathbf{q}},I,\overline{\mathbf{a}},\overline{\mathbf{b}}\right)\right]^2,$$

i.e., the superadditivity property (4.35). □

Remark 4.18. The following refinement of the $(CBS)-$ inequality holds [4, p. 89]

$$\left[\sum_{i\in I}a_i^2\sum_{i\in I}b_i^2-\left(\sum_{i\in I}a_ib_i\right)^2\right]^{\frac{1}{2}} \tag{4.42}$$
$$\geq \left[\sum_{i\in I}a_i^2\sin^2\alpha_i\sum_{i\in I}b_i^2\sin^2\alpha_i-\left(\sum_{i\in I}a_ib_i\sin^2\alpha_i\right)^2\right]^{\frac{1}{2}}$$
$$+\left[\sum_{i\in I}a_i^2\cos^2\alpha_i\sum_{i\in I}b_i^2\cos^2\alpha_i-\left(\sum_{i\in I}a_ib_i\cos^2\alpha_i\right)^2\right]^{\frac{1}{2}} \geq 0$$

for any $\alpha_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$.

4.7. **The Case of Index Set Mapping.** Assume that we are under the hypothesis and notations in Section 4.6. Reconsider the functional $C : S_+(\mathbb{R}) \times \mathcal{P}_f(\mathbb{N}) \times S(\mathbb{R}) \times S(\mathbb{R}) \to \mathbb{R}$ given by

$$C\left(\overline{\mathbf{p}},I,\overline{\mathbf{a}},\overline{\mathbf{b}}\right) := \sum_{i\in I}p_ia_i^2\sum_{i\in I}p_ib_i^2-\left(\sum_{i\in I}p_ia_ib_i\right)^2. \tag{4.43}$$

The following identity holds.

Lemma 4.19. *For any $I,J \in \mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ with $I\cap J \neq \emptyset$ one has the identity:*

$$C\left(\overline{\mathbf{p}},I\cup J,\overline{\mathbf{a}},\overline{\mathbf{b}}\right) = C\left(\overline{\mathbf{p}},I,\overline{\mathbf{a}},\overline{\mathbf{b}}\right)+C\left(\overline{\mathbf{p}},J,\overline{\mathbf{a}},\overline{\mathbf{b}}\right)+\sum_{(i,j)\in I\times J}p_ip_j\left(a_ib_j-a_jb_i\right)^2. \tag{4.44}$$

Proof. Using Lagrange's identity [5, p. 84], we may state

$$C\left(\overline{\mathbf{p}},K,\overline{\mathbf{a}},\overline{\mathbf{b}}\right) = \frac{1}{2}\sum_{(i,j)\in K\times K}p_ip_j\left(a_ib_j-a_jb_i\right)^2, \quad K \in \mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}. \tag{4.45}$$

Thus

$$\begin{aligned}
&C\left(\overline{\mathbf{p}}, I\cup J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\\
&=\frac{1}{2}\sum_{(i,j)\in(I\cup J)\times(I\cup J)} p_i p_j\left(a_i b_j - a_j b_i\right)^2\\
&=\frac{1}{2}\sum_{(i,j)\in I\times I} p_i p_j\left(a_i b_j - a_j b_i\right)^2+\frac{1}{2}\sum_{(i,j)\in I\times J} p_i p_j\left(a_i b_j - a_j b_i\right)^2\\
&\quad+\frac{1}{2}\sum_{(i,j)\in J\times I} p_i p_j\left(a_i b_j - a_j b_i\right)^2+\frac{1}{2}\sum_{(i,j)\in J\times J} p_i p_j\left(a_i b_j - a_j b_i\right)^2\\
&=C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)+C\left(\overline{\mathbf{p}}, J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)+\sum_{(i,j)\in I\times J} p_i p_j\left(a_i b_j - a_j b_i\right)^2
\end{aligned}$$

since, by symmetry,

$$\sum_{(i,j)\in I\times J} p_i p_j\left(a_i b_j - a_j b_i\right)^2=\sum_{(i,j)\in J\times I} p_i p_j\left(a_i b_j - a_j b_i\right)^2.$$

□

Now, if we consider the mapping

$$D\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right):=\left[C\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\right]^{\frac{1}{2}}=\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2-\left(\sum_{i\in I} p_i a_i b_i\right)^2\right]^{\frac{1}{2}},$$

then the following superadditivity property as an index set mapping holds:

Theorem 4.20. *For any $I, J \in \mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ with $I\cap J\neq\emptyset$ one has*

$$D\left(\overline{\mathbf{p}}, I\cup J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\geq D\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)+D\left(\overline{\mathbf{p}}, J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)\geq 0. \tag{4.46}$$

Proof. By Lemma 4.19, we have

$$D^2\left(\overline{\mathbf{p}}, I\cup J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)=D^2\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)+D^2\left(\overline{\mathbf{p}}, J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)+\sum_{(i,j)\in I\times J} p_i p_j\left(a_i b_j - a_j b_i\right)^2 \tag{4.47}$$

To prove (4.46) it is sufficient to show that

$$\sum_{(i,j)\in I\times J} p_i p_j\left(a_i b_j - a_j b_i\right)^2\geq 2D\left(\overline{\mathbf{p}}, I, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right)D\left(\overline{\mathbf{p}}, J, \overline{\mathbf{a}}, \overline{\mathbf{b}}\right). \tag{4.48}$$

Taking the square in (4.48), we must demonstrate that

$$\begin{aligned}
&\left[\sum_{i\in I} p_i a_i^2 \sum_{j\in J} p_j b_j^2+\sum_{j\in J} p_j a_j^2 \sum_{i\in I} p_i b_i^2-2\sum_{i\in I} p_i a_i b_i \sum_{j\in J} p_j a_j b_j\right]^2\\
&\qquad\geq 4\left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2-\left(\sum_{i\in I} p_i a_i b_i\right)^2\right]\\
&\qquad\qquad\times\left[\sum_{j\in J} p_j a_j^2 \sum_{j\in J} p_j b_j^2-\left(\sum_{j\in J} p_j a_j b_j\right)^2\right].
\end{aligned}$$

If we denote

$$a := \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}}, \quad x := \left(\sum_{j\in J} p_j a_j^2\right)^{\frac{1}{2}}, \quad b := \left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}},$$

$$y := \left(\sum_{j\in J} p_j b_j^2\right)^{\frac{1}{2}}, \quad c := \sum_{i\in I} p_i a_i b_i, \quad z := \sum_{j\in J} p_j a_j b_j,$$

then we need to prove

$$\left(a^2y^2 + b^2x^2 - 2cz\right)^2 \geq 4\left(a^2b^2 - c^2\right)\left(x^2y^2 - z^2\right), \tag{4.49}$$

which has been shown in Section 4.6.

This completes the proof. □

Remark 4.21. The following refinement of the (CBS) −inequality holds

$$\left[\sum_{i=1}^{2n} p_i a_i^2 \sum_{i=1}^{2n} p_i b_i^2 - \left(\sum_{i=1}^{2n} p_i a_i b_i\right)^2\right]^{\frac{1}{2}} \geq \left[\sum_{i=1}^{n} p_{2i} a_{2i}^2 \sum_{i=1}^{n} p_{2i} b_{2i}^2 - \left(\sum_{i=1}^{n} p_{2i} a_{2i} b_{2i}\right)^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} p_{2i-1} a_{2i-1}^2 \sum_{i=1}^{n} p_{2i-1} b_{2i-1}^2 - \left(\sum_{i=1}^{n} p_{2i-1} a_{2i-1} b_{2i-1}\right)^2\right]^{\frac{1}{2}} \geq 0.$$

4.8. **Supermultiplicity in Terms of Weights.** Denote by $S_+(\mathbb{R})$ the set of nonnegative sequences. Assume that $A : S_+(\mathbb{R}) \to \mathbb{R}$ is *additive* on $S_+(\mathbb{R})$, i.e.,

$$A(\overline{\mathbf{p}} + \overline{\mathbf{q}}) = A(\overline{\mathbf{p}}) + A(\overline{\mathbf{q}}), \quad \overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R}) \tag{4.50}$$

and $L : S_+(\mathbb{R}) \to \mathbb{R}$ is *superadditive* on $S_+(\mathbb{R})$, i.e.,

$$L(\overline{\mathbf{p}} + \overline{\mathbf{q}}) \geq L(\overline{\mathbf{p}}) + L(\overline{\mathbf{q}}), \quad \overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R}). \tag{4.51}$$

Define the following associated functionals

$$F(\overline{\mathbf{p}}) := \frac{L(\overline{\mathbf{p}})}{A(\overline{\mathbf{p}})} \text{ and } H(\overline{\mathbf{p}}) := [F(\overline{\mathbf{p}})]^{A(\overline{\mathbf{p}})}. \tag{4.52}$$

The following result holds [3, Theorem 2.1].

Lemma 4.22. *With the above assumptions, we have*

$$H(\overline{\mathbf{p}} + \overline{\mathbf{q}}) \geq H(\overline{\mathbf{p}}) H(\overline{\mathbf{q}}); \tag{4.53}$$

for any $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$, *i.e.,* $H(\cdot)$ *is* ***supermultiplicative*** *on* $S_+(\mathbb{R})$.

Proof. We shall follow the proof in [3].

Using the well-known arithmetic mean-geometric mean inequality for real numbers

$$\frac{\alpha x + \beta y}{\alpha + \beta} \geq x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}} \tag{4.54}$$

for any $x, y \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, we have successively

$$
\begin{aligned}
F(\overline{\mathbf{p}}+\overline{\mathbf{q}}) &= \frac{L(\overline{\mathbf{p}}+\overline{\mathbf{q}})}{A(\overline{\mathbf{p}}+\overline{\mathbf{q}})} \\
&= \frac{L(\overline{\mathbf{p}}+\overline{\mathbf{q}})}{A(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})} \\
&\geq \frac{L(\overline{\mathbf{p}})+L(\overline{\mathbf{q}})}{A(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})} \\
&= \frac{A(\overline{\mathbf{p}})\frac{L(\overline{\mathbf{p}})}{A(\overline{\mathbf{p}})}+A(\overline{\mathbf{q}})\frac{L(\overline{\mathbf{q}})}{A(\overline{\mathbf{q}})}}{A(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})} \\
&= \frac{A(\overline{\mathbf{p}})F(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})F(\overline{\mathbf{q}})}{A(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})} \\
&\geq [F(\overline{\mathbf{p}})]^{\frac{A(\overline{\mathbf{p}})}{A(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})}} \cdot [F(\overline{\mathbf{q}})]^{\frac{A(\overline{\mathbf{q}})}{A(\overline{\mathbf{p}})+A(\overline{\mathbf{q}})}}
\end{aligned}
\tag{4.55}
$$

for all $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$. However, $A(\overline{\mathbf{p}}) + A(\overline{\mathbf{q}}) = A(\overline{\mathbf{p}}+\overline{\mathbf{q}})$, and thus (4.55) implies the desired inequality (4.53). □

We are now able to point out the following inequality related to the $(CBS)-$ inequality. The first result is incorporated in the following theorem [3, p. 115].

Theorem 4.23. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$, *and* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$, *one has the inequality*

$$
\begin{aligned}
&\left\{\frac{1}{P_I+Q_I}\left[\sum_{i\in I}(p_i+q_i)a_i^2\sum_{i\in I}(p_i+q_i)b_i^2 - \left(\sum_{i\in I}(p_i+q_i)a_ib_i\right)^2\right]\right\}^{P_I+Q_I} \\
&\qquad \geq \left\{\frac{1}{P_I}\left[\sum_{i\in I}p_ia_i^2\sum_{i\in I}p_ib_i^2 - \left(\sum_{i\in I}p_ia_ib_i\right)^2\right]\right\}^{P_I} \\
&\qquad\qquad \times \left\{\frac{1}{Q_I}\left[\sum_{i\in I}q_ia_i^2\sum_{i\in I}q_ib_i^2 - \left(\sum_{i\in I}q_ia_ib_i\right)^2\right]\right\}^{Q_I} > 0,
\end{aligned}
\tag{4.56}
$$

where $P_I := \sum_{i\in I} p_i > 0$, $Q_I := \sum_{i\in I} q_i > 0$.

Proof. Consider the functionals

$$
A(\overline{\mathbf{p}}) := \sum_{i\in I} p_i = P_I;
$$

$$
C(\overline{\mathbf{p}}) := \sum_{i\in I}p_ia_i^2\sum_{i\in I}p_ib_i^2 - \left(\sum_{i\in I}p_ia_ib_i\right)^2.
$$

Then $A(\cdot)$ is additive and $C(\cdot)$ is superadditive (see for example Lemma 4.16) on $S_+(\mathbb{R})$. Applying Lemma 4.22 we deduce the desired inequality (4.56). □

The following refinement of the $(CBS)-$inequality holds.

Corollary 4.24. *For any* $\overline{\mathbf{a}}, \overline{\mathbf{b}}, \overline{\alpha} \in S(\mathbb{R})$, *one has the inequality*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \geq \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n} \sin^2 \alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n} \sin^2 \alpha_i} \left(\frac{1}{n}\sum_{i=1}^{n} \cos^2 \alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n} \cos^2 \alpha_i}} \times \left[\sum_{i=1}^{n} a_i^2 \sin^2 \alpha_i \sum_{i=1}^{n} b_i^2 \sin^2 \alpha_i - \left(\sum_{i=1}^{n} a_i b_i \sin^2 \alpha_i\right)^2\right]^{\frac{1}{n}\sum_{i=1}^{n} \sin^2 \alpha_i} \times \left[\sum_{i=1}^{n} a_i^2 \cos^2 \alpha_i \sum_{i=1}^{n} b_i^2 \cos^2 \alpha_i - \left(\sum_{i=1}^{n} a_i b_i \cos^2 \alpha_i\right)^2\right]^{\frac{1}{n}\sum_{i=1}^{n} \cos^2 \alpha_i} \geq 0. \tag{4.57}$$

The following result holds [3, p. 116].

Theorem 4.25. *For any* $\overline{\mathbf{p}}, \overline{\mathbf{q}} \in S_+(\mathbb{R})$, *and* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$, *one has the inequality*

$$\left\{\left[\frac{1}{P_I + Q_I}\sum_{i\in I}(p_i + q_i) a_i^2\right]^{\frac{1}{2}} \left[\frac{1}{P_I + Q_I}\sum_{i\in I}(p_i + q_i) b_i^2\right]^{\frac{1}{2}} - \left|\frac{1}{P_I + Q_I}\sum_{i\in I}(p_i + q_i) a_i b_i\right|\right\}^{P_I + Q_I} \geq \left[\left(\frac{1}{P_I}\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}} \left(\frac{1}{P_I}\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\frac{1}{P_I}\sum_{i\in I} p_i a_i b_i\right|\right]^{P_I} \times \left[\left(\frac{1}{Q_I}\sum_{i\in I} q_i a_i^2\right)^{\frac{1}{2}} \left(\frac{1}{Q_I}\sum_{i\in I} q_i b_i^2\right)^{\frac{1}{2}} - \left|\frac{1}{Q_I}\sum_{i\in I} q_i a_i b_i\right|\right]^{Q_I} \geq 0. \tag{4.58}$$

Proof. Follows by Lemma 4.22 on taking into account that the functional

$$B(\overline{\mathbf{p}}) := \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right|$$

is superadditive on $S_+(\mathbb{R})$ (see Section 4.2). □

The following refinement of the (CBS) −inequality holds.

Corollary 4.26. *For any* $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\alpha} \in S(\mathbb{R})$, *one has the inequality*

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i\right| \tag{4.59}$$
$$\geq \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i}} \cdot \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i}}$$
$$\times \left[\left(\sum_{i=1}^{n} a_i^2 \sin^2\alpha_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2 \sin^2\alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i \sin^2\alpha_i\right|\right]^{\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i}$$
$$\times \left[\left(\sum_{i=1}^{n} a_i^2 \cos^2\alpha_i\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2 \cos^2\alpha_i\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n} a_i b_i \cos^2\alpha_i\right|\right]^{\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i} \geq 0.$$

Finally, we may also state [3, p. 117].

Theorem 4.27. *For any* $\bar{\mathbf{p}}, \bar{\mathbf{q}} \in S_+(\mathbb{R})$, *and* $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in S(\mathbb{R})$, *one has the inequality*

$$\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)a_i^2 \cdot \frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)b_i^2 - \left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)a_i b_i\right)^2\right]^{\frac{P_I+Q_I}{2}} \tag{4.60}$$
$$\geq \left[\frac{1}{P_I}\sum_{i\in I}p_i a_i^2 \cdot \frac{1}{P_I}\sum_{i\in I}p_i b_i^2 - \left(\frac{1}{P_I}\sum_{i\in I}p_i a_i b_i\right)^2\right]^{\frac{P_I}{2}}$$
$$\times \left[\frac{1}{Q_I}\sum_{i\in I}q_i a_i^2 \cdot \frac{1}{Q_I}\sum_{i\in I}q_i b_i^2 - \left(\frac{1}{Q_I}\sum_{i\in I}q_i a_i b_i\right)^2\right]^{\frac{Q_I}{2}}.$$

Proof. Follows by Lemma 4.22 on taking into account that the functional

$$D(\bar{\mathbf{p}}) := \left[\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i\right)^2\right]^{\frac{1}{2}}$$

is superadditive on $S_+(\mathbb{R})$ (see Section 4.6). □

The following corollary also holds.

Corollary 4.28. *For any* $\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\alpha} \in S(\mathbb{R})$, *one has the inequality*

$$\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2\right]^{\frac{1}{2}} \tag{4.61}$$
$$\geq \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i}} \cdot \frac{1}{\left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\right)^{\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i}}$$
$$\times \left[\sum_{i=1}^{n} a_i^2\sin^2\alpha_i \sum_{i=1}^{n} b_i^2\sin^2\alpha_i - \left(\sum_{i=1}^{n} a_i b_i \sin^2\alpha_i\right)^2\right]^{\frac{1}{2n}\sum_{i=1}^{n}\sin^2\alpha_i}$$
$$\times \left[\sum_{i=1}^{n} a_i^2\cos^2\alpha_i \sum_{i=1}^{n} b_i^2\cos^2\alpha_i - \left(\sum_{i=1}^{n} a_i b_i \cos^2\alpha_i\right)^2\right]^{\frac{1}{2n}\sum_{i=1}^{n}\cos^2\alpha_i} \geq 0.$$

4.9. **Supermultiplicity as an Index Set Mapping.** Denote by $\mathcal{P}_f(\mathbb{N})$ the set of all finite parts of the natural number set $\mathbb{N}$ and assume that $B : \mathcal{P}_f(\mathbb{N}) \to \mathbb{R}$ is *set-additive* on $\mathcal{P}_f(\mathbb{N})$, i.e.,

$$B(I \cup J) = B(I) + B(J) \quad \text{for any } \; I, J \in \mathcal{P}_f(\mathbb{N}), \quad I \cap J \neq \emptyset, \tag{4.62}$$

and $G : \mathcal{P}_f(\mathbb{N}) \to \mathbb{R}$ is *set-superadditive* on $\mathcal{P}_f(\mathbb{N})$, i.e.,

$$G(I \cup J) \geq G(I) + G(J) \quad \text{for any } \; I, J \in \mathcal{P}_f(\mathbb{N}), \quad I \cap J \neq \emptyset. \tag{4.63}$$

We may define the following associated functionals

$$M(I) := \frac{G(I)}{B(I)} \;\text{ and }\; N(I) := [M(I)]^{A(I)}. \tag{4.64}$$

With these notations we may prove the following lemma that is interesting in itself as well.

Lemma 4.29. *Under the above assumptions one has*

$$N(I \cup J) \geq N(I)\,N(J) \tag{4.65}$$

for any $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *with* $I \cap J \neq \emptyset$, *i.e.,* $N(\cdot)$ *is set-supermultiplicative on* $\mathcal{P}_f(\mathbb{N})$.

Proof. Using the arithmetic mean – geometric mean inequality

$$\frac{\alpha x + \beta y}{\alpha + \beta} \geq x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}} \tag{4.66}$$

for any $x, y \geq 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, we have successively for $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap J \neq \emptyset$ that

$$
\begin{aligned}
M(I \cup J) &= \frac{G(I \cup J)}{B(I \cup J)} \\
&= \frac{G(I \cup J)}{B(I) + B(J)} \\
&\geq \frac{G(I) + G(J)}{B(I) + B(J)} \\
&= \frac{B(I)\frac{G(I)}{B(I)} + B(J)\frac{G(J)}{B(J)}}{B(I) + B(J)} \\
&= \frac{B(I) M(I) + B(J) M(J)}{B(I) + B(J)} \\
&\geq (M(I))^{\frac{B(I)}{B(I)+B(J)}} \cdot (M(J))^{\frac{B(J)}{B(I)+B(J)}} .
\end{aligned} \tag{4.67}
$$

Since $B(I) + B(J) = B(I \cup J)$, we deduce by (4.67) the desired inequality (4.65). □

Now, we are able to point out some set-superadditivity properties for some functionals associates to the $(CBS)-$inequality.

The first result is embodied in the following theorem.

Theorem 4.30. *If $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in S(\mathbb{R})$, $\bar{\mathbf{p}} \in S_+(\mathbb{R})$ and $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ so that $I \cap J \neq \emptyset$, then one has the inequality*

$$
\begin{aligned}
&\left\{ \frac{1}{P_{I\cup J}} \left[\sum_{k\in I\cup J} p_k a_k^2 \sum_{k\in I\cup J} p_k b_k^2 - \left(\sum_{k\in I\cup J} p_k a_k b_k \right)^2 \right] \right\}^{P_{I\cup J}} \\
&\qquad \geq \left\{ \frac{1}{P_I} \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2 \right] \right\}^{P_I} \\
&\qquad\qquad \times \left\{ \frac{1}{P_J} \left[\sum_{j\in J} p_j a_j^2 \sum_{j\in J} p_j b_j^2 - \left(\sum_{j\in J} p_j a_j b_j \right)^2 \right] \right\}^{P_J},
\end{aligned} \tag{4.68}
$$

when $P_J := \sum_{j\in J} p_j$.

Proof. Consider the functionals

$$
B(I) := \sum_{i\in I} p_i;
$$

$$
G(I) := \sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2 .
$$

The functional $B(\cdot)$ is obviously *set-additive* and (see Section 4.7) the functional $G(\cdot)$ is *set-superadditive*. Applying Lemma 4.29 we then deduce the desired inequality (4.68). □

The following corollary is a natural application.

Corollary 4.31. *If* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$ *and* $\overline{\mathbf{p}} \in S_+(\mathbb{R})$, *then for any* $n \geq 1$ *one has the inequality*

$$\left\{\frac{1}{P_{2n}}\left[\sum_{i=1}^{2n} p_i a_i^2 \sum_{i=1}^{2n} p_i b_i^2 - \left(\sum_{i=1}^{2n} p_i a_i b_i\right)^2\right]\right\}^{P_{2n}} \tag{4.69}$$
$$\geq \left\{\frac{1}{\sum_{i=1}^{n} p_{2i}}\left[\sum_{i=1}^{n} p_{2i} a_{2i}^2 \sum_{i=1}^{n} p_{2i} b_{2i}^2 - \left(\sum_{i=1}^{n} p_{2i} a_{2i} b_{2i}\right)^2\right]\right\}^{\sum_{i=1}^{n} p_{2i}}$$
$$\times \left\{\frac{1}{\sum_{i=1}^{n} p_{2i-1}}\left[\sum_{i=1}^{n} p_{2i-1} a_{2i-1}^2 \sum_{i=1}^{n} p_{2i-1} b_{2i-1}^2 - \left(\sum_{i=1}^{n} p_{2i-1} a_{2i-1} b_{2i-1}\right)^2\right]\right\}^{\sum_{i=1}^{n} p_{2i-1}}.$$

The following result also holds.

Theorem 4.32. *If* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$, $\overline{\mathbf{p}} \in S_+(\mathbb{R})$ *and* $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *so that* $I \cap J \neq \emptyset$, *then one has the inequality*

$$\left\{\left[\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J} p_k a_k^2\right]^{\frac{1}{2}}\left[\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J} p_k b_k^2\right]^{\frac{1}{2}} - \left|\frac{1}{P_{I\cup J}}\sum_{k\in I\cup J} p_k a_k b_k\right|\right\}^{P_{I\cup J}} \tag{4.70}$$
$$\geq \left\{\left(\frac{1}{P_I}\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}}\left(\frac{1}{P_I}\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\frac{1}{P_I}\sum_{i\in I} p_i a_i b_i\right|\right\}^{P_I}$$
$$\times \left\{\left(\frac{1}{P_J}\sum_{j\in J} p_j a_j^2\right)^{\frac{1}{2}}\left(\frac{1}{P_J}\sum_{j\in J} p_j b_j^2\right)^{\frac{1}{2}} - \left|\frac{1}{P_J}\sum_{j\in J} p_j a_j b_j\right|\right\}^{P_J}.$$

Proof. Follows by Lemma 4.29 on taking into account that the functional

$$G(I) := \left(\sum_{i\in I} p_i a_i^2\right)^{\frac{1}{2}}\left(\sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right|$$

is *set-superadditive* on $\mathcal{P}_f(\mathbb{N})$. □

The following corollary is a natural application.

Corollary 4.33. *If* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$ *and* $\overline{\mathbf{p}} \in S_+(\mathbb{R})$, *then for any* $n \geq 1$ *one has the inequality*

$$\left[\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i a_i^2\right)^{\frac{1}{2}}\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i b_i^2\right)^{\frac{1}{2}} - \left|\frac{1}{P_{2n}}\sum_{i=1}^{2n} p_i a_i b_i\right|\right]^{P_{2n}} \tag{4.71}$$

$$\geq \left[\left(\frac{1}{\sum_{i=1}^{n} p_{2i}} \sum_{i=1}^{n} p_{2i} a_{2i}^{2} \right)^{\frac{1}{2}} \left(\frac{1}{\sum_{i=1}^{n} p_{2i}} \sum_{i=1}^{n} p_{2i} b_{2i}^{2} \right)^{\frac{1}{2}} - \left| \frac{1}{\sum_{i=1}^{n} p_{2i}} \sum_{i=1}^{n} p_{2i} a_{2i} b_{2i} \right| \right]^{\sum_{i=1}^{n} p_{2i}}$$
$$\times \left[\left(\frac{1}{\sum_{i=1}^{n} p_{2i-1}} \sum_{i=1}^{n} p_{2i-1} a_{2i-1}^{2} \right)^{\frac{1}{2}} \left(\frac{1}{\sum_{i=1}^{n} p_{2i-1}} \sum_{i=1}^{n} p_{2i-1} b_{2i-1}^{2} \right)^{\frac{1}{2}} - \left| \frac{1}{\sum_{i=1}^{n} p_{2i-1}} \sum_{i=1}^{n} p_{2i-1} a_{2i-1} b_{2i-1} \right| \right]^{\sum_{i=1}^{n} p_{2i-1}}.$$

Finally, we may also state:

Theorem 4.34. *If* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$, $\overline{\mathbf{p}} \in S_{+}(\mathbb{R})$ *and* $I, J \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *so that* $I \cap J \neq \emptyset$, *then one has the inequality*

$$\left[\frac{1}{P_{I\cup J}} \sum_{k\in I\cup J} p_k a_k^2 \cdot \frac{1}{P_{I\cup J}} \sum_{k\in I\cup J} p_k b_k^2 - \left(\frac{1}{P_{I\cup J}} \sum_{k\in I\cup J} p_k a_k b_k \right)^2 \right]^{\frac{P_{I\cup J}}{2}} \tag{4.72}$$
$$\geq \left[\frac{1}{P_I} \sum_{i\in I} p_i a_i^2 \cdot \frac{1}{P_I} \sum_{i\in I} p_i b_i^2 - \left(\frac{1}{P_I} \sum_{i\in I} p_i a_i b_i \right)^2 \right]^{\frac{P_I}{2}}$$
$$\times \left[\frac{1}{P_J} \sum_{j\in J} p_j a_j^2 \cdot \frac{1}{P_J} \sum_{j\in J} p_j b_j^2 - \left(\frac{1}{P_J} \sum_{j\in J} p_j a_j b_j \right)^2 \right]^{\frac{P_J}{2}}.$$

Proof. Follows by Lemma 4.29 on taking into account that the functional

$$Q(I) := \left[\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2 - \left(\sum_{i\in I} p_i a_i b_i \right)^2 \right]^{\frac{1}{2}}$$

is *set-superadditive* on $\mathcal{P}_f(\mathbb{N})$ (see Section 4.7). □

The following corollary holds as well.

Corollary 4.35. *If* $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in S(\mathbb{R})$ *and* $\overline{\mathbf{p}} \in S_{+}(\mathbb{R})$, *then for any* $n \geq 1$ *one has the inequality*

$$\left[\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i a_i^2 \cdot \frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i b_i^2 - \left(\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i a_i b_i \right)^2 \right]^{\frac{P_{2n}}{2}} \tag{4.73}$$
$$\geq \left[\left(\frac{1}{\sum_{i=1}^{n} p_{2i}} \sum_{i=1}^{n} p_{2i} a_{2i}^{2} \right) \left(\frac{1}{\sum_{i=1}^{n} p_{2i}} \sum_{i=1}^{n} p_{2i} b_{2i}^{2} \right) - \left(\frac{1}{\sum_{i=1}^{n} p_{2i}} \sum_{i=1}^{n} p_{2i} a_{2i} b_{2i} \right)^2 \right]^{\frac{1}{2}\sum_{i=1}^{n} p_{2i}}$$
$$\times \left[\left(\frac{1}{\sum_{i=1}^{n} p_{2i-1}} \sum_{i=1}^{n} p_{2i-1} a_{2i-1}^{2} \right) \left(\frac{1}{\sum_{i=1}^{n} p_{2i-1}} \sum_{i=1}^{n} p_{2i-1} b_{2i-1}^{2} \right) - \left(\frac{1}{\sum_{i=1}^{n} p_{2i-1}} \sum_{i=1}^{n} p_{2i-1} a_{2i-1} b_{2i-1} \right)^2 \right]^{\frac{1}{2}\sum_{i=1}^{n} p_{2i-1}}.$$

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5. REVERSE INEQUALITIES

5.1. **The Cassels' Inequality.** The following result was proved by J.W.S. Cassels in 1951 (see Appendix 1 of [2] or Appendix of [3]):

Theorem 5.1. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be sequences of positive real numbers and* $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *a sequence of nonnegative real numbers. Suppose that*

$$m = \min_{i=\overline{1,n}} \left\{ \frac{a_i}{b_i} \right\} \quad \text{and} \quad M = \max_{i=\overline{1,n}} \left\{ \frac{a_i}{b_i} \right\}. \tag{5.1}$$

Then one has the inequality

$$\frac{\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2}{\left(\sum_{i=1}^n w_i a_i b_i\right)^2} \le \frac{(m+M)^2}{4mM}. \tag{5.2}$$

The equality holds in (5.2) when $w_1 = \frac{1}{a_1 b_1}$, $w_n = \frac{1}{a_n b_n}$, $w_2 = \dots = w_{n-1} = 0$, $m = \frac{a_n}{b_1}$ *and* $M = \frac{a_1}{b_n}$.

Proof. 1. The original proof by Cassels (1951) is of interest. We shall follow the paper [5] in sketching this proof.

We begin with the assertion that

$$\frac{(1+kw)(1+k^{-1}w)}{(1+w)^2} \le \frac{(1+k)(1+k^{-1})}{4}, \quad k > 0,\ w \ge 0 \tag{5.3}$$

which, being an equivalent form of (5.2) for $n = 2$, shows that it holds for $n = 2$.

To prove that the maximum of (5.2) is obtained when we have more than two w_i's being nonzero, Cassels then notes that if for example, $w_1, w_2, w_3 \neq 0$ lead to an extremum M of $\frac{XY}{2^2}$, then we would have the linear equations

$$a_n^2 X + b_n^2 Y - 2M a_n b_n Z = 0, \quad k = 1, 2, 3.$$

Nontrivial solutions exist if and only if the three vectors $[a_n^2, b_n^2, a_n b_n]$ are linearly dependent. But this will be so only if, for some $i \neq j$ $(i, j = 1, 2, 3)$ $a_i = \gamma a_j$, $b_i = \gamma b_j$. And if that were true, we could, for example, drop the a_i, b_i terms and so deal with the same problem with one less variable. If only one $w_i \neq 0$, then $M = 1$, the lower bound. So we need only examine all pairs $w_i \neq 0$, $w_j \neq 0$. The result (5.2) then quickly follows.

2. We will now use the *barycentric method* of Frucht [1] and Watson [4]. We will follow the paper [5].

We substitute $w_i = \frac{u_i}{b_i^2}$ in the left hand side of (5.2), which may then be expressed as the ratio

$$\frac{N}{D^2}$$

where

$$N = \sum_{i=1}^n \left(\frac{a_i}{b_i}\right)^2 u_i \ \text{ and } \ D = \sum_{i=1}^n \left(\frac{a_i}{b_i}\right) u_i,$$

assuming without loss of generality, that $\sum_{i=1}^n a_i = 1$. But the point with co-ordinates (D, N) must lie within the convex closure of the n points $\left(\frac{a_i}{b_i}, \frac{a_i^2}{b_i^2}\right)$. The value of $\frac{N}{D^2}$ at points on the parabola is one unit. If $m = \min_{i=\overline{1,n}} \left\{\frac{a_i}{b_i}\right\}$ and $M = \max_{i=\overline{1,n}} \left\{\frac{a_i}{b_i}\right\}$, then the minimum must lie on the chord joining the point (m, m^2) and (M, M^2). Some easy calculus then leads to (5.2). □

The following "unweighted" Cassels' inequality holds.

Corollary 5.2. *If* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *satisfy the assumptions in Theorem 5.1, one has the inequality*

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \le \frac{(m+M)^2}{4mM}. \tag{5.4}$$

The following two additive versions of Cassels inequality hold.

Corollary 5.3. *With the assumptions of Theorem 5.1, one has*

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} w_i a_i b_i \\ &\le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=1}^{n} w_i a_i b_i. \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \\ &\le \frac{(M-m)^2}{4mM} \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2. \end{aligned} \tag{5.6}$$

Proof. Taking the square root in (5.2) we get

$$1 \le \frac{\left(\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}}}{\sum_{i=1}^{n} w_i a_i b_i} \le \frac{M+m}{2\sqrt{mM}}.$$

Subtracting 1 on both sides, a simple calculation will lead to (5.5).

The second inequality follows by (5.2) on subtracting 1 and appropriate computation. □

The following additive version of unweighted Cassels inequality also holds.

Corollary 5.4. *With the assumption of Theorem 5.1 for* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *one has the inequalities*

$$0 \le \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} a_i b_i \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=1}^{n} a_i b_i \tag{5.7}$$

and

$$0 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{(M-m)^2}{4mM} \left(\sum_{i=1}^{n} a_i b_i\right)^2. \tag{5.8}$$

5.2. **The Pólya-Szegö Inequality.** The following inequality was proved in 1925 by Pólya and Szegö [6, pp. 57, 213 – 214], [7, pp. 71– 72, 253 – 255].

Theorem 5.5. *Let* $\bar{\mathrm{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathrm{b}} = (b_1, \dots, b_n)$ *be two sequences of positive real numbers. If*

$$0 < a \le a_i \le A < \infty, \quad 0 < b \le b_i \le B < \infty \ \textit{ for each } i \in \{1, \dots, n\}, \tag{5.9}$$

then one has the inequality

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \le \frac{(ab+AB)^2}{4abAB}. \tag{5.10}$$

The equality holds in (5.10) if and only if

$$p = n \cdot \frac{A}{a} \Big/ \left(\frac{A}{a} + \frac{B}{b}\right) \quad \textit{and} \quad q = n \cdot \frac{B}{b} \Big/ \left(\frac{A}{a} + \frac{B}{b}\right)$$

are integers and if p of the numbers $a_1, \dots, a_n$ are equal to a and q of these numbers are equal to A, and if the corresponding numbers b_i are equal to B and b respectively.

Proof. Following [5], we shall present here the original proof of Pólya and Szegö.

We may, without loss of generality, suppose that $a_1 \geq \cdots \geq a_n$, then to maximise the left-hand side of (5.10) we must have that the critical b_i's be reversely ordered (for if $b_k > b_m$ with $k < m$, then we can interchange b_k and b_m such that $b_k^2 + b_m^2 = b_m^2 + b_k^2$ and $a_k b_k + a_m b_m \geq a_k b_m + a_m b_k$), i.e., that $b_1 \leq \cdots \leq b_n$.

Pólya and Szegö then continue by defining nonnegative numbers u_i and v_i for $i = 1, \dots, n-1$ and $n > 2$ such that

$$a_i^2 = u_i a_1^2 + v_i a_n^2 \text{ and } b_i^2 = u_i b_1^2 + v_i b_n^2. \tag{5.11}$$

Since $a_i b_i > u_i a_1 b_1 + v_i a_n b_n$ the left hand side of (5.10),

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \leq \frac{\left(U a_1^2 + V a_n^2\right)\left(U b_1^2 + V b_n^2\right)}{\left(U a_1 b_1 + V a_n b_n\right)^2},$$

where $U = \sum_{i=1}^n u_i$ and $V = \sum_{i=1}^n v_i$.

This reduces the problem to that with $n = 2$, which is solvable by elementary methods, leading to

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \leq \frac{\left(a_1 b_1 + a_n b_n\right)^2}{4 a_1 a_n b_1 b_n}, \tag{5.12}$$

where, since the a_i's and b_i's here are reversely ordered,

$$a_1 = \max_{i=\overline{1,n}} \{a_i\}, \quad a_n = \min_{i=\overline{1,n}} \{a_i\}, \quad b_1 = \min_{i=\overline{1,n}} \{b_i\}, \quad b_n = \max_{i=\overline{1,n}} \{b_i\}. \tag{5.13}$$

If we now assume, as in (5.9), that

$$0 < a \leq a_i \leq A, \;\; 0 < b \leq b_i \leq B, \quad i = (1, \dots, n),$$

then

$$\frac{\left(a_1 b_1 + a_n b_n\right)^2}{4 a_1 a_n b_1 b_n} \leq \frac{(ab + AB)^2}{4abAB}$$

(because $\frac{(k+1)^2}{4k} \leq \frac{(\alpha+1)^2}{4\alpha}$ for $k \leq \alpha$), and the inequality (5.10) is proved. □

Remark 5.6. The inequality (5.10) may also be obtained from the "unweighted" Cassels' inequality

$$\frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i b_i\right)^2} \leq \frac{(m+M)^2}{4mM}, \tag{5.14}$$

where $0 < m \leq \frac{a_i}{b_i} \leq M$ for each $i \in \{1, \dots, n\}$.

The following additive versions of the Pólya-Szegö inequality also hold.

Corollary 5.7. *With the assumptions in Theorem 5.5, one has the inequality*

$$0 \leq \left(\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^n a_i b_i \leq \frac{\left(\sqrt{AB} - \sqrt{ab}\right)^2}{2\sqrt{abAB}} \sum_{i=1}^n a_i b_i \tag{5.15}$$

and

$$0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{(AB - ab)^2}{4abAB} \left(\sum_{i=1}^{n} a_i b_i\right)^2 . \tag{5.16}$$

5.3. **The Greub-Rheinboldt Inequality.** The following weighted version of the Pólya-Szegö inequality was obtained by Greub and Rheinboldt in 1959, [6].

Theorem 5.8. *Let* $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *be two sequences of positive real numbers and* $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ *a sequence of nonnegative real numbers. Suppose that*

$$0 < a \leq a_i \leq A < \infty, \quad 0 < b \leq b_i \leq B < \infty \quad (i = 1, \ldots, n) . \tag{5.17}$$

Then one has the inequality

$$\frac{\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2}{\left(\sum_{i=1}^{n} w_i a_i b_i\right)^2} \leq \frac{(ab + AB)^2}{4abAB}. \tag{5.18}$$

Equality holds in (5.18) when $w_i = \frac{1}{a_1 b_1}$, $w_n = \frac{1}{a_n b_n}$, $w_2 = \cdots = w_{n-1} = 0$, $m = \frac{a_n}{b_1}$, $M = \frac{a_1}{b_n}$ *with* $a_1 = A$, $a_n = a$, $b_1 = b$ *and* $b_n = b$.

Remark 5.9. This inequality follows by Cassels' result which states that

$$\frac{\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2}{\left(\sum_{i=1}^{n} w_i a_i b_i\right)^2} \leq \frac{(m + M)^2}{4mM}, \tag{5.19}$$

provided $0 < m \leq \frac{a_i}{b_i} \leq M < \infty$ for each $i \in \{1, \ldots, n\}$.

The following additive versions of Greub-Rheinboldt also hold.

Corollary 5.10. *With the assumptions in Theorem 5.8, one has the inequalities*

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} w_i a_i b_i \\ &\leq \frac{\left(\sqrt{AB} - \sqrt{ab}\right)^2}{2\sqrt{abAB}} \sum_{i=1}^{n} w_i a_i b_i \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \\ &\leq \frac{(AB - ab)^2}{4abAB} \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 . \end{aligned} \tag{5.21}$$

5.4. **A Cassels' Type Inequality for Complex Numbers.** The following reverse inequality for the (CBS) $-$inequality holds [9].

Theorem 5.11. *Let* $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *such that* $\operatorname{Re}(\bar{a}A) > 0$.

If $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ *are sequences of complex numbers and* $\bar{\mathbf{w}} = (w_1, \ldots, w_n)$ *is a sequence of nonnegative real numbers with the property that*

$$\sum_{i=1}^{n} w_i \operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0, \tag{5.22}$$

then one has the inequality

$$\left[\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2\right]^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right]}{\left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{\left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}}} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|. \tag{5.23}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller one.

Proof. We have, obviously, that

$$\begin{aligned}\Gamma &:= \sum_{i=1}^{n} w_i \operatorname{Re}\left[\left(Ay_i - x_i\right)\left(\bar{x}_i - \bar{a}\bar{y}_i\right)\right] \\ &= \sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right] - \sum_{i=1}^{n} w_i |x_i|^2 - \operatorname{Re}\left(\bar{a}A\right) \sum_{i=1}^{n} w_i |y_i|^2\end{aligned}$$

and then, by (5.22), one has

$$\sum_{i=1}^{n} w_i |x_i|^2 + \operatorname{Re}\left(\bar{a}A\right) \sum_{i=1}^{n} w_i |y_i|^2 \leq \sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right]$$

giving

$$\frac{1}{\left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}}} \sum_{i=1}^{n} w_i |x_i|^2 + \left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}} \sum_{i=1}^{n} w_i |y_i|^2 \leq \frac{\sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right]}{\left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}}}. \tag{5.24}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq$$

holding for any $p, q \geq 0$ and $\alpha > 0$, we deduce

$$2\left(\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2\right)^{\frac{1}{2}} \leq \frac{1}{\left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}}} \sum_{i=1}^{n} w_i |x_i|^2 + \left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}} \sum_{i=1}^{n} w_i |y_i|^2. \tag{5.25}$$

Utilising (5.24) and (5.25), we deduce the first part of (5.23).

The second part is obvious by the fact that for $z \in \mathbb{C}$, $|\operatorname{Re}(z)| \leq |z|$.

Now, assume that the first inequality in (5.23) holds with a constant $c > 0$, i.e.,

$$\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 \leq c \cdot \frac{\sum_{i=1}^{n} w_i \operatorname{Re}\left[A\bar{x}_i y_i + \bar{a} x_i \bar{y}_i\right]}{\left[\operatorname{Re}\left(\bar{a}A\right)\right]^{\frac{1}{2}}}, \tag{5.26}$$

where $a, A, \bar{\mathbf{x}}, \bar{\mathbf{y}}$ satisfy (5.22).

If we choose $a = A = 1$, $y = x \neq 0$, then obviously (5.23) holds and from (5.26) we may get

$$\sum_{i=1}^{n} w_i |x_i|^2 \leq 2c \sum_{i=1}^{n} w_i |x_i|^2,$$

giving $c \geq \frac{1}{2}$.

The theorem is completely proved. □

The following corollary is a natural consequence of the above theorem.

Corollary 5.12. *Let $m, M > 0$ and $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{w}}$ be as in Theorem 5.11 and with the property that*

$$\sum_{i=1}^{n} w_i \operatorname{Re}\left[\left(My_i - x_i\right)\left(\bar{x}_i - m\bar{y}_i\right)\right] \geq 0, \tag{5.27}$$

then one has the inequality

$$\begin{aligned} \left[\sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2\right]^{\frac{1}{2}} &\leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right) \\ &\leq \frac{1}{2} \cdot \frac{M+m}{\sqrt{mM}} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|. \end{aligned} \tag{5.28}$$

The following corollary also holds.

Corollary 5.13. *With the assumptions in Corollary 5.12, then one has the following inequality:*

$$\begin{aligned} 0 &\leq \left[\sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2\right]^{\frac{1}{2}} - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right| \\ &\leq \left[\sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2\right]^{\frac{1}{2}} - \sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right) \\ &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right) \\ &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right| \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2 - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 \\ &\leq \sum_{i=1}^{n} w_i \left|x_i\right|^2 \sum_{i=1}^{n} w_i \left|y_i\right|^2 - \left[\sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right)\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left[\sum_{i=1}^{n} w_i \operatorname{Re}\left(x_i \bar{y}_i\right)\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2. \end{aligned} \tag{5.30}$$

5.5. **A Reverse Inequality for Real Numbers.** The following result holds [10, Proposition 5.1].

Theorem 5.14. *Let $a, A \in \mathbb{R}$ and $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ be two sequences with the property that:*

$$ay_i \leq x_i \leq Ay_i \quad \textit{for each} \quad i \in \{1, \ldots, n\}. \tag{5.31}$$

Then for any $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *a sequence of positive real numbers, one has the inequality*

$$0 \le \sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i x_i y_i \right)^2 \le \frac{1}{4} (A - a)^2 \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2 . \tag{5.32}$$

The constant $\frac{1}{4}$ *is sharp in (5.32).*

Proof. Let us define

$$I_1 := \left(A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i \right) \left(\sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2 \right)$$

and

$$I_2 := \left(\sum_{i=1}^{n} w_i y_i^2 \right) \sum_{i=1}^{n} (A y_i - x_i)(x_i - a y_i) w_i .$$

Then

$$I_1 = (a + A) \sum_{i=1}^{n} w_i y_i^2 \sum_{i=1}^{n} w_i x_i y_i - \left(\sum_{i=1}^{n} w_i x_i y_i \right)^2 - aA \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2$$

and

$$I_2 = (a + A) \sum_{i=1}^{n} w_i y_i^2 \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - aA \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2$$

giving

$$I_1 - I_2 = \sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2 . \tag{5.33}$$

If (5.31) holds, then $(A y_i - x_i)(x_i - a y_i) \ge 0$ for each $i \in \{1, \dots, n\}$ and thus $I_2 \ge 0$ giving

$$\sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2 \le \left[\left(A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i \right) \left(\sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2 \right) \right] . \tag{5.34}$$

If we use the elementary inequality for real numbers $u, v \in \mathbb{R}$

$$uv \le \frac{1}{4} (u + v)^2 , \tag{5.35}$$

then we have for

$$u := A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i, \quad v := \sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2$$

that

$$\left(A \sum_{i=1}^{n} w_i y_i^2 - \sum_{i=1}^{n} w_i x_i y_i \right) \left(\sum_{i=1}^{n} w_i x_i y_i - a \sum_{i=1}^{n} w_i y_i^2 \right) \le \frac{1}{4} (A - a)^2 \left(\sum_{i=1}^{n} w_i y_i^2 \right)^2$$

and the inequality (5.32) is proved.

Now, assume that (5.32) holds with a constant $c > 0$, i.e.,

$$\sum_{i=1}^{n} w_i x_i^2 \sum_{i=1}^{n} w_i y_i^2 - \left(\sum_{i=1}^{n} w_i x_i y_i\right)^2 \le c\,(A-a)^2 \left(\sum_{i=1}^{n} w_i y_i^2\right)^2, \tag{5.36}$$

where $a, A, \bar{\mathbf{x}}, \bar{\mathbf{y}}$ satisfy (5.31).

We choose $n = 2$, $w_1 = w_2 = 1$ and let $a, A, y_1, y_2, x, \alpha \in \mathbb{R}$ such that

$$ay_1 < x_1 = Ay_1,$$
$$ay_2 = x_2 < Ay_2.$$

With these choices, we get from (5.36) that

$$\left(a^2 y_1^2 + a^2 y_2^2\right)\left(y_1^2 + y_2^2\right) - \left(A^2 y_1^2 + a^2 y_2^2\right)^2 \le c\,(A-a)^2 \left(y_1^2 + y_2^2\right)^2,$$

which is equivalent to

$$(A-a)^2\, y_1^2 y_2^2 \le c\,(A-a)^2 \left(y_1^2 + y_2^2\right)^2.$$

Since we may choose $a \neq A$, we deduce

$$y_1^2 y_2^2 \le c\left(y_1^2 + y_2^2\right)^2,$$

giving, for $y_1 = y_2 = 1$, $c \ge \frac{1}{4}$. □

The following corollary is obvious.

Corollary 5.15. *With the above assumptions for a, A, $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, we have the inequality*

$$0 \le \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \frac{1}{4}(A-a)^2 \left(\sum_{i=1}^{n} y_i^2\right)^2. \tag{5.37}$$

Remark 5.16. Condition (5.31) may be replaced by the weaker condition

$$\sum_{i=1}^{n} w_i \,(Ay_i - x_i)\,(x_i - ay_i) \ge 0 \tag{5.38}$$

and the conclusion in Theorem 5.14 will still be valid, i.e., the inequality (5.32) holds.

For (5.37) to be true it suffices that

$$\sum_{i=1}^{n} (Ay_i - x_i)\,(x_i - ay_i) \ge 0 \tag{5.39}$$

holds true.

5.6. **A Reverse Inequality for Complex Numbers.** The following result holds [10, Proposition 5.1].

Theorem 5.17. *Let $a, A \in \mathbb{C}$ and $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in \mathbb{C}^n$, $\bar{\mathbf{w}} = (w_1, \dots, w_n) \in \mathbb{R}_+^n$. If*

$$\sum_{i=1}^{n} w_i \operatorname{Re}\left[(Ay_i - x_i)\,(\bar{x}_i - \bar{a}\bar{y}_i)\right] \ge 0, \tag{5.40}$$

then one has the inequality

$$0 \le \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \bar{y}_i \right|^2 \le \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{n} w_i |y_i|^2 \right)^2. \tag{5.41}$$

The constant $\frac{1}{4}$ is sharp in (5.41).

Proof. Consider

$$A_1 := \operatorname{Re}\left[\left(A \sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i \right) \left(\sum_{i=1}^{n} w_i \bar{x}_i y_i - \bar{a} \sum_{i=1}^{n} w_i |y_i|^2 \right) \right]$$

and

$$A_2 := \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}\left[\sum_{i=1}^{n} w_i (A y_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i) \right].$$

Then

$$A_1 = \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}\left[A \sum_{i=1}^{n} w_i \bar{x}_i y_i + \bar{a} \sum_{i=1}^{n} w_i x_i \bar{y}_i \right] - \left| \sum_{i=1}^{n} w_i x_i \bar{y}_i \right|^2 - \operatorname{Re}(\bar{a}A) \left(\sum_{i=1}^{n} w_i |y_i|^2 \right)^2$$

and

$$A_2 = \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}\left[A \sum_{i=1}^{n} w_i \bar{x}_i y_i + \bar{a} \sum_{i=1}^{n} w_i x_i \bar{y}_i \right] - \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \operatorname{Re}(\bar{a}A) \left(\sum_{i=1}^{n} w_i |y_i|^2 \right)^2$$

giving

$$A_1 - A_2 = \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \bar{y}_i \right|^2. \tag{5.42}$$

If (5.40) holds, then $A_2 \ge 0$ and thus

$$\sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \bar{y}_i \right|^2 \le \operatorname{Re}\left[\left(A \sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i \right) \left(\sum_{i=1}^{n} w_i \bar{x}_i y_i - \bar{a} \sum_{i=1}^{n} w_i |y_i|^2 \right) \right]. \tag{5.43}$$

If we use the elementary inequality for complex numbers $z, t \in \mathbb{C}$

$$\operatorname{Re}[z\bar{t}] \le \frac{1}{4} |z-t|^2, \tag{5.44}$$

then we have for

$$z := A\sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i,$$

$$t := \sum_{i=1}^{n} w_i x_i \bar{y}_i - a\sum_{i=1}^{n} w_i |y_i|^2$$

that

$$\text{(5.45)} \quad \operatorname{Re}\left[\left(A\sum_{i=1}^{n} w_i |y_i|^2 - \sum_{i=1}^{n} w_i x_i \bar{y}_i\right)\left(\sum_{i=1}^{n} w_i \bar{x}_i y_i - \bar{a}\sum_{i=1}^{n} w_i |y_i|^2\right)\right] \leq \frac{1}{4}|A-a|^2\left(\sum_{i=1}^{n} w_i |y_i|^2\right)^2$$

and the inequality (5.41) is proved.

Now, assume that (5.41) holds with a constant $c > 0$, i.e.,

$$\text{(5.46)} \qquad \sum_{i=1}^{n} w_i |x_i|^2 \sum_{i=1}^{n} w_i |y_i|^2 - \left|\sum_{i=1}^{n} w_i x_i \bar{y}_i\right|^2 \leq c\,|A-a|^2\left(\sum_{i=1}^{n} w_i |y_i|^2\right)^2,$$

where $\bar{\mathbf{x}}, \bar{\mathbf{y}}, a, A$ satisfy (5.40).

Consider $\bar{\mathbf{y}} \in \mathbb{C}^n$, $\sum_{i=1}^{n} |y_i|^2 w_i = 1$, $a \neq A$, $\bar{\mathbf{m}} \in \mathbb{C}^n$, $\sum_{i=1}^{n} w_i |m_i|^2 = 1$ with $\sum_{i=1}^{n} w_i y_i m_i = 0$. Define

$$x_i := \frac{A+a}{2} y_i + \frac{A+a}{2} m_i, \quad i \in \{1, \dots, n\}.$$

Then

$$\sum_{i=1}^{n} w_i (Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i) = \left|\frac{A-a}{2}\right|^2 \sum_{i=1}^{n} w_i (y_i - m_i)(\bar{y}_i - \bar{m}_i) = 0$$

and thus the condition (5.40) is fulfilled.

From (5.46) we deduce

$$\sum_{i=1}^{n}\left|\frac{A+a}{2} y_i + \frac{A-a}{2} m_i\right|^2 w_i - \left|\sum_{i=1}^{n}\left(\frac{A+a}{2} y_i + \frac{A-a}{2} m_i\right)\bar{y}_i w_i\right|^2 \leq c\,|A-a|^2$$

and since

$$\sum_{i=1}^{n} w_i \left|\left(\frac{A+a}{2} y_i + \frac{A-a}{2} m_i\right)\right|^2 = \left|\frac{A+a}{2}\right|^2 - \left|\frac{A-a}{2}\right|^2$$

and

$$\left|\sum_{i=1}^{n}\left(\frac{A+a}{2} y_i + \frac{A-a}{2} m_i\right)\bar{y}_i w_i\right|^2 = \left|\frac{A+a}{2}\right|^2$$

then by (5.46) we get

$$\frac{|A-a|^2}{4} \leq c\,|A-a|^2$$

giving $c \geq \frac{1}{4}$ and the theorem is completely proved. □

The following corollary holds.

Corollary 5.18. *Let $a, A \in \mathbb{C}$ and $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{y}} = (y_1, \dots, y_n) \in \mathbb{C}^n$ be with the property that*

$$\sum_{i=1}^{n} \operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0, \tag{5.47}$$

then one has the inequality

$$0 \leq \sum_{i=1}^{n} |x_i|^2 \sum_{i=1}^{n} |y_i|^2 - \left|\sum_{i=1}^{n} x_i \bar{y}_i\right|^2 \leq \frac{1}{4} |A - a|^2 \left(\sum_{i=1}^{n} |y_i|^2\right)^2. \tag{5.48}$$

The constant $\frac{1}{4}$ is best in (5.48).

Remark 5.19. A sufficient condition for both (5.40) and (5.47) to hold is

$$\operatorname{Re}\left[(Ay_i - x_i)(\bar{x}_i - \bar{a}\bar{y}_i)\right] \geq 0 \tag{5.49}$$

for any $i \in \{1, \dots, n\}$.

5.7. **Shisha-Mond Type Inequalities.** As some particular case for bounds on differences of means, O. Shisha and B. Mond obtained in 1967 (see [23]) the following reverse of $(CBS)-$inequality:

Theorem 5.20. *Assume that $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are such that there exists $a, A, b, B > 0$ with the property that:*

$$a \leq a_j \leq A \text{ and } b \leq b_j \leq B \text{ for any } j \in \{1, \dots, n\} \tag{5.50}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2. \tag{5.51}$$

The equality holds in (5.51) if and only if there exists a subsequence $(k_1, \dots, k_p)$ of $(1, 2, \dots, n)$ such that

$$\frac{n}{p} = 1 + \left(\frac{A}{a}\right)^{\frac{1}{2}} \left(\frac{B}{b}\right)^{\frac{3}{2}},$$

$a_{k_\mu} = A$, $b_{k_\mu} = b$ $(\mu = 1, \dots, p)$ and $a_k = a$, $b_k = B$ for every k distinct from all k_μ.

Using another result stated for weighted means in [23], we may prove the following reverse of the $(CBS)-$inequality.

Theorem 5.21. *Assume that $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$ are positive sequences and there exists $\gamma, \Gamma > 0$ with the property that*

$$0 < \gamma \leq \frac{a_i}{b_i} \leq \Gamma < \infty \text{ for any } i \in \{1, \dots, n\}. \tag{5.52}$$

Then we have the inequality

$$0 \leq \left(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} - \sum_{i=1}^{n} a_i b_i \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{5.53}$$

The equality holds in (5.53) if and only if there exists a subsequence $(k_1, \dots, k_p)$ of $(1, 2, \dots, n)$ such that

$$\sum_{m=1}^{p} b_{k_m}^2 = \frac{\Gamma + 3\gamma}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2, \quad \frac{a_{k_m}}{b_{k_m}} = \Gamma \quad (m = 1, \dots, p) \text{ and } \frac{a_k}{b_k} = \gamma$$

for every k distinct from all k_m.

Proof. In [23, p. 301], Shisha and Mond have proved the following weighted inequality

$$0 \leq \left(\sum_{j=1}^{n} q_j x_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} q_j x_j \leq \frac{(C-c)^2}{4(c+C)}, \tag{5.54}$$

provided $q_j \geq 0$ $(j = 1, \dots, n)$ with $\sum_{j=1}^{n} q_j = 1$ and $0 < c \leq x_j < C < \infty$ for any $j \in \{1, \dots, n\}$.

Equality holds in (5.54) if and only if there exists a subsequence $(k_1, \dots, k_p)$ of $(1, 2, \dots, n)$ such that

$$\sum_{m=1}^{p} q_{k_m} = \frac{C+3c}{4(c+C)}, \tag{5.55}$$

$x_{k_m} = C$ $(m = 1, 2, \dots, p)$ and $x_k = c$ for every k distinct from all k_m.

If in (5.54) we choose

$$x_j = \frac{a_j}{b_j}, \quad q_j = \frac{b_j^2}{\sum_{k=1}^{n} b_k^2}, \quad j \in \{1, \dots, n\};$$

then we get

$$\left(\frac{\sum_{j=1}^{n} a_j^2}{\sum_{k=1}^{n} b_k^2} \right)^{\frac{1}{2}} - \frac{\sum_{j=1}^{n} a_j b_j}{\sum_{k=1}^{n} b_k^2} \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)},$$

giving the desired inequality (5.53).

The case of equality follows by the similar case in (5.54) and we omit the details. □

5.8. **Zagier Type Inequalities.** The following result was obtained by D. Zagier in 1995, [24].

Lemma 5.22. *Let $f, g : [0, \infty) \to \mathbb{R}$ be monotone decreasing nonnegative functions on $[0, \infty)$. Then*

$$\int_0^\infty f(x) g(x) dx \geq \frac{\int_0^\infty f(x) F(x) dx \int_0^\infty g(x) G(x) dx}{\max \left\{ \int_0^\infty F(x) dx, \int_0^\infty G(x) dx \right\}}, \tag{5.56}$$

for any integrable functions $F, G : [0, \infty) \to [0, 1]$.

Proof. We will follow the proof in [24].

For all $x \geq 0$ we have

$$\begin{aligned} \int_0^\infty f(t) F(t) dt &= f(x) \int_0^\infty F(t) dt + \int_0^\infty [f(t) - f(x)] F(t) dt \\ &\leq f(x) \int_0^\infty F(t) dt + \int_0^x [f(t) - f(x)] dt \end{aligned}$$

and hence, since $\int_0^x G(t) dt$ is bounded from above by both x and $\int_0^\infty G(t) dt$,

$$\begin{aligned} &\int_0^\infty f(t) F(t) dt \cdot \int_0^x G(t) dt \\ &\leq x f(x) \int_0^\infty F(t) dt + \int_0^\infty G(t) dt \cdot \int_0^x [f(t) - f(x)] dt \\ &\leq \max \left\{ \int_0^\infty F(t) dt, \int_0^\infty G(t) dt \right\} \cdot \int_0^x f(t) dt. \end{aligned}$$

Now, we multiply by $-dg(x)$ and integrate by parts from 0 to ∞. The left hand side gives $\int_{-\infty}^{\infty} f(t) F(t) \, dt \cdot \int_{-\infty}^{\infty} g(t) G(t) \, dt$, the right hand side gives

$$\max \left\{ \int_0^{\infty} F(t) \, dt, \int_0^{\infty} G(t) \, dt \right\} \cdot \int_0^{\infty} f(t) g(t) \, dt,$$

and the inequality remains true because the measure $-dg(x)$ is nonnegative. □

The following particular case is a reverse of the (CBS) −integral inequality obtained by D. Zagier in 1977, [25].

Corollary 5.23. *If* $f, g : [0, \infty) \to [0, \infty)$ *are decreasing function on* $[0, \infty)$, *then*

$$\max \left[f(0) \int_0^{\infty} g(t) \, dt, g(0) \int_0^{\infty} f(t) \, dt \right] \cdot \int_0^{\infty} f(t) g(t) \, dt \geq \int_0^{\infty} f^2(t) \, dt \int_0^{\infty} g^2(t) \, dt. \tag{5.57}$$

Remark 5.24. The following weighted version of (5.56) may be proved in a similar way, as noted by D. Zagier in [25]

$$\int_0^{\infty} w(t) f(t) g(t) \, dt \geq \frac{\int_0^{\infty} w(t) f(t) F(t) \, dt \int_0^{\infty} w(t) f(t) G(t) \, dt}{\max \left\{ \int_0^{\infty} w(t) F(t) \, dt, \int_0^{\infty} w(t) G(t) \, dt \right\}}, \tag{5.58}$$

provided $w(t) > 0$ on $[0, \infty)$, $f, g : [0, \infty) \to [0, \infty)$ are monotonic decreasing and $F, G : [0, \infty) \to [0, 1]$ are integrable on $[0, \infty)$.

We may state and prove the following discrete inequality.

Theorem 5.25. *Consider the sequences of real numbers* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$, $\bar{\mathbf{q}} = (q_1, \dots, q_n)$ *and* $\bar{\mathbf{w}} = (w_1, \dots, w_n)$.

If

(i) $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *are decreasing and nonnegative;*

(ii) $p_i, q_i \in [0, 1]$ *and* $w_i \geq 0$ *for any* $i \in \{1, \dots, n\}$,

then we have the inequality

$$\sum_{i=1}^{n} w_i a_i b_i \geq \frac{\sum_{i=1}^{n} w_i p_i a_i \sum_{i=1}^{n} w_i q_i b_i}{\max \left\{ \sum_{i=1}^{n} w_i p_i, \sum_{i=1}^{n} w_i q_i \right\}}. \tag{5.59}$$

Proof. Consider the functions $f, g, F, G, W : [0, \infty) \to \mathbb{R}$ given by

$$f(t) = \begin{cases} a_1, & t \in [0, 1) \\ a_2, & t \in [1, 2) \\ \vdots \\ a_n, & t \in [n-1, n) \\ 0 & t \in [n, \infty) \end{cases}, \qquad g(t) = \begin{cases} b_1, & t \in [0, 1) \\ b_2, & t \in [1, 2) \\ \vdots \\ b_n, & t \in [n-1, n) \\ 0 & t \in [n, \infty) \end{cases},$$

$$F(t) = \begin{cases} p_1, & t \in [0, 1) \\ p_2, & t \in [1, 2) \\ \vdots \\ p_n, & t \in [n-1, n) \\ 0 & t \in [n, \infty) \end{cases}, \qquad G(t) = \begin{cases} q_1, & t \in [0, 1) \\ q_2, & t \in [1, 2) \\ \vdots \\ q_n, & t \in [n-1, n) \\ 0 & t \in [n, \infty) \end{cases},$$

and

$$W(t) = \begin{cases} w_1, & t \in [0,1) \\ w_2, & t \in [1,2) \\ \vdots & \\ w_n, & t \in [n-1,n) \\ 0 & t \in [n,\infty) \end{cases}.$$

We observe that, the above functions satisfy the hypothesis of Remark 5.24 and since, for example,

$$\int_0^\infty w(t) f(t) g(t)\, dt = \sum_{i=1}^n \int_{i-1}^i w(t) f(t) g(t)\, dt + \int_n^\infty w(t) f(t) g(t)\, dt$$
$$= \sum_{k=1}^n w_k a_k b_k,$$

then by (5.58) we deduce the desired inequality (5.59). □

Remark 5.26. A similar inequality for sequences under some monotonicity assumptions for $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ was obtained in 1995 by J. Pečarić in [26].

The following reverse of the (CBS) −discrete inequality holds.

Theorem 5.27. *Assume that* $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$ *are decreasing nonnegative sequences with* $a_1, b_1 \neq 0$ *and* $\bar{\mathbf{w}}$ *a nonnegative sequence. Then*

$$\sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2 \le \max\left\{ b_1 \sum_{i=1}^n w_i a_i, a_1 \sum_{i=1}^n w_i b_i \right\} \sum_{i=1}^n w_i a_i b_i. \tag{5.60}$$

The proof follows by Theorem 5.25 on choosing $p_i = \frac{a_i}{a_1} \in [0,1]$, $q_i = \frac{b_i}{b_1} \in [0,1]$, $i \in \{1,\dots,n\}$. We omit the details.

Remark 5.28. When $w_i = 1$, we recapture Alzer's result from 1992, [27].

5.9. **A Reverse Inequality in Terms of the** $\sup$ −**Norm.** The following result has been proved in [11].

Lemma 5.29. *Let* $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ *and* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *be sequences of complex numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *a sequence of nonnegative real numbers such that* $\sum_{i=1}^n p_i = 1$. *Then one has the inequality*

$$\left| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right| \le \max_{i=\overline{1,n-1}} |\Delta\alpha_i| \max_{i=\overline{1,n-1}} |\Delta x_i| \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i \right)^2 \right], \tag{5.61}$$

where $\Delta\alpha_i$ *is the forward difference, i.e.,* $\Delta\alpha_i := \alpha_{i+1} - \alpha_i$.

Inequality (5.61) is sharp in the sense that the constant $C = 1$ *in the right membership cannot be replaced be a smaller one.*

Proof. We shall follow the proof in [11]. We start with the following identity

$$\sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \left(\alpha_i - \alpha_j\right)\left(x_i - x_j\right)$$
$$= \sum_{1 \le i < j \le n} p_i p_j \left(\alpha_i - \alpha_j\right)\left(x_i - x_j\right).$$

As $i < j$, we can write that

$$\alpha_j - \alpha_i = \sum_{k=i}^{j-1} \Delta \alpha_k$$

and

$$x_j - x_i = \sum_{k=i}^{j-1} \Delta x_k.$$

Using the generalised triangle inequality, we have successively

$$\left|\sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i\right| = \left|\sum_{1 \le i < j \le n} p_i p_j \sum_{k=i}^{j-1} \Delta \alpha_k \sum_{k=i}^{j-1} \Delta x_k\right|$$
$$\le \sum_{1 \le i < j \le n} p_i p_j \left|\sum_{k=i}^{j-1} \Delta \alpha_k\right| \left|\sum_{k=i}^{j-1} \Delta x_k\right|$$
$$\le \sum_{1 \le i < j \le n} p_i p_j \sum_{k=i}^{j-1} |\Delta \alpha_k| \sum_{k=i}^{j-1} |\Delta x_k|$$
$$:= A.$$

Note that

$$|\Delta \alpha_k| \le \max_{1 \le s \le n-1} |\Delta \alpha_s|$$

and

$$|\Delta x_k| \le \max_{1 \le s \le n-1} |\Delta x_s|$$

for all $k = i, \dots, j-1$ and then by summation

$$\sum_{k=i}^{j-1} |\Delta \alpha_k| \le (j-i) \max_{1 \le s \le n-1} |\Delta \alpha_s|$$

and

$$\sum_{k=i}^{j-1} |\Delta x_k| \le (j-i) \max_{1 \le s \le n-1} |\Delta x_s|.$$

Taking into account the above estimations, we can write

$$A \le \left[\sum_{1 \le i < j \le n} p_i p_j (j-i)^2\right] \max_{1 \le s \le n-1} |\Delta \alpha_s| \max_{1 \le s \le n-1} |\Delta x_s|.$$

As a simple calculation shows that

$$\sum_{1 \le i < j \le n} p_i p_j (j-i)^2 = \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2,$$

inequality (5.61) is proved.

To prove the sharpness of the constant, let us assume that (5.61) holds with a constant $C > 0$, i.e.,

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq C \max_{i=\overline{1,n-1}} |\Delta\alpha_i| \max_{i=\overline{1,n-1}} |\Delta x_i| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]. \tag{5.62}$$

Now, choose the sequences $\alpha_k = \alpha + k\beta \ \ (\beta \neq 0)$ and $x_k = x + ky \ \ (y \neq 0)$, $k \in \{1, \dots, n\}$ to get

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| = \frac{1}{2}\left|\sum_{i,j=1}^{n} p_i p_j (i-j)\beta y\right| = |\beta|\,|y| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]$$

and

$$\max_{i=\overline{1,n-1}} |\Delta\alpha_i| \max_{i=\overline{1,n-1}} |\Delta x_i| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right] = |\beta|\,|y| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right]$$

and then, by (5.62), we get $C \geq 1$. □

The following reverse of the (CBS) −inequality holds [12].

Theorem 5.30. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0$, $(i = 1, \dots, n)$. *Then one has the inequality*

$$0 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \max_{k=\overline{1,n-1}} \left\{\Delta\left(\frac{b_k}{a_k}\right)\right\}^2 \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} i^2 a_i^2 - \left(\sum_{i=1}^{n} i a_i^2\right)^2\right].$$

The constant $C = 1$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. Follows by Lemma 5.29 on choosing

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad \alpha_i = \frac{b_i}{a_i}, \quad x_i = \frac{b_i}{a_i}, \quad i \in \{1, \dots, n\}$$

and performing some elementary calculations.

We omit the details. □

5.10. A Reverse Inequality in Terms of the $1-$Norm.

The following result has been obtained in [13].

Lemma 5.31. *Let* $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ *and* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *be sequences of complex numbers and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *a sequence of nonnegative real numbers such that* $\sum_{i=1}^{n} p_i = 1$. *Then one* *has the inequality*

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq \frac{1}{2}\sum_{i=1}^{n} p_i\left(1-p_i\right)\sum_{i=1}^{n-1}\left|\Delta\alpha_i\right|\sum_{i=1}^{n-1}\left|\Delta x_i\right|, \tag{5.63}$$

where $\Delta\alpha_i := \alpha_{i+1} - \alpha_i$ *is the forward difference.*

The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. We shall follow the proof in [13].

As in the proof of Lemma 5.29 in Section 5.9, we have

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq \sum_{1\leq i<j\leq n} p_i p_j \sum_{k=i}^{j-1}\left|\Delta\alpha_k\right|\sum_{l=i}^{j-1}\left|\Delta x_l\right| := A. \tag{5.64}$$

It is obvious that for all $1 \leq i < j \leq n-1$, we have that

$$\sum_{k=i}^{j-1}\left|\Delta\alpha_k\right| \leq \sum_{k=1}^{n-1}\left|\Delta\alpha_k\right|$$

and

$$\sum_{l=i}^{j-1}\left|\Delta x_l\right| \leq \sum_{l=1}^{n-1}\left|\Delta x_l\right|.$$

Utilising these and the definition of A, we conclude that

$$A \leq \sum_{k=1}^{n-1}\left|\Delta\alpha_k\right|\sum_{l=1}^{n-1}\left|\Delta x_l\right|\sum_{1\leq i<j\leq n} p_i p_j. \tag{5.65}$$

Now, let us observe that

$$\begin{aligned} \sum_{1\leq i<j\leq n} p_i p_j &= \frac{1}{2}\left[\sum_{i,j=1}^{n} p_i p_j - \sum_{i=j} p_i p_j\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i^2\right] \\ &= \frac{1}{2}\sum_{i=1}^{n} p_i\left(1-p_i\right). \end{aligned} \tag{5.66}$$

Making use of (5.64) – (5.66), we deduce the desired inequality (5.63).

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (5.63) holds with a constant $C > 0$. That is

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \leq C\sum_{i=1}^{n} p_i\left(1-p_i\right)\sum_{i=1}^{n-1}\left|\Delta\alpha_i\right|\sum_{i=1}^{n-1}\left|\Delta x_i\right| \tag{5.67}$$

for all α_i, x_i, p_i $(i = 1, \ldots, n)$ as above and $n \geq 1$.

Choose in (5.63) $n = 2$ and compute

$$\begin{aligned}\sum_{i=1}^{2} p_i \alpha_i x_i - \sum_{i=1}^{2} p_i \alpha_i \sum_{i=1}^{2} p_i x_i &= \frac{1}{2} \sum_{i,j=1}^{2} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= \sum_{1 \le i < j \le 2} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= p_1 p_2 (\alpha_1 - \alpha_2)(x_1 - x_2).\end{aligned}$$

Also

$$\sum_{i=1}^{2} p_i (1 - p_i) \sum_{i=1}^{2} |\Delta \alpha_i| \sum_{i=1}^{2} |\Delta x_i| = (p_1 p_2 + p_1 p_2) |\alpha_1 - \alpha_2| |x_1 - x_2|.$$

Substituting in (5.67), we obtain

$$p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2| \le 2C p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|.$$

If we assume that $p_1, p_2 > 0$, $\alpha_1 \neq \alpha_2$, $x_1 \neq x_2$, then we obtain $C \ge \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. □

We are now able to state the following reverse of the (CBS) −inequality [12].

Theorem 5.32. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0 \ \ (i = 1, \dots, n)$. *Then one has the inequality*

$$\begin{aligned}0 &\le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \\ &\le \left[\sum_{k=1}^{n-1} \left| \Delta \left(\frac{b_k}{a_k} \right) \right| \right]^2 \sum_{1 \le i < j \le n} a_i^2 a_j^2.\end{aligned} \tag{5.68}$$

The constant $C = 1$ *is sharp in (5.68), in the sense that it cannot be replaced by a smaller constant.*

Proof. We choose

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad \alpha_i = x_i = \frac{b_i}{a_i}, \ \ i \in \{1, \dots, n\}$$

in (5.63) to get

$$\begin{aligned}0 &\le \frac{\sum_{i=1}^{n} b_i^2}{\sum_{k=1}^{n} a_k^2} - \frac{\left(\sum_{i=1}^{n} a_i b_i \right)^2}{\left(\sum_{k=1}^{n} a_k^2 \right)^2} \\ &\le \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} a_i^2 \left(1 - \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \right)}{\sum_{k=1}^{n} a_k^2} \left(\sum_{j=1}^{n-1} \left| \Delta \left(\frac{b_j}{a_j} \right) \right| \right)^2 \\ &= \frac{1}{2} \cdot \frac{\sum_{i=1}^{n} a_i^2 \left(\sum_{k=1}^{n} a_k^2 - a_i^2 \right)}{\left(\sum_{k=1}^{n} a_k^2 \right)^2} \left(\sum_{j=1}^{n-1} \left| \Delta \left(\frac{b_j}{a_j} \right) \right| \right)^2\end{aligned}$$

which is clearly equivalent to

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \\ &\leq \frac{1}{2} \left[\left(\sum_{k=1}^{n} a_k^2 \right)^2 - \sum_{i=1}^{n} a_i^4 \right] \left(\sum_{j=1}^{n} \left| \Delta \left(\frac{b_j}{a_j} \right) \right| \right)^2 . \end{aligned}$$

Since

$$\frac{1}{2} \left[\left(\sum_{k=1}^{n} a_k^2 \right)^2 - \sum_{i=1}^{n} a_i^4 \right] = \sum_{1 \leq i < j \leq n} a_i^2 a_j^2$$

the inequality (5.68) is thus proved. □

5.11. **A Reverse Inequality in Terms of the $p-$Norm.** The following result has been obtained in [14].

Lemma 5.33. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ be sequences of complex numbers and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ a sequence of nonnegative real numbers such that $\sum_{i=1}^{n} p_i = 1$. Then one has the inequality*

$$\begin{aligned} &\left| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right| \\ &\qquad \leq \sum_{1 \leq j < i \leq n} (i-j)\, p_i p_j \left(\sum_{k=1}^{n-1} |\Delta \alpha_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} |\Delta x_k|^q \right)^{\frac{1}{q}}, \end{aligned} \tag{5.69}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $C = 1$ in the right hand side of (5.69) is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. We shall follow the proof in [14].

As in the proof of Lemma 5.29 in Section 5.9, we have

$$\left| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right| \leq \sum_{1 \leq j < i \leq n} p_i p_j \sum_{k=j}^{i-1} |\Delta \alpha_k| \sum_{l=j}^{i-1} |\Delta x_l| := A. \tag{5.70}$$

Using Hölder's discrete inequality, we can state that

$$\sum_{k=j}^{i-1} |\Delta \alpha_k| \leq (i-j)^{\frac{1}{q}} \left(\sum_{k=j}^{i-1} |\Delta \alpha_k|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{l=j}^{i-1} |\Delta x_l| \leq (i-j)^{\frac{1}{p}} \left(\sum_{l=j}^{i-1} |\Delta x_l|^q \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and then we get

$$A \leq \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left(\sum_{k=j}^{i-1} |\Delta \alpha_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} |\Delta x_k|^q \right)^{\frac{1}{q}}. \tag{5.71}$$

Since

$$\sum_{k=j}^{i-1} |\Delta\alpha_k|^p \le \sum_{k=1}^{n-1} |\Delta\alpha_k|^p$$

and

$$\sum_{k=j}^{i-1} |\Delta x_k|^q \le \sum_{k=1}^{n-1} |\Delta x_k|^q ,$$

for all $1 \le j < i \le n$, then by (5.70) and (5.71) we deduce the desired inequality (5.69).

To prove the sharpness of the constant, let us assume that (5.69) holds with a constant $C > 0$. That is,

$$\left|\sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \sum_{i=1}^{n} p_i x_i\right| \le C \sum_{1\le j<i\le n} (i-j)\, p_i p_j \left(\sum_{k=1}^{n-1} |\Delta\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} |\Delta x_k|^q\right)^{\frac{1}{q}} . \tag{5.72}$$

Note that, for $n = 2$, we have

$$\left|\sum_{i=1}^{2} p_i\alpha_i x_i - \sum_{i=1}^{2} p_i\alpha_i \sum_{i=1}^{2} p_i x_i\right| = p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|$$

and

$$\sum_{1\le j<i\le 2} (i-j)\, p_i p_j \left(\sum_{k=1}^{1} |\Delta\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{1} |\Delta x_k|^q\right)^{\frac{1}{q}} = p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2| .$$

Therefore, from (5.72), we obtain

$$p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2| \le C p_1 p_2 |\alpha_1 - \alpha_2| |x_1 - x_2|$$

for all $\alpha_1 \ne \alpha_2$, $x_1 \ne x_2$, $p_1 p_2 > 0$, giving $C \ge 1$. □

We are able now to state the following reverse of the (CBS) −inequality.

Theorem 5.34. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \ne 0$, $(i = 1, \dots, n)$ *. Then one has the inequality*

$$0 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{k=1}^{n-1} \left|\Delta\left(\frac{b_k}{a_k}\right)\right|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \left|\Delta\left(\frac{b_k}{a_k}\right)\right|^q\right)^{\frac{1}{q}} \sum_{1\le j<i\le n} (i-j)\, a_i^2 a_j^2,$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The constant $C = 1$ *is sharp in the above sense.*

Proof. Follows by Lemma 5.33 for

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad \alpha_i = x_i = \frac{b_i}{a_i}, \quad i \in \{1, \dots, n\} .$$

□

The following corollary is a natural consequence of Theorem 5.34 for $p = q = 2$.

Corollary 5.35. *With the assumptions of Theorem 5.34 for* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$, *we have*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \\ &\leq \sum_{k=1}^{n-1} \left| \Delta \left(\frac{b_k}{a_k} \right) \right|^2 \sum_{1 \leq j < i \leq n} (i-j) \, a_i^2 a_j^2. \end{aligned}$$

5.12. **A Reverse Inequality Via an Andrica-Badea Result.** The following result is due to Andrica and Badea [15, p. 16].

Lemma 5.36. *Let* $\bar{\mathbf{x}} = (x_1, \dots, x_n) \in I^n = [m, M]^n$ *be a sequence of real numbers and let* S *be the subset of* $\{1, \dots, n\}$ *that minimises the expression*

$$\left| \sum_{i \in S} p_i - \frac{1}{2} P_n \right|, \tag{5.73}$$

where $P_n := \sum_{i=1}^{n} p_i > 0$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *is a sequence of nonnegative real numbers. Then*

$$\max_{\bar{\mathbf{x}} \in I^n} \left[\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right)^2 \right] = \sum_{i \in S} p_i \left(P_n - \sum_{i \in S} p_i \right) \frac{(M-m)^2}{P_n^2}. \tag{5.74}$$

Proof. We shall follow the proof in [15, p. 161].

Define

$$\begin{aligned} D_n(\bar{\mathbf{x}}, \bar{\mathbf{p}}) &:= \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right)^2 \\ &= \frac{1}{P_n} \sum_{1 \leq i < j \leq n} p_i p_j (x_i - x_j)^2. \end{aligned}$$

Keeping in mind the convexity of the quadratic function, we have

$$\begin{aligned} & D_n(\alpha \bar{\mathbf{x}} + (1-\alpha) \bar{\mathbf{y}}, \bar{\mathbf{p}}) \\ &\quad = \frac{1}{P_n^2} \sum_{1 \leq i < j \leq n} p_i p_j \left[\alpha x_i + (1-\alpha) y_i - \alpha x_j - (1-\alpha) y_j \right]^2 \\ &\quad = \frac{1}{P_n^2} \sum_{1 \leq i < j \leq n} p_i p_j \left[\alpha (x_i - x_j) + (1-\alpha)(y_i - y_j) \right]^2 \\ &\quad \leq \frac{1}{P_n^2} \sum_{1 \leq i < j \leq n} p_i p_j \left[\alpha (x_i - x_j)^2 + (1-\alpha)(y_i - y_j)^2 \right] \\ &\quad = \alpha D_n(\bar{\mathbf{x}}, \bar{\mathbf{p}}) + (1-\alpha) D_n(\bar{\mathbf{y}}, \bar{\mathbf{p}}), \end{aligned}$$

hence $D_n(\cdot, \bar{\mathbf{p}})$ is a convex function on I^n.

Using a well known theorem (see for instance [16, p. 124]), we get that the maximum of $D_n(\cdot, \bar{\mathbf{p}})$ is attained on the boundary of I^n.

Let $\left(S, \bar{S}\right)$ be the partition of $\{1, \dots, n\}$ such that the maximum of $D_n\left(\cdot, \bar{\mathbf{p}}\right)$ is obtained for $\bar{\mathbf{x}}_0 = \left(x_1^0, \dots, x_n^0\right)$, where $x_i^0 = m$ if $i \in \bar{S}$ and $x_i^0 = M$ if $i \in S$. In this case we have

$$D_n\left(\bar{\mathbf{x}}_0, \bar{\mathbf{p}}\right) = \frac{1}{P_n^2} \sum_{1 \le i < j \le n} p_i p_j \left(x_i - x_j\right)^2 = \frac{\left(M - m\right)^2}{P_n^2} \sum_{i \in S} p_i \left(P_n - \sum_{i \in S} p_i\right). \tag{5.75}$$

The expression

$$\sum_{i \in S} p_i \left(P_n - \sum_{i \in S} p_i\right)$$

is a maximum when the set S minimises the expression

$$\left|\sum_{i \in S} p_i - \frac{1}{2} P_n\right|.$$

From (5.75) it follows that $D_n\left(\bar{\mathbf{x}}, \bar{\mathbf{p}}\right)$ is also a maximum and the proof of the above lemma is complete. □

The following reverse result of the (CBS) −inequality holds.

Theorem 5.37. *Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ and $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ be two sequences of real numbers with $a_i \neq 0 \ \ (i = 1, \dots, n)$ and*

$$-\infty < m \le \frac{b_i}{a_i} \le M < \infty \ \text{ for each } \ i \in \{1, \dots, n\}. \tag{5.76}$$

Let S be the subset of $\{1, \dots, n\}$ that minimizes the expression

$$\left|\sum_{i \in S} a_i^2 - \frac{1}{2} \sum_{i=1}^n a_i^2\right|, \tag{5.77}$$

and denote $\bar{S} := \{1, \dots, n\} \setminus S$. Then we have the inequality

$$0 \le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(M - m\right)^2 \sum_{i \in S} a_i^2 \sum_{i \in \bar{S}} a_i^2 \le \frac{1}{4} \left(M - m\right)^2 \left(\sum_{i=1}^n a_i^2\right)^2. \tag{5.78}$$

Proof. The proof of the second inequality in (5.78) follows by Lemma 5.36 on choosing $p_i = a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \dots, n\}$.

The third inequality is obvious as

$$\sum_{i\in S} a_i^2 \sum_{i\in \bar{S}} a_i^2 = \sum_{i\in S} a_i^2 \left(\sum_{j=1}^{n} a_j^2 - \sum_{i\in S} a_i^2\right) \leq \frac{1}{4}\left(\sum_{i\in S} a_i^2 + \sum_{j=1}^{n} a_j^2 - \sum_{i\in S} a_i^2\right)^2 = \frac{1}{4}\left(\sum_{j=1}^{n} a_j^2\right)^2.$$

□

5.13. **A Refinement of Cassels' Inequality.** In 1914, P. Schweitzer [18] proved the following result.

Theorem 5.38. *If* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *is a sequence of real numbers such that* $0 < m \leq a_i \leq M < \infty$ $(i \in \{1, \dots, n\})$, *then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{a_i}\right) \leq \frac{(M+m)^2}{4mM}. \tag{5.79}$$

In 1972, A. Lupaş [17] proved the following refinement of Schweitzer's result which gives the best bound for n odd as well.

Theorem 5.39. *With the assumptions in Theorem 5.38, one has*

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \leq \frac{\left(\left[\frac{n}{2}\right] M + \left[\frac{n+1}{2}\right] m\right)\left(\left[\frac{n+1}{2}\right] M + \left[\frac{n}{2}\right] m\right)}{Mm}, \tag{5.80}$$

where $[\cdot]$ *is the integer part.*

In 1988, Andrica and Badea [15] established a weighted version of Schweitzer and Lupaş inequalities via the use of the following weighted version of the Grüss inequality [15, Theorem 2].

Theorem 5.40. *If* $m_1 \leq a_i \leq M_1$, $m_2 \leq b_i \leq M_2$ $(i \in \{1, \dots, n\})$ *and* S *is the subset of* $\{1, \dots, n\}$ *which minimises the expression*

$$\left|\sum_{i\in S} p_i - \frac{1}{2}P_n\right|, \tag{5.81}$$

where $P_n := \sum_{i=1}^{n} p_i > 0$, *then*

$$\left|P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i b_i\right| \leq (M_1 - m_1)(M_2 - m_2) \sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \leq \frac{1}{4} P_n^2 (M_1 - m_1)(M_2 - m_2). \tag{5.82}$$

Proof. Using the result in Lemma 5.36, Section 5.12, we have

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)^2 \le \frac{(M_1 - m_1)^2}{P_n^2}\sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \tag{5.83}$$

and

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)^2 \le \frac{(M_2 - m_2)^2}{P_n^2}\sum_{i\in S} p_i \left(P_n - \sum_{i\in S} p_i\right) \tag{5.84}$$

and since

$$\begin{aligned}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)^2 \\ \le \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)^2\right]\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)^2\right],\end{aligned} \tag{5.85}$$

the first part of (5.82) holds true.

The second part follows by the elementary inequality

$$ab \le \frac{1}{4}(a+b)^2, \quad a, b \in \mathbb{R}$$

for the choices $a := \sum_{i\in S} p_i$, $b := P_n - \sum_{i\in S} p_i$. □

We are now able to state and prove the result of Andrica and Badea [15, Theorem 4], which is related to Schweitzer's inequality.

Theorem 5.41. *If* $0 < m \le a_i \le M < \infty$, $i \in \{1, \dots, n\}$ *and* S *is a subset of* $\{1, \dots, n\}$ *that minimises the expression*

$$\left|\sum_{i\in S} p_i - \frac{P_n}{2}\right|,$$

then we have the inequality

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} \frac{p_i}{a_i}\right) &\le P_n^2 + \frac{(M-m)^2}{Mm}\sum_{i\in S} p_i\left(P_n - \sum_{i\in S} p_i\right) \\ &\le \frac{(M+m)^2}{4Mm} P_n^2.\end{aligned} \tag{5.86}$$

Proof. We shall follow the proof in [15]. We obtain from Theorem 5.39 with $b_i = \frac{1}{a_i}$, $m_1 = m$, $M_1 = m$, $m_2 = \frac{1}{M}$, $M_2 = \frac{1}{m}$, the following estimate

$$\left|P_n^2 - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i}\right| \le (M-m)\left(\frac{1}{m} - \frac{1}{M}\right)\sum_{i\in S} p_i\left(P_n - \sum_{i\in S} p_i\right),$$

that leads, in a simple manner, to (5.86). □

We may now prove the following reverse result for the weighted $(CBS)-$ inequality that improves the additive version of Cassels' inequality.

Theorem 5.42. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of positive real numbers with the property that*

$$0 < m \le \frac{b_i}{a_i} \le M < \infty \text{ for each } i \in \{1, \dots, n\}, \tag{5.87}$$

and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *a sequence of nonnegative real numbers such that* $P_n := \sum_{i=1}^n p_i > 0$. *If* S *is a subset of* $\{1, \dots, n\}$ *that minimises the expression*

$$\left| \sum_{i \in S} p_i a_i b_i - \frac{1}{2} \sum_{i=1}^n p_i a_i b_i \right| \tag{5.88}$$

then one has the inequality

$$\begin{aligned} \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \left(\sum_{i=1}^n p_i a_i b_i \right)^2 &\le \frac{(M-m)^2}{Mm} \sum_{i \in S} p_i a_i b_i \left(\sum_{i=1}^n p_i a_i b_i - \sum_{i \in S} p_i a_i b_i \right) \\ &\le \frac{(M-m)^2}{4Mm} \left(\sum_{i=1}^n p_i a_i b_i \right)^2. \end{aligned} \tag{5.89}$$

Proof. Applying Theorem 5.41 for $a_i = x_i$, $p_i = q_i x_i$ we may deduce the inequality

$$\sum_{i=1}^n q_i x_i^2 \sum_{i=1}^n q_i - \left(\sum_{i=1}^n q_i x_i \right)^2 \le \frac{(M-m)^2}{Mm} \sum_{i \in S} q_i x_i \left(\sum_{i=1}^n q_i x_i - \sum_{i \in S} q_i x_i \right), \tag{5.90}$$

provided $q_i \ge 0$, $\sum_{i=1}^n q_i > 0$, $0 < m \le x_i \le M < \infty$, for $i \in \{1, \dots, n\}$ and S is a subset of $\{1, \dots, n\}$ that minimises the expression

$$\left| \sum_{i \in S} q_i x_i - \frac{1}{2} \sum_{i=1}^n q_i x_i \right|. \tag{5.91}$$

Now, if in (5.90) we choose $q_i = p_i a_i^2$, $x_i = \frac{b_i}{a_i} \in [m, M]$ for $i \in \{1, \dots, n\}$, we deduce the desired result (5.89). □

The following corollary provides a refinement of Cassels' inequality.

Corollary 5.43. *With the assumptions of Theorem 5.42, we have the inequality*

$$\begin{aligned} 1 &\le \frac{\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2}{\left(\sum_{i=1}^n p_i a_i b_i \right)^2} \\ &\le 1 + \frac{(M-m)^2}{Mm} \cdot \frac{\sum_{i \in S} p_i a_i b_i}{\sum_{i=1}^n p_i a_i b_i} \left(1 - \frac{\sum_{i \in S} p_i a_i b_i}{\sum_{i=1}^n p_i a_i b_i} \right) \\ &\le \frac{(M+m)^2}{4Mm}. \end{aligned} \tag{5.92}$$

The case of the "unweighted" Cassels' inequality is embodied in the following corollary as well.

Corollary 5.44. *Assume that* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *satisfy (5.88). If* S *is a subset of* $\{1, \dots, n\}$ *that minimises the expression*

$$\left|\sum_{i\in S} a_i b_i - \frac{1}{2}\sum_{i=1}^{n} a_i b_i\right| \tag{5.93}$$

then one has the inequality

$$\begin{aligned} 1 &\leq \frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \\ &\leq 1 + \frac{(M-m)^2}{Mm} \cdot \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i}\left(1 - \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i}\right) \\ &\leq \frac{(M+m)^2}{4Mm}. \end{aligned} \tag{5.94}$$

In particular, we may obtain the following refinement of the Pólya-Szegö's inequality.

Corollary 5.45. *Assume that*

$$0 < a \leq a_i \leq A < \infty, \ \ 0 < b \leq b_i \leq B < \infty \ \text{ for } \ i \in \{1, \dots, n\}. \tag{5.95}$$

If S *is a subset of* $\{1, \dots, n\}$ *that minimises the expression (5.93), then one has the inequality*

$$\begin{aligned} 1 &\leq \frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{\left(\sum_{i=1}^{n} a_i b_i\right)^2} \\ &\leq 1 + \frac{(AB-ab)^2}{abAB} \cdot \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i}\left(1 - \frac{\sum_{i\in S} a_i b_i}{\sum_{i=1}^{n} a_i b_i}\right) \\ &\leq \frac{(AB+ab)^2}{4abAB}. \end{aligned} \tag{5.96}$$

5.14. **Two Reverse Results Via Diaz-Metcalf Results.** In [19], J.B. Diaz and F.T. Metcalf proved the following inequality for sequences of complex numbers.

Lemma 5.46. *Let* $\bar{\mathrm{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathrm{b}} = (b_1, \dots, b_n)$ *be sequences of complex numbers such that* $a_k \neq 0$, $k \in \{1, \dots, n\}$ *and*

$$m \leq \operatorname{Re}\left(\frac{b_k}{a_k}\right) + \operatorname{Im}\left(\frac{b_k}{a_k}\right) \leq M, \ \ m \leq \operatorname{Re}\left(\frac{b_k}{a_k}\right) - \operatorname{Im}\left(\frac{b_k}{a_k}\right) \leq M, \tag{5.97}$$

where $m, M \in \mathbb{R}$ *and* $k \in \{1, \dots, n\}$. *Then one has the inequality*

$$\begin{aligned} \sum_{k=1}^{n} |b_k|^2 + mM\sum_{k=1}^{n} |a_k|^2 &\leq (m+M)\operatorname{Re}\left[\sum_{k=1}^{n} a_k \bar{b}_k\right] \\ &\leq |M+m|\left|\sum_{k=1}^{n} a_k \bar{b}_k\right|. \end{aligned} \tag{5.98}$$

Using the above result we may state and prove the following reverse inequality.

Theorem 5.47. *If* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *are as in (5.97) and* $m, M > 0$*, then one has the inequality*

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \leq \frac{(M+m)^2}{4mM} \left(\operatorname{Re} \sum_{k=1}^{n} a_k \bar{b}_k \right)^2 \leq \frac{(M+m)^2}{4mM} \left| \sum_{k=1}^{n} a_k \bar{b}_k \right|^2 . \tag{5.99}$$

Proof. Using the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0, \; p, q \geq 0$$

we have

$$\sqrt{mM} \sum_{k=1}^{n} |a_k|^2 + \frac{1}{\sqrt{mM}} \sum_{k=1}^{n} |b_k|^2 \geq 2 \left(\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \right)^{\frac{1}{2}} . \tag{5.100}$$

On the other hand, by (5.98), we have

$$\frac{1}{\sqrt{mM}} \sum_{k=1}^{n} |b_k|^2 + \sqrt{mM} \sum_{k=1}^{n} |a_k|^2 \leq \frac{(M+m)}{\sqrt{mM}} \operatorname{Re} \left[\sum_{k=1}^{n} a_k \bar{b}_k \right] \leq \frac{M+m}{\sqrt{mM}} \left| \sum_{k=1}^{n} a_k \bar{b}_k \right| . \tag{5.101}$$

Combining (5.100) and (5.101), we deduce the desired result (5.99). □

The following corollary is a natural consequence of the above lemma.

Corollary 5.48. *If* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *and* m, M *satisfy the hypothesis of Theorem 5.47, then*

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}} - \left| \sum_{i=1}^{n} a_i \bar{b}_i \right| \\ &\leq \left(\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left(\sum_{k=1}^{n} a_k \bar{b}_k \right) \right| \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m} \right)^2}{2\sqrt{mM}} \left| \operatorname{Re} \left(\sum_{k=1}^{n} a_k \bar{b}_k \right) \right| \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m} \right)^2}{2\sqrt{mM}} \left| \sum_{k=1}^{n} a_k \bar{b}_k \right| \end{aligned} \tag{5.102}$$

and

$$0 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} a_i \bar{b}_i \right|^2 \tag{5.103}$$

$$\leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \operatorname{Re} \left(\sum_{i=1}^{n} a_i \bar{b}_i \right) \right|^2 \tag{5.104}$$

$$\leq \frac{(M-m)^2}{4mM} \left| \operatorname{Re} \left(\sum_{i=1}^{n} a_i \bar{b}_i \right) \right|^2 \tag{5.105}$$

$$\leq \frac{(M-m)^2}{4mM} \left| \sum_{i=1}^{n} a_i \bar{b}_i \right|^2 . \tag{5.106}$$

Another result obtained by Diaz and Metcalf in [19] is the following one.

Lemma 5.49. *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, m *and* M *be complex numbers such that*

$$\operatorname{Re}(m) + \operatorname{Im}(m) \leq \operatorname{Re}\left(\frac{b_k}{a_k}\right) + \operatorname{Im}\left(\frac{b_k}{a_k}\right) \leq \operatorname{Re}(M) + \operatorname{Im}(M); \tag{5.107}$$

$$\operatorname{Re}(m) - \operatorname{Im}(m) \leq \operatorname{Re}\left(\frac{b_k}{a_k}\right) - \operatorname{Im}\left(\frac{b_k}{a_k}\right) \leq \operatorname{Re}(M) - \operatorname{Im}(M);$$

for each $k \in \{1, \dots, n\}$ *. Then*

$$\sum_{k=1}^{n} |b_k|^2 + \operatorname{Re}\left(m\bar{M}\right) \sum_{k=1}^{n} |a_k|^2 \leq \operatorname{Re}\left[(M+m) \sum_{k=1}^{n} a_k \bar{b}_k\right] \tag{5.108}$$

$$\leq |M+m| \left| \sum_{k=1}^{n} a_k \bar{b}_k \right| .$$

The following reverse result for the (CBS) $-$inequality may be stated as well.

Theorem 5.50. *With the assumptions in Lemma 5.49, and if* $\operatorname{Re}\left(m\bar{M}\right) > 0$, *then we have the inequality:*

$$\left[\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2\right]^{\frac{1}{2}} \leq \frac{\operatorname{Re}\left[(M+m)\sum_{k=1}^{n} a_k \bar{b}_k\right]}{2\left[\operatorname{Re}\left(m\bar{M}\right)\right]^{\frac{1}{2}}} \tag{5.109}$$

$$\leq \frac{|M+m| \left|\sum_{k=1}^{n} a_k \bar{b}_k\right|}{2\left[\operatorname{Re}\left(m\bar{M}\right)\right]^{\frac{1}{2}}}.$$

The proof is similar to the one in Theorem 5.47 and we omit the details.

Remark 5.51. Similar additive versions may be stated. They are left as an exercise for the interested reader.

5.15. **Some Reverse Results Via the Čebyšev Functional.** For $\bar{x} = (x_1, \dots, x_n)$, $\bar{y} = (y_1, \dots, y_n)$ two sequences of real numbers and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ a sequence of nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$, define the *Čebyšev functional*

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}}) := \sum_{i=1}^{n} p_i x_i y_i - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i y_i. \tag{5.110}$$

For $\bar{x}$ and $\bar{\mathbf{p}}$ as above consider the norms:

$$\|\bar{\mathbf{x}}\|_{\infty} := \max_{i=\overline{1,n}} |x_i|$$

$$\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha} := \left(\sum_{i=1}^{n} p_i |x_i|^{\alpha} \right)^{\frac{1}{\alpha}}, \quad \alpha \in [1, \infty).$$

The following result holds [20].

Theorem 5.52. *Let $\bar{x}, \bar{y}, \bar{\mathbf{p}}$ be as above and $\bar{c} = (c, \dots, c)$ a constant sequence with $c \in \mathbb{R}$. Then one has the inequalities*

$$0 \le |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| \tag{5.111}$$

$$\le \begin{cases} \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\bar{\mathbf{p}},1} \cdot \inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\infty}; \\ \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \cdot \inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},\alpha}, \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\infty} \cdot \inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},1}; \end{cases}$$

$$\le \begin{cases} \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\bar{\mathbf{p}},1} \min\left\{\|\bar{\mathbf{x}}\|_{\infty}, \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty}\right\}; \\ \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \min\left\{\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha}, \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha}\right\}, \\ \qquad\qquad \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\infty} \cdot \min\left\{\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},1}, \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1}\right\}; \end{cases}$$

where

$$x_{\mu,p} := \sum_{i=1}^{n} p_i x_i, \qquad y_{\mu,p} := \sum_{i=1}^{n} p_i y_i$$

and $\bar{x}_{\mu,p}, \bar{y}_{\mu,p}$ are the sequences with all components equal to $x_{\mu,p}, y_{\mu,p}$.

Proof. Firstly, let us observe that for any $c \in \mathbb{R}$, one has Sonin's identity

$$\begin{aligned} T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}}) &= T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}} - \bar{\mathbf{c}}, \bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}) \\ &= \sum_{i=1}^{n} p_i (x_i - c) \left(y_i - \sum_{j=1}^{n} p_j y_j \right). \end{aligned} \tag{5.112}$$

Taking the modulus and using Hölder's inequality, we have

$$|T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}, \bar{\mathbf{y}})| \le \sum_{i=1}^{n} p_i |x_i - c| |y_i - y_{\mu,p}| \tag{5.113}$$

$$\leq \begin{cases} \max\limits_{i=\overline{1,n}} |x_i - c| \sum_{i=1}^{n} p_i |y_i - y_{\mu,p}| \\ \\ \left(\sum_{i=1}^{n} p_i |x_i - c|^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} p_i |y_i - y_{\mu,p}|^{\beta}\right)^{\frac{1}{\beta}}, \\ \hfill \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \sum_{i=1}^{n} p_i |x_i - c| \max\limits_{i=\overline{1,n}} |y_i - y_{\mu,p}| \end{cases}$$

$$= \begin{cases} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\infty} \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\bar{\mathbf{p}},1}; \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},\alpha} \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta}, \quad \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},1} \|\bar{\mathbf{y}} - \bar{\mathbf{y}}_{\mu,p}\|_{\infty}. \end{cases}$$

Taking the $\inf$ over $c \in \mathbb{R}$ in (5.113), we deduce the second inequality in (5.111).

Since

$$\inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},\alpha} \leq \begin{cases} \|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha}, \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha} \end{cases} \text{for any} \quad \alpha \in [1, \infty]$$

the last part of (5.110) is also proved. □

For $\bar{\mathbf{p}}$ and $\bar{x}$ as above, define

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}) := \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2.$$

The following corollary holds [20].

Corollary 5.53. *With the above assumptions we have*

$$(5.114) \qquad 0 \leq |T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}})|$$

$$\leq \begin{cases} \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1} \cdot \inf\limits_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\infty}; \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \cdot \inf\limits_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},\alpha}, \ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty} \cdot \inf\limits_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},1}; \end{cases}$$

$$\leq \begin{cases} \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1} \min\left\{\|\bar{\mathbf{x}}\|_{\infty}, \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty}\right\}; \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \min\left\{\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha}, \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha}\right\}, \\ \hfill \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty} \cdot \min\left\{\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},1}, \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1}\right\}. \end{cases}$$

Remark 5.54. If $p_i := \frac{1}{n}$, $i = 1, \dots, n$, then from Theorem 5.52 and Corollary 5.53 we recapture the results in [22].

The following reverse of the (CBS) $-$inequality holds [20].

Theorem 5.55. *Let* $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ *be two sequences of real numbers with* $a_i \neq 0,\ i \in \{1, \dots, n\}$. *Then one has the inequality*

$$0 \le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \tag{5.115}$$
$$\le \inf_{c\in\mathbb{R}} \left[\max_{i=\overline{1,n}} \left|\frac{b_i}{a_i} - c\right|\right] \sum_{i=1}^n \left[|a_i| \left|\sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|\right]$$
$$\le \sum_{i=1}^n \left[|a_i| \left|\sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|\right] \times \begin{cases} \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\right| \\ \\ \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n a_k^2}\right|. \end{cases}$$

Proof. By Corollary 5.53, we may state that

$$0 \le T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}) \tag{5.116}$$
$$\le \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1} \cdot \inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_\infty$$
$$\le \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1} \times \begin{cases} \|\bar{\mathbf{x}}\|_\infty, \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_\infty. \end{cases}$$

For the choices

$$p_i = \frac{a_i^2}{\sum_{k=1}^n a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i = 1, \dots, n;$$

we get

$$T_n(\bar{\mathbf{p}}; \bar{\mathbf{x}}) = \frac{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2}{\left(\sum_{k=1}^n a_k^2\right)^2},$$

$$\begin{aligned}
\|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1} &= \sum_{i=1}^n p_i \left|x_i - \sum_{j=1}^n p_j x_j\right| \\
&= \frac{1}{\sum_{k=1}^n a_k^2} \sum_{i=1}^n a_i^2 \left|\frac{b_i}{a_i} - \frac{1}{\sum_{k=1}^n a_k^2} \sum_{j=1}^n a_j b_j\right| \\
&= \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^2} \sum_{i=1}^n \left|a_i b_i \sum_{k=1}^n a_k^2 - a_i^2 \sum_{j=1}^n a_j b_j\right| \\
&= \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^2} \sum_{i=1}^n |a_i| \left|\sum_{k=1}^n a_k (a_k b_i - a_i b_k)\right| \\
&= \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^2} \sum_{i=1}^n |a_i| \left|\sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|,
\end{aligned}$$

$$\|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_\infty = \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i} - c\right|, \quad \|\bar{\mathbf{x}}\|_\infty = \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\right|$$

and

$$\|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_\infty = \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i} - \frac{\sum_{j=1}^n a_j b_j}{\sum_{k=1}^n a_k^2}\right|.$$

Utilising the inequality (5.116) we deduce the desired result (5.115). □

The following result also holds [20].

Theorem 5.56. *With the assumption in Theorem 5.55 and if* $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, *then we have the inequality:*

$$0 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \tag{5.117}$$
$$\le \left(\sum_{i=1}^{n} \left[|a_i|^{2-\beta} \left|\sum_{k=1}^{n} a_k \left|\begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array}\right|\right|^{\beta}\right]\right)^{\frac{1}{\beta}} \inf_{c\in\mathbb{R}} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} |b_i - ca_i|^{\alpha}\right)^{\frac{1}{\alpha}}$$
$$\le \left(\sum_{i=1}^{n} \left[|a_i|^{2-\beta} \left|\sum_{k=1}^{n} a_k \left|\begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array}\right|\right|^{\beta}\right]\right)^{\frac{1}{\beta}}$$
$$\times \begin{cases} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} |b_i|^{\alpha}\right)^{\frac{1}{\alpha}} \\ \\ \frac{1}{\sum_{k=1}^{n} a_k^2} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} \left|\sum_{k=1}^{n} a_k \left|\begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array}\right|\right|^{\alpha}\right)^{\frac{1}{\alpha}}. \end{cases}$$

Proof. By Corollary 5.53, we may state that

$$0 \le T_n(\bar{\mathbf{p}};\bar{\mathbf{x}}) \tag{5.118}$$
$$\le \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \cdot \inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},\alpha}$$
$$\le \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} \times \begin{cases} \|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha}, \\ \\ \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha}, \end{cases}$$

for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

For the choices

$$p_i = \frac{a_i^2}{\sum_{k=1}^{n} a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i = 1, \dots, n;$$

we get

$$\|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\beta} = \left(\sum_{i=1}^{n} p_i \left|x_i - \sum_{j=1}^{n} p_j x_j\right|^{\beta}\right)^{\frac{1}{\beta}}$$
$$= \left(\sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \left|\frac{b_i \sum_{k=1}^{n} a_k^2 - a_i \sum_{j=1}^{n} a_j b_j}{a_i \sum_{k=1}^{n} a_k^2}\right|^{\beta}\right)^{\frac{1}{\beta}}$$
$$= \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^{1+\frac{1}{\beta}}} \left(\sum_{i=1}^{n} |a_i|^{2-\beta} \left|\sum_{k=1}^{n} a_k \left|\begin{array}{cc} a_k & a_i \\ b_k & b_i \end{array}\right|\right|^{\beta}\right)^{\frac{1}{\beta}},$$

$$\|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},\alpha} = \left(\sum_{i=1}^{n} p_i |x_i - c|^{\alpha}\right)^{\frac{1}{\alpha}} = \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{\alpha}}} \left(\sum_{i=1}^{n} |a_i|^{2-\alpha} |b_i - ca_i|^{\alpha}\right)^{\frac{1}{\alpha}},$$

$$\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},\alpha} = \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{\alpha}}} \left(\sum_{i=1}^n |a_i|^{2-\alpha} |b_i|^{\alpha}\right)^{\frac{1}{\alpha}}$$

and

$$\|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},\alpha} = \frac{1}{\left(\sum_{k=1}^n a_k^2\right)^{1+\frac{1}{\alpha}}} \left(\sum_{i=1}^n |a_i|^{2-\alpha} \left|\sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|^{\alpha}\right)^{\frac{1}{\alpha}}.$$

Utilising the inequality (5.118), we deduce the desired result (5.117). □

Finally, the following result also holds [20].

Theorem 5.57. *With the assumptions in Theorem 5.55 we have the following reverse of the* (CBS) *−inequality:*

$$\begin{aligned}
0 &\le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 \\
&\le \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\sum_{k=1}^n a_k^2 - \sum_{j=1}^n a_j b_j\right| \inf_{c\in\mathbb{R}} \left[\sum_{i=1}^n |a_i|\,|b_i - c a_i|\right] \\
&\le \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\sum_{k=1}^n a_k^2 - \sum_{j=1}^n a_j b_j\right| \\
&\quad\times \begin{cases} \sum_{i=1}^n |a_i b_i| \\[2ex] \frac{1}{\sum_{k=1}^n a_k^2} \sum_{i=1}^n |a_i| \left|\sum_{k=1}^n a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix}\right|. \end{cases}
\end{aligned} \tag{5.119}$$

Proof. By Corollary 5.53, we may state that

$$\begin{aligned}
0 &\le T_n(\bar{\mathbf{p}};\bar{\mathbf{x}}) \\
&\le \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty} \cdot \inf_{c\in\mathbb{R}} \|\bar{\mathbf{x}} - \bar{\mathbf{c}}\|_{\bar{\mathbf{p}},1} \\
&\le \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty} \begin{cases} \|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},1}, \\[2ex] \|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1}. \end{cases}
\end{aligned} \tag{5.120}$$

For the choices

$$p_i = \frac{a_i^2}{\sum_{k=1}^n a_k^2}, \quad x_i = \frac{b_i}{a_i}, \quad i = 1,\dots,n;$$

we get

$$\begin{aligned}
\|\bar{\mathbf{x}} - \bar{\mathbf{x}}_{\mu,p}\|_{\infty} &= \max_{i=\overline{1,n}} \left|x_i - \sum_{j=1}^n p_j x_j\right| \\
&= \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i} - \frac{\sum_{j=1}^n a_j b_j}{\sum_{k=1}^n a_k^2}\right| \\
&= \frac{1}{\sum_{k=1}^n a_k^2} \max_{i=\overline{1,n}} \left|\frac{b_i}{a_i}\sum_{k=1}^n a_k^2 - \sum_{j=1}^n a_j b_j\right|,
\end{aligned}$$

$$\begin{aligned}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_{\bar{\mathbf{p}},1} &= \sum_{i=1}^{n} p_i |x_i - c| \\ &= \sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \left| \frac{b_i}{a_i} - c \right| \\ &= \frac{1}{\sum_{k=1}^{n} a_k^2} \sum_{i=1}^{n} |a_i| |b_i - c a_i| ,\end{aligned}$$

$$\|\bar{\mathbf{x}}\|_{\bar{\mathbf{p}},1} = \sum_{i=1}^{n} p_i |x_i| = \sum_{i=1}^{n} \frac{a_i^2}{\sum_{k=1}^{n} a_k^2} \left| \frac{b_i}{a_i} \right| = \frac{1}{\sum_{k=1}^{n} a_k^2} \sum_{i=1}^{n} |a_i b_i|$$

and

$$\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_{\mu,p}\|_{\bar{\mathbf{p}},1} = \frac{1}{\left(\sum_{k=1}^{n} a_k^2\right)^2} \sum_{i=1}^{n} |a_i| \left| \sum_{k=1}^{n} a_k \begin{vmatrix} a_k & a_i \\ b_k & b_i \end{vmatrix} \right| .$$

Utilising the inequality (5.120) we deduce (5.119). □

5.16. **Another Reverse Result via a Grüss Type Result.** The following Grüss type inequality has been obtained in [21].

Lemma 5.58. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers and assume that there are* $\gamma, \Gamma \in \mathbb{R}$ *such that*

(5.121) $$-\infty < \gamma \le a_i \le \Gamma < \infty \text{ for each } i \in \{1, \dots, n\} .$$

Then for any $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *a nonnegative sequence with the property that* $\sum_{i=1}^{n} p_i = 1$, *one has the inequality*

(5.122) $$\left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| \le \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} p_i \left| b_i - \sum_{k=1}^{n} p_k b_k \right| .$$

The constant $\frac{1}{2}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

Proof. We will give here a simpler direct proof based on Sonin's identity. A simple calculation shows that:

(5.123) $$\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i = \sum_{i=1}^{n} p_i \left(a_i - \frac{\gamma + \Gamma}{2} \right) \left(b_i - \sum_{k=1}^{n} p_k b_k \right) .$$

By (5.121) we have

$$\left| a_i - \frac{\gamma + \Gamma}{2} \right| \le \frac{\Gamma - \gamma}{2} \quad \text{for all} \quad i \in \{1, \dots, n\}$$

and thus, by (5.123), on taking the modulus, we get

$$\begin{aligned}\left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| &\le \sum_{i=1}^{n} p_i \left| a_i - \frac{\gamma + \Gamma}{2} \right| \left| b_i - \sum_{k=1}^{n} p_k b_k \right| \\ &\le \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} p_i \left| b_i - \sum_{k=1}^{n} p_k b_k \right| .\end{aligned}$$

To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (5.122) holds with a constant $c > 0$, i.e.,

$$\left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leq c\left(\Gamma - \gamma\right) \sum_{i=1}^{n} p_i \left|b_i - \sum_{k=1}^{n} p_k b_k\right|. \tag{5.124}$$

provided a_i satisfies (5.121).

If we choose $n = 2$ in (5.124) and take into account that

$$\sum_{i=1}^{2} p_i a_i b_i - \sum_{i=1}^{2} p_i a_i \sum_{i=1}^{2} p_i b_i = p_1 p_2 \left(a_1 - a_2\right)\left(b_1 - b_2\right)$$

provided $p_1 + p_2 = 1$, $p_1, p_2 \in [0, 1]$, and since

$$\begin{aligned}\sum_{i=1}^{2} p_i \left|b_i - \sum_{k=1}^{2} p_k b_k\right| &= p_1 \left|\left(p_1 + p_2\right) b_1 - p_1 b_1 - p_2 b_2\right| + p_2 \left|\left(p_1 + p_2\right) b_2 - p_1 b_1 - p_2 b_2\right| \\ &= 2 p_1 p_2 \left|b_1 - b_2\right|\end{aligned}$$

we deduce by (5.124)

$$p_1 p_2 \left|a_1 - a_2\right| \left|b_1 - b_2\right| \leq 2c\left(\Gamma - \gamma\right) \left|b_1 - b_2\right| p_1 p_2. \tag{5.125}$$

If we assume that $p_1, p_2 \neq 0$, $b_1 \neq b_2$ and $a_1 = \Gamma$, $a_2 = \gamma$, then by (5.125) we deduce $c \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. □

The following corollary is a natural consequence of the above lemma.

Corollary 5.59. *Assume that* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *satisfies the assumption (5.121) and* $\bar{\mathbf{p}}$ *is a probability sequence. Then*

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2 \leq \frac{1}{2}\left(\Gamma - \gamma\right) \sum_{i=1}^{n} p_i \left|a_i - \sum_{k=1}^{n} p_k a_k\right|. \tag{5.126}$$

The constant $\frac{1}{2}$ *is best possible in the sense mentioned above.*

The following reverse of the (CBS) −inequality may be stated.

Theorem 5.60. *Assume that* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *and* $\bar{\mathbf{y}} = (y_1, \dots, y_n)$ *are sequences of real numbers with* $y_i \neq 0$ $(i = 1, \dots, n)$. *If there exists the real numbers* m, M *such that*

$$m \leq \frac{x_i}{y_i} \leq M \text{ for each } i \in \{1, \dots, n\}, \tag{5.127}$$

then we have the inequality

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2 \\ &\leq \frac{1}{2}\left(M - m\right) \sum_{i=1}^{n} \left|y_i\right| \left|\sum_{k=1}^{n} y_k \cdot \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix}\right|. \end{aligned} \tag{5.128}$$

Proof. If we choose $p_i = \frac{y_i^2}{\sum_{k=1}^n y_k^2}$, $a_i = \frac{x_i}{y_i}$ for $i = 1, \dots, n$ and $\gamma = m$, $\Gamma = M$ in (5.126), we deduce

$$\begin{aligned}
&\frac{\sum_{i=1}^n x_i^2}{\sum_{k=1}^n y_k^2} - \left(\frac{1}{\sum_{k=1}^n y_k^2} \sum_{i=1}^n x_i y_i \right)^2 \\
&\qquad \leq \frac{1}{2} (M - m) \frac{1}{\sum_{k=1}^n y_k^2} \sum_{i=1}^n y_i^2 \left| \frac{x_i}{y_i} - \frac{1}{\sum_{k=1}^n y_k^2} \sum_{k=1}^n x_k y_k \right| \\
&\qquad = \frac{1}{2} (M - m) \frac{1}{\left(\sum_{k=1}^n y_k^2\right)^2} \sum_{i=1}^n |y_i| \left| x_i \sum_{k=1}^n y_k^2 - y_i \sum_{k=1}^n x_k y_k \right| \\
&\qquad = \frac{1}{2} (M - m) \frac{1}{\left(\sum_{k=1}^n y_k^2\right)^2} \sum_{i=1}^n |y_i| \left| \sum_{k=1}^n y_k \cdot \left| \begin{array}{cc} x_i & y_i \\ x_k & y_k \end{array} \right| \right| .
\end{aligned}$$

giving the desired inequality (5.128). □

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6. RELATED INEQUALITIES

6.1. **Ostrowski's Inequality for Real Sequences.** In 1951, A.M. Ostrowski [2, p. 289] gave the following result related to the (CBS) −inequality for real sequences (see also [1, p. 92]).

Theorem 6.1. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$ *and* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two non-proportional sequences of real numbers. Let* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *be a sequence of real numbers such that*

$$\sum_{i=1}^{n} a_i x_i = 0 \text{ and } \sum_{i=1}^{n} b_i x_i = 1. \tag{6.1}$$

Then

$$\sum_{i=1}^{n} x_i^2 \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \tag{6.2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2} \tag{6.3}$$

for any $k \in \{1, \dots, n\}$.

Proof. We shall follow the proof in [1, p. 93 – p. 94].

Let

$$A = \sum_{i=1}^{n} a_i^2, \quad B = \sum_{i=1}^{n} b_i^2, \quad C = \sum_{i=1}^{n} a_i b_i \tag{6.4}$$

and

$$y_i = \frac{Ab_i - Ca_i}{AB - C^2} \text{ for any } i \in \{1, \dots, n\}. \tag{6.5}$$

It is easy to see that the sequence $\bar{\mathbf{y}} = (y_1, \dots, y_n)$ as defined by (6.5) satisfies (6.1).

Any sequence $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ that satisfies (6.1) fulfills the equality

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i \cdot \frac{(Ab_i - Ca_i)}{AB - C^2} = \frac{A}{AB - C^2};$$

so, in particular

$$\sum_{i=1}^{n} y_i^2 = \frac{A}{AB - C^2}.$$

Any sequence $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ that satisfies (6.1) therefore satisfies

$$\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} (x_i - y_i)^2 \geq 0, \tag{6.6}$$

and thus

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} y_i^2 = \frac{A}{AB - C^2}$$

and the inequality (6.2) is proved.

From (6.6) it follows that equality holds in (6.1) iff $x_i = y_i$ for each $i \in \{1, \dots, n\}$, and the theorem is completely proved. □

6.2. **Ostrowski's Inequality for Complex Sequences.** The following result that points out a natural generalisation of Ostrowski's inequality for complex numbers holds [3].

Theorem 6.2. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *and* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *be sequences of complex numbers. If* $\bar{\mathbf{a}}$ *and* $\bar{\bar{\mathbf{b}}}$*, where* $\bar{\bar{\mathbf{b}}} = \left(\bar{b}_1, \dots, \bar{b}_n\right)$, *are not proportional and*

$$\sum_{i=1}^{n} x_i \bar{a}_i = 0; \tag{6.7}$$

$$\left| \sum_{i=1}^{n} x_i \bar{b}_i \right| = 1, \tag{6.8}$$

then one has the inequality

$$\sum_{i=1}^{n} |x_i|^2 \geq \frac{\sum_{i=1}^{n} |a_i|^2}{\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \left| \sum_{i=1}^{n} a_i \bar{b}_i \right|^2} \tag{6.9}$$

with equality iff

$$x_i = \mu \left[b_i - \frac{\sum_{k=1}^{n} b_k \bar{a}_k}{\sum_{k=1}^{n} |a_k|^2} \cdot a_i \right], \quad i \in \{1, \dots, n\} \tag{6.10}$$

and $\mu \in \mathbb{C}$ *with*

$$|\mu| = \frac{\sum_{k=1}^{n} |a_k|^2}{\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left| \sum_{k=1}^{n} a_k \bar{b}_k \right|^2}. \tag{6.11}$$

Proof. Recall the (CBS) $-$inequality for complex sequences

$$\sum_{k=1}^{n} |u_k|^2 \sum_{k=1}^{n} |v_k|^2 \geq \left| \sum_{k=1}^{n} u_k \bar{v}_k \right|^2 \tag{6.12}$$

with equality iff there is a complex number $\alpha \in \mathbb{C}$ such that

$$u_k = \alpha v_k, \quad k = 1, \dots, n. \tag{6.13}$$

If we apply (6.12) for

$$u_k = z_k - \frac{\sum_{i=1}^{n} z_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k,$$
$$v_k = d_k - \frac{\sum_{i=1}^{n} d_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k, \quad \text{where } \bar{\mathbf{c}} \neq 0 \text{ and } \bar{\mathbf{c}}, \bar{\mathbf{d}}, \bar{\mathbf{z}} \in \mathbb{C}^n,$$

we have

$$\sum_{k=1}^{n} \left| z_k - \frac{\sum_{i=1}^{n} z_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k \right|^2 \sum_{k=1}^{n} \left| d_k - \frac{\sum_{i=1}^{n} d_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k \right|^2 \geq \left| \sum_{k=1}^{n} \left(z_k - \frac{\sum_{i=1}^{n} z_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k \right) \left(\overline{d_k - \frac{\sum_{i=1}^{n} d_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k} \right) \right|^2 \tag{6.14}$$

with equality iff there is a $\beta \in \mathbb{C}$ such that

$$z_k = \frac{\sum_{i=1}^{n} z_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k + \beta \left(d_k - \frac{\sum_{i=1}^{n} d_i \bar{c}_i}{\sum_{i=1}^{n} |c_i|^2} \cdot c_k \right). \tag{6.15}$$

Since a simple calculation shows that

$$\sum_{k=1}^{n}\left|z_k-\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right|^2=\frac{\sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}z_k\bar{c}_k\right|^2}{\left(\sum_{k=1}^{n}|c_k|^2\right)^2},$$

$$\sum_{k=1}^{n}\left|d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right|^2=\frac{\sum_{k=1}^{n}|d_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}d_k\bar{c}_k\right|^2}{\left(\sum_{k=1}^{n}|c_k|^2\right)^2}$$

and

$$\sum_{k=1}^{n}\left(z_k-\frac{\sum_{i=1}^{n}z_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k\right)\left(\overline{d_k-\frac{\sum_{i=1}^{n}d_i\bar{c}_i}{\sum_{i=1}^{n}|c_i|^2}\cdot c_k}\right)$$
$$=\frac{\sum_{k=1}^{n}z_k\bar{d}_k\cdot\sum_{k=1}^{n}|c_k|^2-\sum_{k=1}^{n}z_k\bar{c}_k\cdot\sum_{k=1}^{n}c_k\bar{d}_k}{\left(\sum_{i=1}^{n}|c_i|^2\right)^2}$$

then by (6.12) we deduce

$$\left[\sum_{k=1}^{n}|z_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}z_k\bar{c}_k\right|^2\right]\left[\sum_{k=1}^{n}|d_k|^2\sum_{k=1}^{n}|c_k|^2-\left|\sum_{k=1}^{n}d_k\bar{c}_k\right|^2\right]$$
$$\geq\left|\sum_{k=1}^{n}z_k\bar{d}_k\cdot\sum_{k=1}^{n}|c_k|^2-\sum_{k=1}^{n}z_k\bar{c}_k\sum_{k=1}^{n}c_k\bar{d}_k\right|^2 \tag{6.16}$$

with equality iff there is a $\beta\in\mathbb{C}$ such that (6.15) holds.

If $\bar{\mathbf{a}},\bar{\mathbf{x}},\bar{\mathbf{b}}$ satisfy (6.7) and (6.8), then by (6.16) and (6.15) for the choices $\bar{\mathbf{z}}=\bar{\mathbf{x}}$, $\bar{\mathbf{c}}=\bar{\mathbf{a}}$ and $\bar{\mathbf{d}}=\bar{\mathbf{b}}$, we deduce (6.9) with equality iff there is a $\mu\in\mathbb{C}$ such that

$$x_k=\mu\left(b_k-\frac{\overline{\sum_{i=1}^{n}a_i\bar{b}_i}}{\sum_{i=1}^{n}|a_i|^2}\cdot a_k\right),$$

and, by (6.8),

$$\left|\mu\sum_{k=1}^{n}\left(b_k-\frac{\sum_{i=1}^{n}a_i\bar{b}_i}{\sum_{i=1}^{n}|a_i|^2}\cdot a_k\right)\cdot\bar{b}_k\right|=1. \tag{6.17}$$

Since (6.17) is clearly equivalent to (6.15), the theorem is completely proved. □

6.3. **Another Ostrowski's Inequality.** In his book from 1951, [2, p. 130], A.M. Ostrowski proved the following inequality as well (see also [1, p. 94]).

Theorem 6.3. *Let* $\bar{\mathrm{a}},\bar{\mathrm{b}},\bar{\mathrm{x}}$ *be sequences of real numbers so that* $\bar{\mathrm{a}}\neq 0$ *and*

$$\sum_{k=1}^{n}x_k^2=1 \tag{6.18}$$

$$\sum_{k=1}^{n}a_kx_k=0. \tag{6.19}$$

Then

$$\frac{\sum_{k=1}^{n}a_k^2\sum_{k=1}^{n}b_n^2-\left(\sum_{k=1}^{n}a_kb_k\right)^2}{\sum_{k=1}^{n}a_k^2}\geq\left(\sum_{k=1}^{n}b_kx_k\right)^2. \tag{6.20}$$

If $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *are non-proportional, then equality holds in (6.20) iff*

$$x_k = q \cdot \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n a_i b_i}{\left(\sum_{k=1}^n a_k^2\right)^{\frac{1}{2}} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2\right]^{\frac{1}{2}}}, \quad k \in \{1, \dots, n\}, \ q \in \{-1, 1\}. \tag{6.21}$$

We may extend this result for sequences of complex numbers as follows [4].

Theorem 6.4. *Let* $\bar{\mathrm{a}}, \bar{\mathrm{b}}, \bar{\mathrm{x}}$ *be sequences of complex numbers so that* $\bar{\mathrm{a}} \neq 0$, $\bar{\mathrm{a}}, \bar{\mathrm{b}}$ *are not proportional, and*

$$\sum_{k=1}^n |x_k|^2 = 1 \tag{6.22}$$

$$\sum_{k=1}^n x_k \bar{a}_k = 0. \tag{6.23}$$

Then

$$\frac{\sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 - \left|\sum_{k=1}^n a_k \bar{b}_k\right|^2}{\sum_{k=1}^n |a_k|^2} \geq \left|\sum_{k=1}^n x_k \bar{b}_k\right|^2. \tag{6.24}$$

The equality holds in (6.24) iff

$$x_k = \beta \left(b_k - \frac{\sum_{i=1}^n b_i \bar{a}_i}{\sum_{i=1}^n |a_i|^2} \cdot a_k\right), \quad k \in \{1, \dots, n\}; \tag{6.25}$$

where $\beta \in \mathbb{C}$ *is such that*

$$|\beta| = \frac{\left(\sum_{k=1}^n |a_k|^2\right)^{\frac{1}{2}}}{\left(\sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 - \left|\sum_{k=1}^n a_k \bar{b}_k\right|^2\right)^{\frac{1}{2}}}. \tag{6.26}$$

Proof. In Subsection 6.2, we proved the following inequality:

$$\left[\sum_{k=1}^n |z_k|^2 \sum_{k=1}^n |c_k|^2 - \left|\sum_{k=1}^n z_k \bar{c}_k\right|^2\right] \left[\sum_{k=1}^n |d_k|^2 \sum_{k=1}^n |c_k|^2 - \left|\sum_{k=1}^n d_k \bar{c}_k\right|^2\right] \geq \left|\sum_{k=1}^n z_k \bar{d}_k \cdot \sum_{k=1}^n |c_k|^2 - \sum_{k=1}^n z_k \bar{c}_k \sum_{k=1}^n c_k \bar{d}_k\right|^2 \tag{6.27}$$

for any $\bar{\mathbf{z}}, \bar{\mathbf{c}}, \bar{\mathbf{d}}$ sequences of complex numbers, with equality iff there is a $\beta \in \mathbb{C}$ such that

$$z_k = \frac{\sum_{i=1}^n z_i \bar{c}_i}{\sum_{i=1}^n |c_i|^2} \cdot c_k + \beta \left(d_k - \frac{\sum_{i=1}^n d_i \bar{c}_i}{\sum_{i=1}^n |c_i|^2} \cdot c_k\right) \tag{6.28}$$

for each $k \in \{1, \dots, n\}$.

If in (6.27) we choose $\bar{\mathbf{z}} = \bar{\mathbf{x}}$, $\bar{\mathbf{c}} = \bar{\mathbf{a}}$ and $\bar{\mathbf{d}} = \bar{\mathbf{b}}$ and take into consideration that (6.22) and (6.23) hold, then we get

$$\sum_{k=1}^n |x_k|^2 \sum_{k=1}^n |a_k|^2 \left[\sum_{k=1}^n |b_k|^2 \sum_{k=1}^n |a_k|^2 - \left|\sum_{k=1}^n b_k \bar{a}_k\right|^2\right] \geq \left|\sum_{k=1}^n x_k \bar{b}_k\right|^2 \left(\sum_{k=1}^n |a_k|^2\right)^2$$

which is clearly equivalent to (6.24).

By (6.28) the equality holds in (6.24) iff

$$x_k = \beta \left(b_k - \frac{\sum_{i=1}^n b_i \bar{a}_i}{\sum_{i=1}^n |a_i|^2} \cdot a_k \right), \quad k \in \{1, \dots, n\}.$$

Since $\bar{\mathrm{x}}$ should satisfy (6.22), we get

$$\begin{aligned} 1 &= \sum_{k=1}^n |x_k|^2 \\ &= |\beta|^2 \sum_{k=1}^n \left| b_k - \frac{\sum_{i=1}^n b_i \bar{a}_i}{\sum_{i=1}^n |a_i|^2} \cdot a_k \right|^2 \\ &= |\beta|^2 \left[\sum_{k=1}^n |b_k|^2 - \frac{\left|\sum_{k=1}^n b_k \bar{a}_k\right|^2}{\sum_{k=1}^n |a_k|^2} \right] \end{aligned}$$

from where we deduce that β satisfies (6.26). □

6.4. **Fan and Todd Inequalities.** In 1955, K. Fan and J. Todd [5] proved the following inequality (see also [1, p. 94]).

Theorem 6.5. *Let $\bar{\mathrm{a}} = (a_1, \dots, a_n)$ and $\bar{\mathrm{b}} = (b_1, \dots, b_n)$ be two sequences of real numbers such that $a_i b_j \neq a_j b_i$ for $i \neq j$. Then*

$$\frac{\sum_{i=1}^n a_i^2}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_i b_i\right)^2} \le \binom{n}{2} \sum_{i=1}^n \left(\sum_{\substack{j=1 \\ j\neq i}}^n \frac{a_j}{a_j b_i - a_i b_j} \right)^2. \tag{6.29}$$

Proof. We shall follow the proof in [1, p. 94 – p. 95].

Define

$$x_i := \binom{n}{2}^{-1} \sum_{j \neq i} \frac{a_j}{a_j b_i - a_i b_j} \qquad (1 \le i \le n).$$

The terms in the sum on the right-hand side

$$\sum_{i=1}^n x_i a_i = \binom{n}{2}^{-1} \sum_{i=1}^n \left(\sum_{\substack{j=1 \\ j\neq i}}^n \frac{a_i a_j}{a_j b_i - a_i b_j} \right)$$

can be grouped in pairs of the form

$$\binom{n}{2}^{-1} \left(\frac{a_i a_j}{a_j b_i - a_i b_j} + \frac{a_j a_i}{a_i b_j - a_j b_i} \right) \quad (i \neq j)$$

and the sum of each such pair vanishes.

Hence, we deduce

$$\sum_{i=1}^n a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^n b_i x_i = 1.$$

Applying Ostrowski's inequality (see Section 6.1) we deduce the desired result (6.29). □

A weighted version of the result is also due to K. Fan and J. Todd [5] (see also [1, p. 95]). We may state the result as follows.

Theorem 6.6. *Let* p_{ij} $(i,j \in \{1,\dots,n\},\ i \neq j)$ *be real numbers such that*

$$p_{ij} = p_{ji}, \quad \text{for any } i,j \in \{1,\dots,n\} \quad \text{with } i \neq j. \tag{6.30}$$

Denote $P := \sum_{1 \leq i < j \leq n} p_{ij}$ *and assume that* $P \neq 0$. *Then for any two sequences of real numbers* $\bar{\mathbf{a}} = (a_1,\dots,a_n)$ *and* $\bar{\mathbf{b}} = (b_1,\dots,b_n)$ *satisfying* $a_i b_j \neq a_j b_i$ $(i \neq j)$, *we have*

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2} \leq \frac{1}{P^2} \sum_{i=1}^n \left(\sum_{\substack{j=1 \\ j \neq i}}^n \frac{p_{ij} a_j}{a_j b_i - a_i b_j} \right)^2. \tag{6.31}$$

6.5. **Some Results for Asynchronous Sequences.** If $S(\mathbb{R})$ is the linear space of real sequences, $S_+(\mathbb{R})$ is the subset of nonnegative sequences and $\mathcal{P}_f(\mathbb{N})$ denotes the set of finite parts of $\mathbb{N}$, then for the functional $T : \mathcal{P}_f(\mathbb{N}) \times S_+(\mathbb{R}) \times S^2(\mathbb{R}) \to \mathbb{R}$,

$$T\left(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) := \left(\sum_{i \in I} p_i a_i^2 \sum_{i \in I} p_i b_i^2 \right)^{\frac{1}{2}} - \left| \sum_{i \in I} p_i a_i b_i \right| \tag{6.32}$$

we may state the following result [6, Theorem 3].

Theorem 6.7. *If* $|\bar{\mathbf{a}}| = (|a_i|)_{i \in \mathbb{N}}$ *and* $\left|\bar{\mathbf{b}}\right| = (|b_i|)_{i \in \mathbb{N}}$ *are asynchronous, i.e.,*

$$(|a_i| - |a_j|)(|b_i| - |b_j|) \leq 0$$

for all $i,j \in \mathbb{N}$, *then*

$$T\left(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \geq \frac{\sum_{i \in I} p_i |a_i| \sum_{i \in I} p_i |b_i|}{\sum_{i \in I} p_i} - \sum_{i \in I} p_i |a_i b_i| \geq 0. \tag{6.33}$$

Proof. We shall follow the proof in [6].

Consider the inequalities

$$\left(\sum_{i \in I} p_i \sum_{i \in I} p_i a_i^2 \right)^{\frac{1}{2}} \geq \sum_{i \in I} p_i |a_i|$$

and

$$\left(\sum_{i \in I} p_i \sum_{i \in I} p_i b_i^2 \right)^{\frac{1}{2}} \geq \sum_{i \in I} p_i |b_i|$$

which by multiplication give

$$\left(\sum_{i \in I} p_i a_i^2 \sum_{i \in I} p_i b_i^2 \right)^{\frac{1}{2}} \geq \frac{\sum_{i \in I} p_i |a_i| \sum_{i \in I} p_i |b_i|}{\sum_{i \in I} p_i}.$$

Now, by the definition of T and by Čebyšev's inequality for asynchronous sequences, we have

$$\begin{aligned} T\left(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) &\geq \frac{\sum_{i \in I} p_i |a_i| \sum_{i \in I} p_i |b_i|}{\sum_{i \in I} p_i} - \left| \sum_{i \in I} p_i a_i b_i \right| \\ &\geq \frac{\sum_{i \in I} p_i |a_i| \sum_{i \in I} p_i |b_i|}{\sum_{i \in I} p_i} - \sum_{i \in I} p_i |a_i| |b_i| \\ &\geq 0 \end{aligned}$$

and the theorem is proved. □

The following result also holds [6, Theorem 4].

Theorem 6.8. *If $|\bar{\mathbf{a}}|$ and $|\bar{\mathbf{b}}|$ are synchronous, i.e., $(|a_i| - |a_j|)(|b_i| - |b_j|) \geq 0$ for all $i, j \in \mathbb{N}$, then one has the inequality*

$$0 \leq T(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) \leq T(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}\bar{\mathbf{b}}, \mathbf{1}), \tag{6.34}$$

where $\mathbf{1} = (e_i)_{i\in\mathbb{N}}$, $e_i = 1,\ i \in \mathbb{N}$.

Proof. We have, by Čebyšev's inequality for the synchronous sequences $\bar{\mathbf{a}}^2 = (a_i^2)_{i\in\mathbb{N}}$ and $\bar{\mathbf{b}}^2 = (b_i^2)_{i\in\mathbb{N}}$, that

$$\begin{aligned} T(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) &= \left(\sum_{i\in I} p_i a_i^2 \sum_{i\in I} p_i b_i^2\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right| \\ &\leq \left(\sum_{i\in I} p_i a_i^2 b_i^2 \sum_{i\in I} p_i\right)^{\frac{1}{2}} - \left|\sum_{i\in I} p_i a_i b_i\right| \\ &= T(I, \bar{\mathbf{p}}, \bar{\mathbf{a}}\bar{\mathbf{b}}, \mathbf{1}) \end{aligned}$$

and the theorem is proved. □

6.6. **An Inequality via $A - G - H$ Mean Inequality.** The following result holds [6, Theorem 5].

Theorem 6.9. *Let $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ be sequences of positive real numbers. Define*

$$\Delta_i = \begin{vmatrix} a_i^2 & b_i^2 \\ \sum_{i\in I} a_i^2 & \sum_{i\in I} b_i^2 \end{vmatrix} \tag{6.35}$$

where $i \in I$ and I is a finite part of $\mathbb{N}$. Then one has the inequality

$$\begin{aligned} \frac{\left(\sum_{i\in I} a_i b_i\right)^2}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2} &\geq \left[\prod_{i\in I}\left(\frac{a_i}{b_i}\right)^{\Delta_i}\right]^{\frac{1}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}} \\ &\geq \frac{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}{\sum_{i\in I} \frac{a_i^3}{b_i} \sum_{i\in I} \frac{b_i^3}{a_i}}. \end{aligned} \tag{6.36}$$

The equality holds in all the inequalities from (6.36) iff there exists a positive number $k > 0$ such that $a_i = kb_i$ for all $i \in I$.

Proof. We shall follow the proof in [6].

We will use the $AGH-$inequality

$$\frac{1}{P_I} \sum_{i\in I} p_i x_i \geq \left(\prod_{i\in I} x_i^{p_i}\right)^{\frac{1}{P_I}} \geq \frac{P_I}{\sum_{i\in I} \frac{p_i}{x_i}}, \tag{6.37}$$

where $p_i > 0$, $x_i \geq 0$ for all $i \in I$, where $P_I := \sum_{i\in I} p_i > 0$.

We remark that the equality holds in (6.37) iff $x_i = x_j$ for each $i, j \in I$.

Choosing $p_i = a_i^2$ and $x_i = \frac{b_i}{a_i}$ $(i \in I)$ in (6.37), then we get

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} a_i^2} \geq \prod_{i\in I}\left(\frac{b_i}{a_i}\right)^{\frac{a_i^2}{\sum_{i\in I} a_i^2}} \geq \frac{\sum_{i\in I} a_i^2}{\sum_{i\in I} \frac{a_i^3}{b_i}} \tag{6.38}$$

and by $p_i = b_i^2$ and $x_i = \frac{a_i}{b_i}$, we also have

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} b_i^2} \geq \prod_{i\in I} \left(\frac{a_i}{b_i}\right)^{\frac{b_i^2}{\sum_{i\in I} b_i^2}} \geq \frac{\sum_{i\in I} b_i^2}{\sum_{i\in I} \frac{b_i^3}{a_i}}. \tag{6.39}$$

If we multiply (6.38) with (6.39) we easily deduce the desired inequality (6.36).

The case of equality follows by the same case in the arithmetic mean – geometric mean – harmonic mean inequality. We omit the details. □

The following corollary holds [6, Corollary 5.1].

Corollary 6.10. *With* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *as above, one has the inequality*

$$\left[\frac{\sum_{i\in I} \frac{a_i^3}{b_i} \sum_{i\in I} \frac{b_i^3}{a_i}}{\left(\sum_{i\in I} a_i b_i\right)^2}\right]^{\frac{1}{2}} \geq \frac{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}{\left(\sum_{i\in I} a_i b_i\right)^2}. \tag{6.40}$$

The equality holds in (6.40) iff there is a $k > 0$ *such that* $a_i = kb_i$, $i \in \{1, \dots, n\}$.

6.7. **A Related Result via Jensen's Inequality for Power Functions.** The following result also holds [6, Theorem 6].

Theorem 6.11. *Let* $\bar{\mathrm{a}}$ *and* $\bar{\mathrm{b}}$ *be sequences of positive real numbers and* $p \geq 1$. *If* I *is a finite part of* $\mathbb{N}$, *then one has the inequality*

$$\frac{\left(\sum_{i\in I} a_i b_i\right)^2}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2} \leq \left[\frac{\sum_{i\in I} a_i^{2-p} b_i^p \sum_{i\in I} a_i^p b_i^{2-p}}{\sum_{i\in I} a_i^2 \sum_{i\in I} b_i^2}\right]^{\frac{1}{p}}. \tag{6.41}$$

The equality holds in (6.41) if and only if there exists a $k > 0$ *such that* $a_i = kb_i$ *for all* $i \in I$.

If $p \in (0, 1)$, *the inequality in (6.41) reverses.*

Proof. We shall follow the proof in [6].

By Jensen's inequality for the convex mapping $f : \mathbb{R}_+ \to \mathbb{R}$,

$$f(x) = x^p, \quad p \geq 1$$

one has

$$\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)^p \leq \frac{\sum_{i\in I} p_i x_i^p}{P_I}, \tag{6.42}$$

where $P_I := \sum_{i\in I} p_i$, $p_i > 0$, $x_i \geq 0$, $i \in I$. The equality holds in (6.42) iff $x_i = x_j$ for all $i, j \in I$.

Now, choosing in (6.42) $p_i = a_i^2$, $x_i = \frac{b_i}{a_i}$, we get

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} a_i^2} \leq \left(\frac{\sum_{i\in I} a_i^{2-p} b_i^p}{\sum_{i\in I} a_i^2}\right)^{\frac{1}{p}} \tag{6.43}$$

and for $p_i = b_i^2$, $x_i = \frac{a_i}{b_i}$, the inequality (6.42) also gives

$$\frac{\sum_{i\in I} a_i b_i}{\sum_{i\in I} b_i^2} \leq \left(\frac{\sum_{i\in I} a_i^p b_i^{2-p}}{\sum_{i\in I} b_i^2}\right)^{\frac{1}{p}}. \tag{6.44}$$

By multiplying the inequalities (6.43) and (6.44), we deduce the desired result from (6.42).

The case of equality follows by the fact that in (6.42) the equality holds iff $(x_i)_{i\in I}$ is constant.

If $p \in (0, 1)$, then a reverse inequality holds in (6.42) giving the corresponding result in (6.41). □

Remark 6.12. If $p=2$, then (6.41) becomes the $(CBS)-$inequality.

6.8. **Inequalities Derived from the Double Sums Case.** Let $A=(a_{ij})_{i,j=\overline{1,n}}$ and $B=(b_{ij})_{i,j=\overline{1,n}}$ be two matrices of real numbers. The following inequality is known as the $(CBS)-$inequality for double sums

$$\left(\sum_{i,j=1}^{n} a_{ij}b_{ij}\right)^2 \le \sum_{i,j=1}^{n} a_{ij}^2 \sum_{i,j=1}^{n} b_{ij}^2 \tag{6.45}$$

with equality iff there is a real number r such that $a_{ij}=rb_{ij}$ for any $i,j\in\{1,\dots,n\}$.

The following inequality holds [7, Theorem 5.2].

Theorem 6.13. *Let* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$ *and* $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ *be sequences of real numbers. Then*

$$\left|\left(\sum_{k=1}^{n} a_k\right)^2+\left(\sum_{k=1}^{n} b_k\right)^2-2n\sum_{k=1}^{n} a_kb_k\right| \le n\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)-2\sum_{k=1}^{n} a_k\sum_{k=1}^{n} b_k. \tag{6.46}$$

Proof. We shall follow the proof from [7].

Applying (6.45) for $a_{ij}=a_i-b_j$, $b_{ij}=b_i-a_j$ and taking into account that

$$\sum_{i,j=1}^{n}(a_i-b_j)(b_i-a_j)=2n\sum_{k=1}^{n} a_kb_k-\left(\sum_{k=1}^{n} a_k\right)^2-\left(\sum_{k=1}^{n} b_k\right)^2$$

$$\sum_{i,j=1}^{n}(a_i-b_j)^2=n\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)-2\sum_{k=1}^{n} a_k\sum_{k=1}^{n} b_k$$

and

$$\sum_{i,j=1}^{n}(b_i-a_j)^2=n\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)-2\sum_{k=1}^{n} a_k\sum_{k=1}^{n} b_k,$$

we may deduce the desired inequality (6.46). □

The following result also holds [7, Theorem 5.3].

Theorem 6.14. *Let* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{c}}=(c_1,\dots,c_n)$ *and* $\bar{\mathbf{d}}=(d_1,\dots,d_n)$ *be sequences of real numbers. Then one has the inequality:*

$$\left[\det\begin{bmatrix}\sum_{i=1}^n a_ic_i & \sum_{i=1}^n a_id_i\\ \sum_{i=1}^n b_ic_i & \sum_{i=1}^n b_id_i\end{bmatrix}\right]^2 \le \det\begin{bmatrix}\sum_{i=1}^n a_i^2 & \sum_{i=1}^n a_ib_i\\ \sum_{i=1}^n a_ib_i & \sum_{i=1}^n b_i^2\end{bmatrix}\times\det\begin{bmatrix}\sum_{i=1}^n c_i^2 & \sum_{i=1}^n c_id_i\\ \sum_{i=1}^n c_id_i & \sum_{i=1}^n d_i^2\end{bmatrix}. \tag{6.47}$$

Proof. We shall follow the proof in [7].

Applying (6.45) for $a_{ij}=a_ib_j-a_jb_i$, $b_{ij}=c_id_j-c_jd_i$ and using Cauchy-Binet's identity [1, p. 85]

$$\frac{1}{2}\sum_{i,j=1}^{n}(a_ib_j-a_jb_i)(c_id_j-c_jd_i)=\sum_{i=1}^{n} a_ic_i\sum_{i=1}^{n} b_id_i-\left(\sum_{i=1}^{n} a_id_i\right)\left(\sum_{i=1}^{n} b_ic_i\right) \tag{6.48}$$

and Lagrange's identity [1, p. 84]

$$\frac{1}{2}\sum_{i,j=1}^{n}\left(a_ib_j-a_jb_i\right)^2=\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2, \tag{6.49}$$

we deduce the desired result (6.47). □

6.9. **A Functional Generalisation for Double Sums.** The following result holds [7, Theorem 5.5].

Theorem 6.15. *Let A be a subset of real numbers $\mathbb{R}$, $f : A \to \mathbb{R}$ and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ sequences of real numbers with the property that*

(i) $a_kb_i, a_i^2, b_k^2 \in A$ *for any* $i, k \in \{1, \dots, n\}$;
(ii) $f\left(a_k^2\right), f\left(b_k^2\right) \geq 0$ *for any* $k \in \{1, \dots, n\}$;
(iii) $f^2\left(a_kb_i\right) \leq f\left(a_k^2\right) f\left(b_i^2\right)$ *for any* $i, k \in \{1, \dots, n\}$.

Then one has the inequality

$$\left[\sum_{k,i=1}^{n} f\left(a_kb_i\right)\right]^2 \leq n^2\sum_{k=1}^{n} f\left(a_k^2\right)\sum_{k=1}^{n} f\left(b_k^2\right). \tag{6.50}$$

Proof. We will follow the proof in [7].

Using the assumption (iii) and the (CBS) $-$inequality for double sums, we have

$$\begin{aligned}\left|\sum_{k,i=1}^{n} f\left(a_kb_i\right)\right| &\leq \sum_{k,i=1}^{n}\left|f\left(a_kb_i\right)\right| \\ &\leq \sum_{k,i=1}^{n}\left[f\left(a_k^2\right)f\left(b_i^2\right)\right]^{\frac{1}{2}} \\ &\leq \left\{\left(\sum_{k,i=1}^{n}\left[f\left(a_k^2\right)\right]^{\frac{1}{2}}\right)^2\left(\sum_{k,i=1}^{n}\left[f\left(b_i^2\right)\right]^{\frac{1}{2}}\right)^2\right\}^{\frac{1}{2}} \\ &= \left[\sum_{k,i=1}^{n} f\left(a_k^2\right)\sum_{k,i=1}^{n} f\left(b_i^2\right)\right]^{\frac{1}{2}} \\ &= n\left[\sum_{k=1}^{n} f\left(a_k^2\right)\right]^{\frac{1}{2}}\left[\sum_{k=1}^{n} f\left(b_k^2\right)\right]^{\frac{1}{2}}\end{aligned} \tag{6.51}$$

which is clearly equivalent to (6.50). □

The following corollary is a natural consequence of the above theorem [7, Corollary 5.6].

Corollary 6.16. *Let A, f and $\bar{\mathbf{a}}$ be as above. If*

(i) $a_ka_i \in A$ *for any* $i, k \in \{1, \dots, n\}$;
(ii) $f\left(a_k^2\right) \geq 0$ *for any* $k \in \{1, \dots, n\}$;
(iii) $f^2\left(a_ka_i\right) \leq f\left(a_k^2\right) f\left(a_i^2\right)$ *for any* $i, k \in \{1, \dots, n\}$,

then one has the inequality

$$\left|\sum_{k,i=1}^{n} f\left(a_ka_i\right)\right| \leq n\sum_{k=1}^{n} f\left(a_k^2\right). \tag{6.52}$$

The following particular inequalities also hold [7, p. 23].

(1) If $\varphi : \mathbb{N} \to \mathbb{N}$ is Euler's indicator and $s(n)$ denotes the sum of all relatively prime numbers including and less than n, then for any $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ sequences of natural numbers, one has the inequalities

$$\left[\sum_{k,i=1}^{n} \varphi(a_k b_i)\right]^2 \le n^2 \sum_{k=1}^{n} \varphi\left(a_k^2\right) \sum_{k=1}^{n} \varphi\left(b_k^2\right); \tag{6.53}$$

$$\sum_{k,i=1}^{n} \varphi(a_k a_i) \le n \sum_{k=1}^{n} \varphi\left(a_k^2\right); \tag{6.54}$$

$$\left[\sum_{k,i=1}^{n} s(a_k b_i)\right]^2 \le n^2 \sum_{k=1}^{n} s\left(a_k^2\right) \sum_{k=1}^{n} s\left(b_k^2\right); \tag{6.55}$$

$$\sum_{k,i=1}^{n} s(a_k a_i) \le n \sum_{k=1}^{n} s\left(a_k^2\right). \tag{6.56}$$

(2) If $a > 1$ and $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ are sequences of real numbers, then

$$\left[\sum_{k,i=1}^{n} \exp_a(a_k b_i)\right]^2 \le n^2 \sum_{k=1}^{n} \exp_a\left(a_k^2\right) \sum_{k=1}^{n} \exp_a\left(b_k^2\right); \tag{6.57}$$

$$\sum_{k,i=1}^{n} \exp_a(a_k a_i) \le n \sum_{k=1}^{n} \exp_a\left(a_k^2\right); \tag{6.58}$$

(3) If $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are sequences of real numbers such that $a_k, b_k \in (-1, 1)$ $(k \in \{1, \dots, n\})$, then one has the inequalities:

$$\left[\sum_{k,i=1}^{n} \frac{1}{(1 - a_k b_i)^m}\right]^2 \le n^2 \sum_{k=1}^{n} \frac{1}{(1 - a_k^2)^m} \sum_{k=1}^{n} \frac{1}{(1 - b_k^2)^m}, \tag{6.59}$$

$$\sum_{k,i=1}^{n} \frac{1}{(1 - a_k a_i)^m} \le n \sum_{k=1}^{n} \frac{1}{(1 - a_k^2)^m}, \tag{6.60}$$

where $m > 0$.

6.10. **A** (CBS) −**Type Result for Lipschitzian Functions.** The following result was obtained in [8, Theorem].

Theorem 6.17. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a Lipschitzian function with the constant M, i.e., it satisfies the condition*

$$|f(x) - f(y)| \le M |x - y| \text{ for any } x, y \in I. \tag{6.61}$$

If $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, $\bar{\mathbf{b}} = (b_1, \ldots, b_n)$ *are sequences of real numbers with* $a_i b_j \in I$ *for any* $i, j \in \{1, \ldots, n\}$, *then*

$$0 \le \left| \sum_{i,j=1}^{n} f(a_i b_j) |f(a_i b_j)| - \sum_{i,j=1}^{n} |f(a_j b_i)| f(a_i b_j) \right| \tag{6.62}$$
$$\le \sum_{i,j=1}^{n} f^2(a_j b_i) - \sum_{i,j=1}^{n} f(a_i b_j) f(a_j b_i)$$
$$\le M^2 \left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \right].$$

Proof. We shall follow the proof in [8].

Since f is Lipschitzian with the constant M, we have

$$0 \le ||f(a_i b_j)| - |f(a_j b_i)|| \le |f(a_i b_j) - f(a_j b_i)| \le M |a_i b_j - a_j b_i| \tag{6.63}$$

for any $i, j \in \{1, \ldots, n\}$, giving

$$0 \le |(|f(a_i b_j)| - |f(a_j b_i)|)(f(a_i b_j) - f(a_j b_i))| \tag{6.64}$$
$$\le (f(a_i b_j) - f(a_j b_i))^2 \le M^2 (a_i b_j - a_j b_i)^2$$

for any $i, j \in \{1, \ldots, n\}$.

The inequality (6.64) is obviously equivalent to

$$\big| |f(a_i b_j)| f(a_i b_j) + |f(a_j b_i)| f(a_j b_i) - |f(a_i b_j)| f(a_i b_j) - |f(a_j b_i)| f(a_j b_i) \big| \tag{6.65}$$
$$\le f^2(a_i b_j) - 2f(a_i b_j) f(a_j b_i) + f^2(a_j b_i)$$
$$\le M^2 \left(a_i^2 b_j^2 - 2a_i b_i a_j b_j + a_j^2 b_i^2 \right)$$

for any $i, j \in \{1, \ldots, n\}$. Summing over i and j from 1 to n in (6.65) and taking into account that:

$$\sum_{i,j=1}^{n} |f(a_i b_j)| f(a_i b_j) = \sum_{i,j=1}^{n} |f(a_j b_i)| f(a_i b_j),$$
$$\sum_{i,j=1}^{n} |f(a_i b_j)| f(a_j b_i) = \sum_{i,j=1}^{n} |f(a_j b_i)| f(a_i b_j),$$
$$\sum_{i,j=1}^{n} f^2(a_i b_j) = \sum_{i,j=1}^{n} f^2(a_j b_i),$$

we deduce the desired inequality. □

The following particular inequalities hold [8, p. 27 – p. 28].

(1) Let $\bar{\mathbf{x}} = (x_1, \ldots, x_n)$, $\bar{\mathbf{y}} = (y_1, \ldots, y_n)$ be sequences of real numbers such that $0 \le |x_i| \le M_1$, $0 \le |y_i| \le M_2$, $i \in \{1, \ldots, n\}$. Then for any $r \ge 1$ one has

$$0 \le \sum_{i=1}^{n} x_i^{2r} \sum_{i=1}^{n} y_i^{2r} - \left(\sum_{i=1}^{n} |x_i y_i|^r \right)^2 \tag{6.66}$$
$$\le r^2 (M_1 M_2)^{2(r-1)} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} |x_i y_i| \right)^2 \right].$$

(2) If $0 < m_1 \leq |x_i|$, $0 < m_2 \leq |y_i|$, $i \in \{1, \dots, n\}$ and $r \in (0,1)$, then

$$0 \leq \sum_{i=1}^{n} x_i^{2r} \sum_{i=1}^{n} y_i^{2r} - \left(\sum_{i=1}^{n} |x_i y_i|^r\right)^2 \leq \frac{r^2}{(m_1 m_2)^{2(r-1)}} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} |x_i y_i|\right)^2\right]. \tag{6.67}$$

(3) If $0 \leq |x_i| \leq M_1$, $0 \leq |y_i| \leq M_2$, $i \in \{1, \dots, n\}$, then for any natural number k one has

$$\begin{aligned} 0 &\leq \left| \sum_{i=1}^{n} x_i^{2k+1} |x_i|^{2k+1} \sum_{i=1}^{n} y_i^{2k+1} |y_i|^{2k+1} \right. \\ &\qquad \left. - \sum_{i=1}^{n} x_i^{2k+1} |y_i|^{2k+1} \sum_{i=1}^{n} y_i^{2k+1} |x_i|^{2k+1} \right| \\ &\leq \sum_{i=1}^{n} x_i^{2(2k+1)} \sum_{i=1}^{n} y_i^{2(2k+1)} - \left(\sum_{i=1}^{n} x_i^{2k+1} y_i^{2k+1}\right)^2 \\ &\leq (2k+1)^2 (M_1 M_2)^{4k} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} |x_i y_i|\right)^2\right]. \end{aligned} \tag{6.68}$$

(4) If $0 < m_1 \leq x_i$, $0 < m_2 \leq y_i$, for any $i \in \{1, \dots, n\}$, then one has the inequality

$$0 \leq n \sum_{i=1}^{n} \left[\ln\left(\frac{x_i}{y_i}\right)\right]^2 - \left[\sum_{i=1}^{n} \ln\left(\frac{x_i}{y_i}\right)\right]^2 \leq \frac{1}{(m_1 m_2)^2} \left[\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2\right]. \tag{6.69}$$

6.11. **An Inequality via Jensen's Discrete Inequality.** The following result holds [9].

Theorem 6.18. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0$, $i \in \{1, \dots, n\}$. *If* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex (concave) function on* I *and* $\frac{b_i}{a_i} \in I$ *for each* $i \in \{1, \dots, n\}$, *and* $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *is a sequence of nonnegative real numbers, then*

$$f\left(\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}\right) \leq (\geq) \frac{\sum_{i=1}^{n} w_i a_i^2 f\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^{n} w_i a_i^2}. \tag{6.70}$$

Proof. We shall use Jensen's discrete inequality for convex (concave) functions

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \leq (\geq) \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i), \tag{6.71}$$

where $p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$ and $x_i \in I$ for each $i \in \{1, \dots, n\}$.

If in (6.71) we choose $x_i = \frac{b_i}{a_i}$ and $p_i = w_i a_i^2$, then by (6.71) we deduce the desired result (6.70). □

The following corollary holds [9].

Corollary 6.19. *Let* $\bar{\mathbf{a}}$ *and* $\bar{\mathbf{b}}$ *be sequences of positive real numbers and assume that* $\bar{\mathbf{w}}$ *is as above. If* $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$, *then one has the inequality*

$$\left(\sum_{i=1}^{n} w_i a_i b_i \right)^p \leq (\geq) \left(\sum_{i=1}^{n} w_i a_i^2 \right)^{p-1} \sum_{i=1}^{n} w_i a_i^{2-p} b_i^p. \tag{6.72}$$

Proof. Follows by Theorem 6.18 applied for convex (concave) function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$, $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$. □

Remark 6.20. If $p = 2$, then by (6.72) we deduce the $(CBS)-$inequality.

6.12. **An Inequality via Lah-Ribarić Inequality.** The following reverse of Jensen's discrete inequality was obtained in 1973 by Lah and Ribarić [10].

Lemma 6.21. *Let* $f : I \to \mathbb{R}$ *be a convex function,* $x_i \in [m, M] \subseteq I$ *for each* $i \in \{1, \dots, n\}$ *and* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *be a positive* $n-$*tuple. Then*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq \frac{M - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i}{M - m} f(m) + \frac{\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i - m}{M - m} f(M). \tag{6.73}$$

Proof. We observe for each $i \in \{1, \dots, n\}$, that

$$x_i = \frac{(M - x_i) m + (x_i - m) M}{M - m}. \tag{6.74}$$

If in the definition of convexity, i.e., $\alpha, \beta \geq 0$, $\alpha + \beta > 0$

$$f\left(\frac{\alpha a + \beta b}{\alpha + \beta} \right) \leq \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} \tag{6.75}$$

we choose $\alpha = M - x_i$, $\beta = x_i - m$, $a = m$ and $b = M$, we deduce, by (6.75), that

$$\begin{aligned} f(x_i) &= f\left(\frac{(M - x_i) m + (x_i - m) M}{M - m} \right) \\ &\leq \frac{(M - x_i) f(m) + (x_i - m) f(M)}{M - m} \end{aligned} \tag{6.76}$$

for each $i \in \{1, \dots, n\}$.

If we multiply (6.76) by $p_i > 0$ and sum over i from 1 to n, we deduce (6.73). □

The following result holds.

Theorem 6.22. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \neq 0$, $i \in \{1, \dots, n\}$. *If* $I \subseteq \mathbb{R} \to \mathbb{R}$ *is a convex (concave) function on* I *and* $\frac{b_i}{a_i} \in [m, M] \subseteq I$ *for each* $i \in \{1, \dots, n\}$ *and* $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *is a sequence of nonnegative real numbers, then*

$$\frac{\sum_{i=1}^{n} w_i a_i^2 f\left(\frac{b_i}{a_i} \right)}{\sum_{i=1}^{n} w_i a_i^2} \leq (\geq) \frac{M - \frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}}{M - m} f(m) + \frac{\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2} - m}{M - m} f(M). \tag{6.77}$$

Proof. Follows by Lemma 6.21 for the choices $p_i = w_i a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \dots, n\}$. □

The following corollary holds.

Corollary 6.23. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of positive real numbers and such that*

$$0 < m \leq \frac{b_i}{a_i} \leq M < \infty \text{ for each } i \in \{1, \dots, n\}. \tag{6.78}$$

If $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *is a sequence of nonnegative real numbers and* $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$ *, then one has the inequality*

$$\sum_{i=1}^{n} w_i a_i^{2-p} b_i^p + \frac{Mm\left(M^{p-1} - m^{p-1}\right)}{M - m} \sum_{i=1}^{n} w_i a_i^2 \le (\ge) \frac{M^p - m^p}{M - m} \sum_{i=1}^{n} w_i a_i b_i. \tag{6.79}$$

Proof. If we write the inequality (6.77) for the convex (concave) function $f(x) = x^p$, $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$, we get

$$\frac{\sum_{i=1}^{n} w_i a_i^{2-p} b_i^p}{\sum_{i=1}^{n} w_i a_i^2} \le (\ge) \frac{M - \frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}}{M - m} \cdot m^p + \frac{\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2} - m}{M - m} \cdot M^p,$$

which, after elementary calculations, is equivalent to (6.79). □

Remark 6.24. For $p = 2$, we get

$$\sum_{i=1}^{n} w_i b_i^2 + Mm \sum_{i=1}^{n} w_i a_i^2 \le (M + m) \sum_{i=1}^{n} w_i a_i b_i, \tag{6.80}$$

which is the well known Diaz-Metcalf inequality [11].

6.13. **An Inequality via Dragomir-Ionescu Inequality.** The following reverse of Jensen's inequality was proved in 1994 by S.S. Dragomir and N.M. Ionescu [12].

Lemma 6.25. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex function on* $\mathring{I}$, $x_i \in \mathring{I}$ $(i \in \{1, \dots, n\})$ *and* $p_i \ge 0$ $(i \in \{1, \dots, n\})$ *such that* $P_n := \sum_{i=1}^{n} p_i > 0$. *Then one has the inequality*

$$\begin{aligned} 0 &\le \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \\ &\le \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i f'(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i f'(x_i) . \end{aligned} \tag{6.81}$$

Proof. Since f is differentiable convex on $\mathring{I}$, one has

$$f(x) - f(y) \ge (x - y) f'(y) , \tag{6.82}$$

for any $x, y \in \mathring{I}$.

If we choose $x = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$ and $y = y_k$, $k \in \{1, \dots, n\}$, we get

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) - f(y_k) \ge \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i - y_k\right) f'(y_k) . \tag{6.83}$$

Multiplying (6.83) by $p_k \ge 0$ and summing over k from 1 to n, we deduce the desired result (6.81). □

The following result holds [9].

Theorem 6.26. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of real numbers with* $a_i \ne 0$, $i \in \{1, \dots, n\}$ *. If* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable convex (concave) function on* $\mathring{I}$ *and* $\frac{b_i}{a_i} \in \mathring{I}$ *for each* $i \in \{1, \dots, n\}$ *, and* $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *is a sequence of nonnegative real* *numbers, then*

$$0 \le \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i^2 f\left(\frac{b_i}{a_i}\right) - \left(\sum_{i=1}^{n} w_i a_i^2\right)^2 \cdot f\left(\frac{\sum_{i=1}^{n} w_i a_i b_i}{\sum_{i=1}^{n} w_i a_i^2}\right) \le \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i b_i f'\left(\frac{b_i}{a_i}\right) - \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2 f'\left(\frac{b_i}{a_i}\right). \tag{6.84}$$

Proof. Follows from Lemma 6.25 on choosing $p_i = w_i a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \dots, n\}$. □

The following corollary holds [9].

Corollary 6.27. *Let* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$*,* $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *be two sequences of positive real numbers with* $a_i \neq 0$, $i \in \{1, \dots, n\}$ *. If* $p \in [1, \infty)$*, then one has the inequality*

$$0 \le \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i^{2-p} b_i^p - \left(\sum_{i=1}^{n} w_i a_i^2\right)^{2-p} \left(\sum_{i=1}^{n} w_i a_i b_i\right)^p \le p\left[\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i^{2-p} b_i - \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^{3-p} b_i^{p-1}\right]. \tag{6.85}$$

If $p \in (0, 1)$ *, then*

$$0 \le \left(\sum_{i=1}^{n} w_i a_i^2\right)^{2-p} \left(\sum_{i=1}^{n} w_i a_i b_i\right) - \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i^{2-p} b_i^p \le p\left[\sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^{3-p} b_i^p - \sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i a_i^{2-p} b_i\right]. \tag{6.86}$$

6.14. **An Inequality via a Refinement of Jensen's Inequality.** We will use the following lemma which contains a refinement of Jensen's inequality obtained in [13].

Lemma 6.28. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval* I *and* $x_i \in I$, $p_i \ge 0$ *with* $P_n := \sum_{i=1}^{n} p_i > 0$*. Then the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n^{k+1}} \sum_{i_1, \dots, i_{k+1}=1}^{n} p_{i_1} \cdots p_{i_k} f\left(\frac{x_{i_1} + \cdots + x_{i_{k+1}}}{k+1}\right) \le \frac{1}{P_n^{k}} \sum_{i_1, \dots, i_{k}=1}^{n} p_{i_1} \cdots p_{i_k} f\left(\frac{x_{i_1} + \cdots + x_{i_{k}}}{k}\right) \le \cdots \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{6.87}$$

where $k \ge 1$, $k \in \mathbb{N}$.

Proof. We shall follow the proof in [13].

The first inequality follows by Jensen's inequality for multiple sums

$$f\left(\frac{\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}}\right) = \frac{\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}}$$

since

$$\frac{\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right)}{\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}}} = P_n^k \sum_{i=1}^{n} p_i x_i$$

and

$$\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} = P_n^{k+1}.$$

Now, applying Jensen's inequality for

$$y_1 = \frac{x_{i_1}+x_{i_2}\cdots+x_{i_{k-1}}+x_{i_k}}{k},\quad y_2 = \frac{x_{i_2}+x_{i_3}\cdots+x_{i_k}+x_{i_{k+1}}}{k},\quad \dots,\quad y_{k+1} = \frac{x_{i_{k+1}}+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}}{k}$$

we have

$$f\left(\frac{y_1+y_2+\cdots+y_k+y_{k+1}}{k+1}\right) \le \frac{f(y_1)+f(y_2)+\cdots+f(y_k)+f(y_{k+1})}{k+1},$$

which is equivalent to

$$f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \le \frac{f\left(\frac{x_{i_1}+x_{i_2}\cdots+x_{i_{k-1}}+x_{i_k}}{k}\right)+\cdots+f\left(\frac{x_{i_{k+1}}+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}}{k}\right)}{k+1}. \tag{6.88}$$

Multiplying (6.88) with the nonnegative real numbers $p_{i_1},\dots,p_{i_{k+1}}$ and summing over $i_1,\dots,i_{k+1}$ from 1 to n we deduce

$$\begin{aligned}\sum_{i_1,\dots,i_{k+1}=1}^{n} & p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \\ &\le \frac{1}{k+1}\left[\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right. \\ &\qquad \left. +\cdots+\sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_{k+1}}+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}}{k}\right)\right] \\ &= P_n \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\end{aligned} \tag{6.89}$$

which proves the second part of (6.87). □

The following result holds.

Theorem 6.29. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ real numbers such that $a_i \neq 0$, $i \in \{1, \dots, n\}$ and $\frac{b_i}{a_i} \in I$, $i \in \{1, \dots, n\}$. If $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ are positive real numbers, then*

$$f\left(\frac{\sum_{i=1}^n w_i a_i b_i}{\sum_{i=1}^n w_i a_i^2}\right) \tag{6.90}$$
$$\leq \frac{1}{\left(\sum_{i=1}^n w_i a_i^2\right)^{k+1}} \sum_{i_1,\dots,i_{k+1}=1}^n w_{i_1} \cdots w_{i_{k+1}} a_{i_1}^2 \cdots a_{i_{k+1}}^2 f\left(\frac{\frac{b_{i_1}}{a_{i_1}} + \cdots + \frac{b_{i_{k+1}}}{a_{i_{k+1}}}}{k+1}\right)$$
$$\leq \frac{1}{\left(\sum_{i=1}^n w_i a_i^2\right)^{k}} \sum_{i_1,\dots,i_k=1}^n w_{i_1} \cdots w_{i_k} a_{i_1}^2 \cdots a_{i_k}^2 f\left(\frac{\frac{b_{i_1}}{a_{i_1}} + \cdots + \frac{b_{i_k}}{a_{i_k}}}{k}\right)$$
$$\leq \cdots \leq \frac{1}{\sum_{i=1}^n w_i a_i^2} \sum_{i=1}^n w_i a_i^2 f\left(\frac{b_i}{a_i}\right).$$

The proof is obvious by Lemma 6.28 applied for $p_i = w_i a_i^2$, $x_i = \frac{b_i}{a_i}$, $i \in \{1, \dots, n\}$.

The following corollary holds.

Corollary 6.30. *Let $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$ and $\bar{\mathbf{w}}$ be sequences of positive real numbers. If $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0, 1))$, then one has the inequalities*

$$\left(\sum_{i=1}^n w_i a_i b_i\right)^p$$
$$\underset{(\geq)}{\leq} \frac{\left(\sum_{i=1}^n w_i a_i^2\right)^{p-k-1}}{(k+1)^p} \sum_{i_1,\dots,i_{k+1}=1}^n w_{i_1} \cdots w_{i_{k+1}} a_{i_1}^2 \cdots a_{i_{k+1}}^2 \left(\frac{b_{i_1}}{a_{i_1}} + \cdots + \frac{b_{i_{k+1}}}{a_{i_{k+1}}}\right)^p$$
$$\underset{(\geq)}{\leq} \frac{\left(\sum_{i=1}^n w_i a_i^2\right)^{p-k}}{k^p} \sum_{i_1,\dots,i_k=1}^n w_{i_1} \cdots w_{i_k} a_{i_1}^2 \cdots a_{i_k}^2 \left(\frac{b_{i_1}}{a_{i_1}} + \cdots + \frac{b_{i_k}}{a_{i_k}}\right)^p$$
$$\underset{(\geq)}{\leq} \cdots \underset{(\geq)}{\leq} \left(\sum_{i=1}^n w_i a_i^2\right)^{p-1} \sum_{i=1}^n w_i a_i^{2-p} b_i^p.$$

Remark 6.31. If $p = 2$, then we deduce the following refinement of the $(CBS)-$ inequality

$$\left(\sum_{i=1}^n w_i a_i b_i\right)^2 \leq \frac{\left(\sum_{i=1}^n w_i a_i^2\right)^{1-k}}{(k+1)^2} \sum_{i_1,\dots,i_{k+1}=1}^n w_{i_1} \cdots w_{i_{k+1}} \left(\sum_{\ell=1}^{k+1} b_{i_\ell} \prod_{\substack{j=1 \\ j \neq \ell}}^{k+1} a_{ij}\right)^2$$
$$\leq \frac{\left(\sum_{i=1}^n w_i a_i^2\right)^{2-k}}{k^2} \sum_{i_1,\dots,i_k=1}^n w_{i_1} \cdots w_{i_k} \left(\sum_{\ell=1}^{k} b_{i_\ell} \prod_{\substack{j=1 \\ j \neq \ell}}^{k} a_{ij}\right)^2$$
$$\leq \cdots \leq \frac{1}{4} \sum_{i,j=1}^n w_i w_j \left(b_i a_j + a_i b_j\right)^2$$
$$\leq \sum_{i=1}^n w_i a_i^2 \sum_{i=1}^n w_i b_i^2.$$

6.15. **Another Refinement via Jensen's Inequality.** The following refinement of Jensen's inequality holds (see [15]).

Lemma 6.32. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable convex function on* (a,b) *and* $x_i\in(a,b)$, $p_i\geq 0$ *with* $P_n:=\sum_{i=1}^n p_i>0$. *Then one has the inequality*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\geq\left|\frac{1}{P_n}\sum_{i=1}^n p_i\left|f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right.$$
$$\left.-\left|f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\cdot\frac{1}{P_n}\sum_{i=1}^n p_i\left|x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right|\right|\geq 0. \tag{6.91}$$

Proof. Since f is differentiable convex on (a,b), then for each $x,y\in(a,b)$, one has the inequality

$$f(x)-f(y)\geq(x-y)f'(y). \tag{6.92}$$

Using the properties of the modulus, we have

$$\begin{aligned} f(x)-f(y)-(x-y)f'(y)&=|f(x)-f(y)-(x-y)f'(y)|\\ &\geq||f(x)-f(y)|-|x-y||f'(y)|| \end{aligned} \tag{6.93}$$

for each $x,y\in(a,b)$.

If we choose $y=\frac{1}{P_n}\sum_{j=1}^n p_j x_j$ and $x=x_i$, $i\in\{1,\dots,n\}$, then we have

$$\begin{aligned} &f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)-\left(x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\\ &\quad\geq\left|\left|f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|-\left|x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right|\left|f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right| \end{aligned} \tag{6.94}$$

for any $i\in\{1,\dots,n\}$.

If we multiply (6.94) by $p_i\geq 0$, sum over i from 1 to n, and divide by $P_n>0$, we deduce

$$\begin{aligned} &\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\\ &\qquad-\frac{1}{P_n}\sum_{i=1}^n p_i\left(x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\\ &\geq\frac{1}{P_n}\sum_{i=1}^n p_i\left|\left|f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right.\\ &\qquad\left.-\left|x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right|\left|f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right|\\ &\geq\left|\frac{1}{P_n}\sum_{i=1}^n p_i\left|f(x_i)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right.\\ &\qquad\left.-\frac{1}{P_n}\sum_{i=1}^n p_i\left|x_i-\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right|\cdot\left|f'\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right)\right|\right|. \end{aligned}$$

Since

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \left(x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j \right) = 0,$$

the inequality (6.91) is proved. □

In particular, we have the following result for unweighted means.

Corollary 6.33. *With the above assumptions for f and x_i, one has the inequality*

$$\begin{aligned}(6.95)\quad & \frac{f(x_1)+\cdots+f(x_n)}{n} - f\left(\frac{x_1+\cdots+x_n}{n}\right) \\ & \geq \left| \frac{1}{n}\sum_{i=1}^{n}\left| x_i - f\left(\frac{x_1+\cdots+x_n}{n}\right)\right| \right. \\ & \qquad \left. - \left| f'\left(\frac{x_1+\cdots+x_n}{n}\right)\right| \cdot \frac{1}{n}\sum_{i=1}^{n}\left| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right| \right| \geq 0.\end{aligned}$$

The following refinement of the (CBS) −inequality holds.

Theorem 6.34. *If $a_i, b_i \in \mathbb{R}$, $i \in \{1,\dots,n\}$, then one has the inequality;*

$$\begin{aligned}(6.96)\quad & \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \\ & \geq \frac{1}{\sum_{i=1}^{n} b_i^2}\left| \sum_{i=1}^{n} \left| \begin{vmatrix} a_i^2 & b_i^2 \\ \left(\sum_{j=1}^{n} a_j b_j\right)^2 & \left(\sum_{j=1}^{n} b_j^2\right)^2 \end{vmatrix} \right| \right. \\ & \qquad \left. -2\left|\sum_{k=1}^{n} a_k b_k\right| \cdot \sum_{i=1}^{n} |b_i| \left| \sum_{j=1}^{n} b_j \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \right| \right| \geq 0.\end{aligned}$$

Proof. If we apply Lemma 6.32 for $f(x) = x^2$, we get

$$\begin{aligned}(6.97)\quad & \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2 \\ & \geq \left| \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| x_i^2 - \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right)^2 \right| \right. \\ & \qquad \left. - 2\left|\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \left| x_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j \right| \right| \geq 0.\end{aligned}$$

If in (6.97), we choose $p_i = b_i^2$, $x_i = \frac{a_i}{b_i}$, $i \in \{1, \dots, n\}$, we get

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n b_i^2} - \frac{\left(\sum_{i=1}^n a_i b_i\right)^2}{\left(\sum_{i=1}^n b_i^2\right)^2} \geq \left| \frac{1}{\sum_{i=1}^n b_i^2} \sum_{i=1}^n b_i^2 \cdot \left| \frac{a_i^2}{b_i^2} - \left(\frac{\sum_{j=1}^n a_j b_j}{\sum_{j=1}^n b_j^2} \right)^2 \right| - 2 \left| \frac{\sum_{k=1}^n a_k b_k}{\sum_{i=1}^n b_i^2} \right| \cdot \frac{\sum_{i=1}^n b_i^2 \left| \frac{a_i}{b_i} - \sum_{j=1}^n a_j b_j \Big/ \sum_{j=1}^n b_j^2 \right|}{\sum_{i=1}^n b_i^2} \right|, \tag{6.98}$$

which is clearly equivalent to (6.96). □

6.16. **An Inequality via Slater's Result.** Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $D^- f(x) \leq D^+ f(x) \leq D^- f(y) \leq D^+ f(y)$ which shows that both $D^- f$ and $D^+ f$ are nondecreasing functions on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the *subdifferential* of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x - a)\varphi(a) \quad \text{for any} \quad x, a \in I. \tag{6.99}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $D^+ f, D^- f \in \partial f$ and if $\varphi \in \partial f$, then

$$D^- f(x) \leq \varphi(x) \leq D^+ f(x) \tag{6.100}$$

for every $x \in \mathring{I}$. In particular, φ is a nondecreasing function.

If f is differentiable convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following inequality is well known in the literature as Slater's inequality [16].

Lemma 6.35. *Let $f : I \to \mathbb{R}$ be a nondecreasing (nonincreasing) convex function, $x_i \in I$, $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $\sum_{i=1}^n p_i \varphi(x_i) \neq 0$ where $\varphi \in \partial f$. Then one has the inequality:*

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \right). \tag{6.101}$$

Proof. Firstly, observe that since, for example, f is nondecreasing, then $\varphi(x) \geq 0$ for any $x \in I$ and thus

$$\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \in I, \tag{6.102}$$

since it is a convex combination of x_i with the positive weights

$$\frac{x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)}, \quad i = 1, \dots, n.$$

A similar argument applies if f is nonincreasing.

Now, if we use the inequality (6.99), we deduce

$$f(x) - f(x_i) \geq (x - x_i)\varphi(x_i) \quad \text{for any } x, x_i \in I,\ i = 1, \dots, n. \tag{6.103}$$

Multiplying (6.103) by $p_i \geq 0$ and summing over i from 1 to n, we deduce

$$f(x) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \geq x \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \varphi(x_i) \tag{6.104}$$

for any $x \in I$.

If in (6.104) we choose

$$x = \frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)},$$

which, we have proved that it belongs to I, we deduce the desired inequality (6.101). □

If one would like to drop the assumption of monotonicity for the function f, then one can state and prove in a similar way the following result.

Lemma 6.36. *Let $f : I \to \mathbb{R}$ be a convex function, $x_i \in I$, $p_i \geq 0$ with $P_n > 0$ and $\sum_{i=1}^{n} p_i \varphi(x_i) \neq 0$, where $\varphi \in \partial f$. If*

$$\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I, \tag{6.105}$$

then the inequality (6.101) holds.

The following result in connection to the $(CBS)-$inequality holds.

Theorem 6.37. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is a convex function on $\mathbb{R}_+ := [0, \infty)$, $a_i, b_i \geq 0$ with $a_i \neq 0$, $i \in \{1, \dots, n\}$ and $\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right) \neq 0$ where $\varphi \in \partial f$.*

(i) *If f is monotonic nondecreasing (nonincreasing) in $[0, \infty)$ then*

$$\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right) \leq \sum_{i=1}^{n} a_i^2 \cdot f\left(\frac{\sum_{i=1}^{n} a_i b_i \varphi\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right)}\right). \tag{6.106}$$

(ii) *If*

$$\frac{\sum_{i=1}^{n} a_i b_i \varphi\left(\frac{b_i}{a_i}\right)}{\sum_{i=1}^{n} a_i^2 \varphi\left(\frac{b_i}{a_i}\right)} \geq 0, \tag{6.107}$$

then (6.106) also holds.

Remark 6.38. Consider the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$, $p \geq 1$. Then f is convex and monotonic nondecreasing and $\varphi(x) = px^{p-1}$. Applying (6.106), we may deduce the following inequality:

$$p\left(\sum_{i=1}^{n} a_i^{3-p} b_i^{p-1}\right)^{p+1} \leq \sum_{i=1}^{n} a_i^2 \left(\sum_{i=1}^{n} a_i^{2-p} b_i^{p}\right)^{p} \tag{6.108}$$

for $p \geq 1$, $a_i, b_i \geq 0$, $i = 1, \dots, n$.

6.17. **An Inequality via an Andrica-Raşa Result.** The following Jensen type inequality has been obtained in [17] by Andrica and Raşa.

Lemma 6.39. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function and assume that*

$$m=\inf_{t\in(a,b)}f''(t)>-\infty \text{ and } M=\sup_{t\in(a,b)}f''(t)<\infty.$$

If $x_i\in[a,b]$ and $p_i\geq 0$ $(i=1,\dots,n)$ with $\sum_{i=1}^n p_i=1$, then one has the inequalities:

$$\frac{1}{2}m\left[\sum_{i=1}^n p_i x_i^2-\left(\sum_{i=1}^n p_i x_i\right)^2\right]\leq\sum_{i=1}^n p_i f(x_i)-f\left(\sum_{i=1}^n p_i x_i\right)\leq\frac{1}{2}M\left[\sum_{i=1}^n p_i x_i^2-\left(\sum_{i=1}^n p_i x_i\right)^2\right]. \tag{6.109}$$

Proof. Consider the auxiliary function $f_m(t):=f(t)-\frac{1}{2}mt^2$. This function is twice differentiable and $f_m''(t)\geq 0$, $t\in(a,b)$, showing that f_m is convex.

Applying Jensen's inequality for f_m, i.e.,

$$\sum_{i=1}^n p_i f_m(x_i)\geq f_m\left(\sum_{i=1}^n p_i x_i\right),$$

we deduce, by a simple calculation, the first inequality in (6.109).

The second inequality follows in a similar way for the auxiliary function $f_M(t)=\frac{1}{2}Mt^2-f(t)$. We omit the details. □

The above result may be naturally used to obtain the following inequality related to the $(CBS)-$inequality.

Theorem 6.40. *Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ be two sequences of real numbers with the property that there exists $\gamma,\Gamma\in\mathbb{R}$ such that*

$$-\infty\leq\gamma\leq\frac{a_i}{b_i}\leq\Gamma\leq\infty, \text{ for each } i\in\{1,\dots,n\}, \tag{6.110}$$

and $b_i\neq 0$, $i=1,\dots,n$. If $f:(\gamma,\Gamma)\to\mathbb{R}$ is twice differentiable and

$$m=\inf_{t\in(\gamma,\Gamma)}f''(t) \text{ and } M=\sup_{t\in(\gamma,\Gamma)}f''(t),$$

then we have the inequality

$$\frac{1}{2}m\left[\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\left(\sum_{i=1}^n a_i b_i\right)^2\right]\leq\sum_{i=1}^n b_i^2\sum_{i=1}^n b_i^2 f\left(\frac{a_i}{b_i}\right)-\left(\sum_{i=1}^n b_i^2\right)^2 f\left(\frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n b_i^2}\right)\leq\frac{1}{2}M\left[\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\left(\sum_{i=1}^n a_i b_i\right)^2\right]. \tag{6.111}$$

Proof. We may apply Lemma 6.39 for the choices $p_i = \frac{b_i^2}{\sum_{k=1}^n b_k^2}$ and $x_i = \frac{a_i}{b_i}$ to get

$$\begin{aligned} &\frac{1}{2} m \left[\frac{\sum_{i=1}^n a_i^2}{\sum_{k=1}^n b_k^2} - \left(\frac{\sum_{i=1}^n a_i b_i}{\sum_{k=1}^n b_k^2} \right)^2 \right] \\ &\qquad \le \frac{\sum_{i=1}^n b_i^2 f\left(\frac{a_i}{b_i}\right)}{\sum_{k=1}^n b_k^2} - f\left(\frac{\sum_{i=1}^n a_i b_i}{\sum_{k=1}^n b_k^2} \right) \\ &\qquad \le \frac{1}{2} M \left[\frac{\sum_{i=1}^n a_i^2}{\sum_{k=1}^n b_k^2} - \left(\frac{\sum_{i=1}^n a_i b_i}{\sum_{k=1}^n b_k^2} \right)^2 \right] \end{aligned}$$

giving the desired result (6.111). □

The following corollary is a natural consequence of the above theorem.

Corollary 6.41. *Assume that* $\bar{\mathrm{a}}, \bar{\mathrm{b}}$ *are sequences of nonnegative real numbers and*

$$0 < \varphi \le \frac{a_i}{b_i} \le \Phi < \infty \quad \textit{for each} \quad i \in \{1, \dots, n\}. \tag{6.112}$$

Then for any $p \in [1, \infty)$ *one has the inequalities*

$$\begin{aligned} &\frac{1}{2} p (p-1) \varphi^{p-2} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 \right] \\ &\qquad \le \sum_{i=1}^n b_i^2 \left(\sum_{i=1}^n a_i^p b_i^{2-p} \right)^p - \left(\sum_{i=1}^n b_i^2 \right)^{2-p} \left(\sum_{i=1}^n a_i b_i \right)^p \\ &\qquad \le \frac{1}{2} p (p-1) \Phi^{p-2} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 \right] \end{aligned} \tag{6.113}$$

if $p \in [2, \infty)$ *and*

$$\begin{aligned} &\frac{1}{2} p (p-1) \Phi^{p-2} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 \right] \\ &\qquad \le \sum_{i=1}^n b_i^2 \left(\sum_{i=1}^n a_i^p b_i^{2-p} \right)^p - \left(\sum_{i=1}^n b_i^2 \right)^{2-p} \left(\sum_{i=1}^n a_i b_i \right)^p \\ &\qquad \le \frac{1}{2} p (p-1) \varphi^{p-2} \left[\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 \right] \end{aligned} \tag{6.114}$$

if $p \in [1, 2]$.

6.18. **An Inequality via Jensen's Result for Double Sums.** The following result for convex functions via Jensen's inequality also holds [18].

Lemma 6.42. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex (concave) function and* $\bar{\mathbf{x}} = (x_1, \dots, x_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *real sequences with the property that* $p_i \ge 0$ $(i = 1, \dots, n)$ *and* $\sum_{i=1}^n p_i = 1$. *Then one has the inequality:*

$$f\left[\frac{\sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i\right)^2}{\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2} \right] \le (\ge) \frac{\sum_{1 \le i < j \le n} p_i p_j \left[\sum_{k,l=i}^{j-1} f\left(\Delta x_k \cdot \Delta x_l\right) \right]}{\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2}, \tag{6.115}$$

where $\Delta x_k := x_{k+1} - x_k \ (k = 1, \dots, n-1)$ *is the forward difference.*

Proof. We have, by Jensen's inequality for multiple sums that

$$f\left[\frac{\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2}{\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2}\right] = f\left[\frac{\sum_{1\le i<j\le n} p_i p_j \left(x_i - x_j\right)^2}{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2}\right] = f\left[\frac{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2 \frac{(x_j - x_i)^2}{(j-i)^2}}{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2}\right] \le \frac{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2 f\left(\frac{(x_j-x_i)^2}{(j-i)^2}\right)}{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2} =: I. \tag{6.116}$$

On the other hand, for $j > i$, one has

$$x_j - x_i = \sum_{k=i}^{j-1} \left(x_{k+1} - x_k\right) = \sum_{k=i}^{j-1} \Delta x_k \tag{6.117}$$

and thus

$$\left(x_j - x_i\right)^2 = \left(\sum_{k=i}^{j-1} \Delta x_k\right)^2 = \sum_{k,l=i}^{j-1} \Delta x_k \cdot \Delta x_l.$$

Applying once more the Jensen inequality for multiple sums, we deduce

$$f\left[\frac{\left(x_j - x_i\right)^2}{\left(j-i\right)^2}\right] = f\left[\frac{\sum_{k,l=i}^{j-1} \Delta x_k \cdot \Delta x_l}{\left(j-i\right)^2}\right] \le (\ge) \frac{\sum_{k,l=i}^{j-1} f\left(\Delta x_k \cdot \Delta x_l\right)}{\left(j-i\right)^2} \tag{6.118}$$

and thus, by (6.118), we deduce

$$I \le (\ge) \frac{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2 \frac{\sum_{k,l=i}^{j-1} f(\Delta x_k \cdot \Delta x_l)}{(j-i)^2}}{\sum_{1\le i<j\le n} p_i p_j \left(j-i\right)^2} = \frac{\sum_{1\le i<j\le n} p_i p_j \left[\sum_{k,l=i}^{j-1} f\left(\Delta x_k \cdot \Delta x_l\right)\right]}{\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2}, \tag{6.119}$$

and then, by (6.116) and (6.119), we deduce the desired inequality (6.115). □

The following inequality connected with the (CBS) −inequality may be stated.

Theorem 6.43. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex (concave) function and* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{w}} = (w_1, \dots, w_n)$ *sequences of real numbers such that* $b_i \ne 0, w_i \ge 0 \ (i = 1, \dots, n)$ *and not all* w_i *are zero. Then one has the inequality*

$$f\left[\frac{\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2}{\sum_{i=1}^{n} w_i b_i^2 \sum_{i=1}^{n} i^2 w_i b_i^2 - \left(\sum_{i=1}^{n} i w_i b_i\right)^2}\right] \le (\ge) \frac{\sum_{1\le i<j\le n} w_i w_j b_i^2 b_j^2 \left[\sum_{k,l=i}^{j-1} f\left(\Delta\left(\frac{a_k}{b_k}\right) \cdot \Delta\left(\frac{a_l}{b_l}\right)\right)\right]}{\sum_{i=1}^{n} w_i b_i^2 \sum_{i=1}^{n} i^2 w_i b_i^2 - \left(\sum_{i=1}^{n} i w_i b_i\right)^2}. \tag{6.120}$$

Proof. Follows by Lemma 6.42 on choosing $p_i = w_i b_i^2$ and $x_i = \frac{a_i}{b_i}$, $i = 1, \dots, n$. We omit the details. □

6.19. **Some Inequalities for the Čebyšev Functional.** For two sequences of real numbers $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ with $p_i \geq 0$ $(i \in \{1, \dots, n\})$ and $\sum_{i=1}^{n} p_i = 1$, consider the Čebyšev functional

$$T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) := \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i. \tag{6.121}$$

By Korkine's identity [1, p. 242] one has the representation

$$T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (a_i - a_j)(b_i - b_j) = \sum_{1 \leq i < j \leq n} p_i p_j (a_j - a_i)(b_j - b_i). \tag{6.122}$$

Using the (CBS) −inequality for double sums one may state the following result

$$\left[T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}})\right]^2 \leq T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) T(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}), \tag{6.123}$$

where, obviously

$$T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (a_i - a_j)^2 = \sum_{1 \leq i < j \leq n} p_i p_j (a_j - a_i)^2. \tag{6.124}$$

The following result holds [14].

Lemma 6.44. *Assume that* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are real numbers such that for each* $i, j \in \{1, \dots, n\}$, $i < j$, *one has*

$$m(b_j - b_i) \leq a_j - a_i \leq M(b_j - b_i), \tag{6.125}$$

where m, M *are given real numbers.*

If $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *is a nonnegative sequence with* $\sum_{i=1}^{n} p_i = 1$, *then one has the inequality*

$$(m + M) T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}) \geq T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) + mMT(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}). \tag{6.126}$$

Proof. If we use the condition (6.125), we get

$$[M(b_i - b_j) - (a_i - a_j)][(a_i - a_j) - m(b_i - b_j)] \geq 0 \tag{6.127}$$

for $i, j \in \{1, \dots, n\}$, $i < j$.

If we multiply in (6.127), then, obviously, for any $i, j \in \{1, \dots, n\}$, $i < j$ we have

$$(a_j - a_i)^2 + mM(b_j - b_i)^2 \leq (m + M)(a_j - a_i)(b_j - b_i). \tag{6.128}$$

Multiplying (6.128) by $p_i p_j \geq 0$, $i, j \in \{1, \dots, n\}$, $i < j$, summing over i and j, $i < j$ from 1 to n and using the identities (6.122) and (6.124), we deduce the required inequality (6.125). □

The following result holds [14].

Theorem 6.45. *If* $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{p}}$ *are as in Lemma 6.44 and* $M \geq m > 0$, *then one has the inequality providing a reverse for (6.123)*

$$\left[T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}})\right]^2 \geq \frac{4mM}{(m + M)^2} T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) T(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}). \tag{6.129}$$

Proof. We use the following elementary inequality

$$\alpha x^2 + \frac{1}{\alpha} y^2 \geq 2xy, \quad x, y \geq 0, \ \alpha > 0 \tag{6.130}$$

to get, for the choices

$$\alpha = \sqrt{mM} > 0, \quad x = \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} \geq 0, \quad y = \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right)\right]^{\frac{1}{2}} \geq 0$$

the following inequality:

$$\sqrt{mM} T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right) + \frac{1}{\sqrt{mM}} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) \geq 2\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right)\right]^{\frac{1}{2}}. \tag{6.131}$$

Using (6.130) and (6.131), we deduce

$$\frac{(m+M)}{2\sqrt{mM}} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \geq \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right)\right]^{\frac{1}{2}}$$

which is clearly equivalent to (6.129). □

The following corollary also holds [14].

Corollary 6.46. *With the assumptions of Theorem 6.45, we have:*

$$\begin{aligned} 0 &\leq \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right)\right]^{\frac{1}{2}} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right)\right]^{\frac{1}{2}} - T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \\ &\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \end{aligned} \tag{6.132}$$

and

$$\begin{aligned} 0 &\leq T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}\right) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right) - \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2. \end{aligned} \tag{6.133}$$

The following result is useful in practical applications [14].

Theorem 6.47. *Let* $f, g : [\alpha, \beta] \to \mathbb{R}$ *be continuous on* $[\alpha, \beta]$ *and differentiable on* (α, β) *with* $g'(x) \neq 0$ *for* $x \in (\alpha, \beta)$. *Assume*

$$-\infty < \gamma = \inf_{x \in (\alpha, \beta)} \frac{f'(x)}{g'(x)}, \quad \sup_{x \in (\alpha, \beta)} \frac{f'(x)}{g'(x)} = \Gamma < \infty. \tag{6.134}$$

If $\bar{\mathbf{x}}$ *is a real sequence with* $x_i \in [\alpha, \beta]$ *and* $x_i \neq x_j$ *for* $i \neq j$ *and if we denote by* $\mathbf{f}(\bar{\mathbf{x}}) := (f(x_1), \dots, f(x_n))$, *then we have the inequality:*

$$(\gamma + \Gamma) T\left(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})\right) \geq T\left(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{f}(\bar{\mathbf{x}})\right) + \gamma\Gamma T\left(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})\right) \tag{6.135}$$

for any $\bar{p}$ *with* $p_i \geq 0$ $(i \in \{1, \dots, n\})$, $\sum_{i=1}^n p_i = 1$.

Proof. Applying the Cauchy Mean-Value Theorem, there exists $\xi_{ij} \in (\alpha, \beta)$ $(i < j)$ such that

$$\frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)} = \frac{f'(\xi_{ij})}{g'(\xi_{ij})} \in [\gamma, \Gamma] \tag{6.136}$$

for $i, j \in \{1, \dots, n\}$, $i < j$. Then

$$\left[\Gamma - \frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)}\right] \left[\frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)} - \gamma\right] \geq 0, \quad 1 \leq i < j \leq n, \tag{6.137}$$

which, by a similar argument to the one in Lemma 6.44 will give the desired result (6.135). □

The following corollary is natural [14].

Corollary 6.48. *With the assumptions in Theorem 6.47 and if* $\Gamma \geq \gamma > 0$, *then one has the inequalities:*

$$[T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}}))]^2 \geq \frac{4\gamma\Gamma}{(\gamma+\Gamma)^2} T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{f}(\bar{\mathbf{x}})) T(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})), \tag{6.138}$$

$$\begin{aligned} 0 &\leq [T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{f}(\bar{\mathbf{x}}))]^{\frac{1}{2}} [T(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}}))]^{\frac{1}{2}} - T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \\ &\leq \frac{\left(\sqrt{\Gamma} - \sqrt{\gamma}\right)^2}{2\sqrt{\gamma\Gamma}} T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \end{aligned} \tag{6.139}$$

and

$$\begin{aligned} 0 &\leq T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{f}(\bar{\mathbf{x}})) T(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) - T^2(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \\ &\leq \frac{(\Gamma-\gamma)^2}{4\gamma\Gamma} T^2(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})). \end{aligned} \tag{6.140}$$

6.20. **Other Inequalities for the Čebyšev Functional.** For two sequences of real numbers $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ and $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ with $p_i \geq 0 \ \ (i \in \{1, \dots, n\})$ and $\sum_{i=1}^n p_i = 1$, consider the Čebyšev functional

$$T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) = \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i. \tag{6.141}$$

By Sonin's identity [1, p. 246] one has the representation

$$T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) = \sum_{i=1}^n p_i (a_i - A_n(\bar{\mathbf{p}}, \bar{\mathbf{a}})) \left(b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)\right), \tag{6.142}$$

where

$$A_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}) := \sum_{j=1}^n p_j a_j, \quad A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right) := \sum_{j=1}^n p_j b_j.$$

Using the (CBS) $-$inequality for weighted sums, we may state the following result

$$\left[T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right)\right]^2 \leq T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) T\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right), \tag{6.143}$$

where, obviously

$$T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) = \sum_{i=1}^n p_i (a_i - A_n(\bar{\mathbf{p}}, \bar{\mathbf{a}}))^2. \tag{6.144}$$

The following result holds [14].

Lemma 6.49. *Assume that* $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$ *are real numbers,* $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ *are nonnegative numbers with* $\sum_{i=1}^n p_i = 1$ *and* $b_i \neq A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)$ *for each* $i \in \{1, \dots, n\}$. *If*

$$-\infty < l \leq \frac{a_i - A_n(\bar{\mathbf{p}}, \bar{\mathbf{a}})}{b_i - A_n\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}\right)} \leq L < \infty \ \text{ for all } i \in \{1, \dots, n\}, \tag{6.145}$$

where l, L *are given real numbers, then one has the inequality*

$$(l+L) T\left(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{b}}\right) \geq T(\bar{\mathbf{p}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}) + LlT\left(\bar{\mathbf{p}}, \bar{\mathbf{b}}, \bar{\mathbf{b}}\right). \tag{6.146}$$

Proof. Using (6.145) we have

$$\left(L-\frac{a_i-A_n(\bar{\mathbf{p}},\bar{\mathbf{a}})}{b_i-A_n(\bar{\mathbf{p}},\bar{\mathbf{b}})}\right)\left(\frac{a_i-A_n(\bar{\mathbf{p}},\bar{\mathbf{a}})}{b_i-A_n(\bar{\mathbf{p}},\bar{\mathbf{b}})}-l\right)\geq 0 \tag{6.147}$$

for each $i\in\{1,\dots,n\}$.

If we multiply (6.147) by $\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2\geq 0$, we get

$$\begin{aligned}(a_i-A_n(\bar{\mathbf{p}},\bar{\mathbf{a}}))^2+Ll\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2&\\ \leq (L+l)\,(a_i-A_n(\bar{\mathbf{p}},\bar{\mathbf{a}}))&\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)\end{aligned} \tag{6.148}$$

for each $i\in\{1,\dots,n\}$.

Finally, if we multiply (6.148) by $p_i\geq 0$, sum over i from 1 to n and use the identity (6.142) and (6.144), we obtain (6.146). □

Using Lemma 6.49 and a similar argument to that in the previous section, we may state the following theorem [14].

Theorem 6.50. *With the assumption of Lemma 6.49 and if $L\geq l>0$, then one has the inequality*

$$\left[T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right]^2\geq\frac{4lL}{(L+l)^2}T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{a}}\right)T\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right). \tag{6.149}$$

The following corollary is natural [14].

Corollary 6.51. *With the assumptions in Theorem 6.50 one has*

$$\begin{aligned}0&\leq\left[T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{a}}\right)\right]^{\frac12}\left[T\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right)\right]^{\frac12}-T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\\&\leq\frac{\left(\sqrt{L}-\sqrt{l}\right)^2}{2\sqrt{lL}}T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right),\end{aligned} \tag{6.150}$$

and

$$\begin{aligned}0&\leq T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{a}}\right)T\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right)-\left[T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right]^2\\&\leq\frac{(L-l)^2}{4lL}\left[T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\right]^2.\end{aligned} \tag{6.151}$$

6.21. **Bounds for the Čebyšev Functional.** The following result holds.

Theorem 6.52. *Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ (with $b_i\neq b_j$ for $i\neq j$) be two sequences of real numbers with the property that there exists the real constants m,M such that for any $1\leq i<j\leq n$ one has*

$$m\leq\frac{a_j-a_i}{b_j-b_i}\leq M. \tag{6.152}$$

Then we have the inequality

$$mT\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right)\leq T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\leq MT\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right), \tag{6.153}$$

for any nonnegative sequence $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ with $\sum_{i=1}^n p_i=1$.

Proof. From (6.152), by multiplying with $(b_j - b_i)^2 > 0$, one has

$$m(b_j - b_i)^2 \le (a_j - a_i)(b_j - b_i) \le M(b_j - b_i)^2$$

for any $1 \le i < j \le n$, giving by multiplication with $p_i p_j \ge 0$, that

$$m \sum_{1 \le i < j \le n} p_i p_j (b_i - b_j)^2 \le \sum_{1 \le i < j \le n} p_i p_j (a_j - a_i)(b_j - b_i) \le M \sum_{1 \le i < j \le n} p_i p_j (b_i - b_j)^2 .$$

Using Korkine's identity (see for example Subsection 6.19), we deduce the desired result (6.153). □

The following corollary is natural.

Corollary 6.53. *Assume that the sequence* $\bar{\mathbf{b}}$ *in Theorem 6.52 is strictly increasing and there exists* m, M *such that*

$$m \le \frac{\Delta a_k}{\Delta b_k} \le M, \quad k = 1, \dots, n-1; \tag{6.154}$$

where $\Delta a_k := a_{k+1} - a_k$ *is the forward difference, then (6.153) holds true.*

Proof. Follows from Theorem 6.52 on taking into account that for $j > i$ and from (6.154) one has

$$m \sum_{k=i}^{j-1} \Delta b_k \le \sum_{k=i}^{j-1} \Delta a_k \le M \sum_{k=i}^{j-1} \Delta b_k,$$

giving $m(b_j - b_i) \le a_j - a_i \le M(b_j - b_i)$. □

Another possibility is to use functions that generate similar inequalities.

Theorem 6.54. *Let* $f, g : [\alpha, \beta] \to \mathbb{R}$ *be continuous on* $[\alpha, \beta]$ *and differentiable on* (α, β) *with* $g'(x) \ne 0$ *for* $x \in (\alpha, \beta)$. *Assume that*

$$-\infty < m = \inf_{x \in (\alpha, \beta)} \frac{f'(x)}{g'(x)}, \quad \sup_{x \in (\alpha, \beta)} \frac{f'(x)}{g'(x)} = M < \infty.$$

If $\bar{\mathbf{x}} = (x_1, \dots, x_n)$ *is a real sequence with* $x_i \in [\alpha, \beta]$ *and* $x_i \ne x_j$ *for* $i \ne j$ *and if we denote* $\mathbf{f}(\bar{\mathbf{x}}) := (f(x_1), \dots, f(x_n))$, *then we have the inequality*

$$mT(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \le T(\bar{\mathbf{p}}, \mathbf{f}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})) \le MT(\bar{\mathbf{p}}, \mathbf{g}(\bar{\mathbf{x}}), \mathbf{g}(\bar{\mathbf{x}})). \tag{6.155}$$

Proof. Applying the Cauchy Mean-Value Theorem, for any $j > i$ there exists $\xi_{ij} \in (\alpha, \beta)$ such that

$$\frac{f(x_j) - f(x_i)}{g(x_j) - g(x_i)} = \frac{f'(\xi_{ij})}{g'(\xi_{ij})} \in [m, M].$$

Then, by Theorem 6.52 applied for $a_i = f(x_i)$, $b_i = g(x_i)$, we deduce the desired inequality (6.155). □

The following inequality related to the (CBS) −inequality holds.

Theorem 6.55. *Let* $\bar{\mathbf{a}}$, $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$ *be sequences of real numbers such that* $x_i \ne 0$ *and* $\frac{y_i}{x_i} \ne \frac{y_j}{x_j}$ *for* $i \ne j$, $(i, j \in \{1, \dots, n\})$. *If there exist the real numbers* γ, Γ *such that*

$$\gamma \le \frac{a_j - a_i}{\frac{y_j}{x_j} - \frac{y_i}{x_i}} \le \Gamma \quad \text{for} \quad 1 \le i < j \le n, \tag{6.156}$$

then we have the inequality

$$\gamma\left[\sum_{i=1}^{n}x_i^2\sum_{i=1}^{n}y_i^2-\left(\sum_{i=1}^{n}x_iy_i\right)^2\right]\le\sum_{i=1}^{n}x_i^2\sum_{i=1}^{n}a_ix_iy_i-\sum_{i=1}^{n}a_ix_i^2\sum_{i=1}^{n}x_iy_i \le\Gamma\left[\sum_{i=1}^{n}x_i^2\sum_{i=1}^{n}y_i^2-\left(\sum_{i=1}^{n}x_iy_i\right)^2\right]. \tag{6.157}$$

Proof. Follows by Theorem 6.52 on choosing $p_i=\frac{x_i^2}{\sum_{k=1}^n x_k^2}$, $b_i=\frac{y_i}{x_i}$, $m=\gamma$ and $M=\Gamma$. We omit the details. □

The following different approach may be considered as well.

Theorem 6.56. *Assume that* $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$ *are sequences of real numbers,* $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ *is a sequence of nonnegative real numbers with* $\sum_{i=1}^n p_i=1$ *and* $b_i\neq A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right):=\sum_{i=1}^n p_ib_i$. *If*

$$-\infty<l\le\frac{a_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{a}}\right)}{b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)}\le L<\infty \ \text{ for each } \ i\in\{1,\dots,n\}, \tag{6.158}$$

where l,L *are given real numbers, then one has the inequality*

$$lT\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right)\le T\left(\bar{\mathbf{p}},\bar{\mathbf{a}},\bar{\mathbf{b}}\right)\le LT\left(\bar{\mathbf{p}},\bar{\mathbf{b}},\bar{\mathbf{b}}\right). \tag{6.159}$$

Proof. From (6.158), by multiplying with $\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2>0$, we deduce

$$l\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2\le\left(a_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{a}}\right)\right)\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right) \le L\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2,$$

for any $i\in\{1,\dots,n\}$.

By multiplying with $p_i\ge0$, and summing over i from 1 to n, we deduce

$$l\sum_{i=1}^{n}p_i\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2\le\sum_{i=1}^{n}p_i\left(a_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{a}}\right)\right)\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right) \le L\sum_{i=1}^{n}p_i\left(b_i-A_n\left(\bar{\mathbf{p}},\bar{\mathbf{b}}\right)\right)^2.$$

Using Sonin's identity (see for example Section 6.20), we deduce the desired result (6.159). □

The following result in connection with the (CBS) $-$inequality may be stated.

Theorem 6.57. *Let* $\bar{\mathbf{a}},\bar{\mathbf{x}},\bar{\mathbf{b}}$ *be sequences of real numbers such that* $x_i\neq0$ *and*

$$\frac{y_i}{x_i}\neq\frac{1}{\sum_{i=1}^n x_i^2}A_n\left(\overline{\mathbf{x}^2},\frac{\overline{\mathbf{y}}}{\mathbf{x}}\right)$$

for $i\in\{1,\dots,n\}$. *If there exists the real numbers* ϕ,Φ *such that*

$$\phi\le\frac{a_i-\frac{1}{\sum_{i=1}^n x_i^2}A_n\left(\overline{\mathbf{x}^2},\bar{\mathbf{a}}\right)}{\frac{y_i}{x_i}-\frac{1}{\sum_{i=1}^n x_i^2}A_n\left(\overline{\mathbf{x}^2},\frac{\overline{\mathbf{y}}}{\mathbf{x}}\right)}\le\Phi, \tag{6.160}$$

where $\overline{\mathbf{x}^2}=(x_1^2,\dots,x_n^2)$ *and* $\frac{\overline{\mathbf{y}}}{\mathbf{x}}=\left(\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}\right)$, *then one has the inequality (6.157).*

Proof. Follows by Theorem 6.56 on choosing $p_i = \frac{x_i^2}{\sum_{i=1}^n x_i^2}$, $b_i = \frac{y_i}{x_i}$, $l = \phi$ and $L = \Phi$. We omit the details. □

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