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A GENERALIZATION OF AN INEQUALITY OF JIA AND CAU

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Abstract

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Abstract

Let L , H_r , and A_s stand for the logarithmic mean, the Heronian mean of order r , and the power mean of order s , of two positive variables. A generalization of the inequality

$$L \leq H_r \leq A_s$$

($1/2 \leq r \leq 3s/2$), of G. Jia and J. Cao ([3]), is obtained.

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1. Introduction and Definitions

Let x and y be positive numbers. The Heronian mean of order $a \in \mathbb{R}$ of x and y , denoted by $H_a \equiv H_a(x, y)$, is defined as

$$H_a = \begin{cases} \left(\frac{x^a + (xy)^{a/2} + y^a}{3} \right)^{\frac{1}{a}}, & a \neq 0 \\ G, & a = 0, \end{cases}$$

where $G = \sqrt{xy}$ is the geometric mean of x and y . When $a = 1$, we will write H instead of H_1 . Let us note that $H = (2A + G)/3$, where $A = (x + y)/2$ is the arithmetic mean of x and y . The logarithmic mean L of x and y and the power mean A_a of order a of x and y are defined as

$$L = \begin{cases} \frac{x - y}{\ln x - \ln y}, & x \neq y \\ x, & x = y, \end{cases}$$

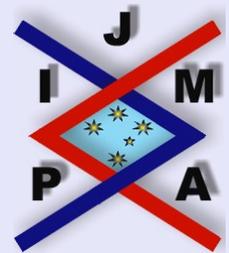
and

$$A_a = \begin{cases} \left(\frac{x^a + y^a}{2} \right)^{\frac{1}{a}}, & a \neq 0 \\ G, & a = 0, \end{cases}$$

respectively. Throughout the sequel the means of order one will be denoted by a single letter with the subscript 1 being omitted.

In the recent paper [3] the authors have established the following result. Let $\frac{1}{2} \leq r \leq \frac{3}{2}s$. Then

$$(1.1) \quad L \leq H_r \leq A_s.$$



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All the means mentioned earlier in this section belong to the large family of means introduced by K.B. Stolarsky in [8]. This two-parameter class of means, denoted by $\mathcal{D}_{a,b}$, is defined as follows

$$(1.2) \quad \mathcal{D}_{a,b} = \begin{cases} \left(\frac{b}{a} \cdot \frac{x^a - y^a}{x^b - y^b} \right)^{\frac{1}{(a-b)}}, & ab(a-b) \neq 0 \\ \exp \left(-\frac{1}{a} + \frac{x^a \ln x - y^a \ln y}{x^a - y^a} \right), & a = b \neq 0 \\ \left[\frac{x^a - y^a}{a(\ln x - \ln y)} \right]^{\frac{1}{a}}, & a \neq 0, b = 0 \\ G, & a = b = 0. \end{cases}$$

For later use let us record some formulas which follow from (1.2). We have

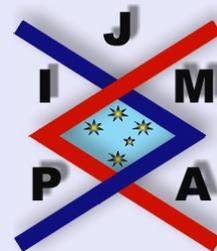
$$(1.3) \quad H_r = \mathcal{D}_{3r/2, r/2}, \quad A_s = \mathcal{D}_{2s, s}, \quad L_p = \mathcal{D}_{p, 0}, \quad I_t = \mathcal{D}_{t, t}.$$

Here L_p is the logarithmic mean of order p and I_t is called the identric mean of order t .

The inequalities (1.1) can be written in terms of the Stolarsky means as

$$(1.1') \quad \mathcal{D}_{1,0} \leq \mathcal{D}_{3r/2, r/2} \leq \mathcal{D}_{2s, s}.$$

The goal of this note is to provide a short proof of a general inequality (see (2.1)) which contains (1.1) as a special case.



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2. Main Result

For the reader's convenience, we recall the Comparison Theorem for the Stolarsky means. Two functions

$$k(p, q) = \begin{cases} \frac{|p| - |q|}{p - q}, & p \neq q \\ \text{sign}(p), & p = q \end{cases}$$

and

$$l(p, q) = \begin{cases} L(p, q), & p > 0, q > 0 \\ 0, & p \cdot q = 0 \end{cases}$$

play a crucial role in the Comparison Theorem which has been established by E.B. Leach and M.C. Sholander [4] and also by Zs. Páles [6].

Theorem 2.1 (Comparison Theorem). *Let $a, b, c, d \in \mathbb{R}$. Then the comparison inequality*

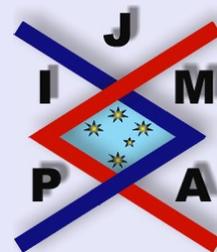
$$\mathcal{D}_{a,b} \leq \mathcal{D}_{c,d}$$

holds true if and only if $a + b \leq c + d$ and

$$\begin{aligned} l(a, b) &\leq l(c, d) && \text{if } 0 \leq \min(a, b, c, d), \\ k(a, b) &\leq k(c, d) && \text{if } \min(a, b, c, d) < 0 < \max(a, b, c, d), \\ -l(-a, -b) &\leq -l(-c, -d) && \text{if } \max(a, b, c, d) \leq 0. \end{aligned}$$

In what follows the symbols \mathbb{R}_+ and \mathbb{R}_- will stand for the nonnegative semi-axis and the nonpositive semi-axis, respectively.

The main result of this note reads as follows.



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Theorem 2.2. Let $p, q, r, s, t \in \mathbb{R}_+$. Then the inequalities

$$(2.1) \quad \mathcal{D}_{p,q} \leq H_r \leq \mathcal{D}_{s,t}$$

hold true if and only if

$$(2.2) \quad \max \left\{ \frac{p+q}{2}, (\ln 3)l(p, q) \right\} \leq r \leq \min \left\{ \frac{s+t}{2}, (\ln 3)l(s, t) \right\}.$$

If $p, q, r, s, t \in \mathbb{R}_-$, then the inequalities (2.1) are reversed if and only if

$$(2.3) \quad \max \left\{ \frac{s+t}{2}, (-\ln 3)l(-s, -t) \right\} \\ \leq r \leq \min \left\{ \frac{p+q}{2}, (-\ln 3)l(-p, -q) \right\}.$$

Proof. We shall establish the first part of the assertion only. Using the Comparison Theorem we see that the inequalities

$$(2.4) \quad \mathcal{D}_{p,q} \leq \mathcal{D}_{3r/2, r/2} \leq \mathcal{D}_{s,t}$$

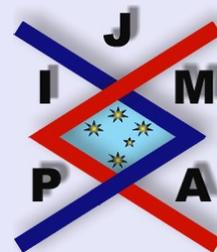
hold true if and only if

$$(2.5) \quad p+q \leq 2r \leq s+t$$

and

$$(2.6) \quad l(p, q) \leq \frac{r}{\ln 3} \leq l(s, t).$$

Solving the inequalities for r we obtain (2.2). Since the middle term in (2.4) equals to H_r (see (1.3)), the assertion follows. \square



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Remark 2.1. Letting $p = 1$, $q = 0$, $s := 2s$ and $t = s$ in (2.1) and next using (1.1') we obtain the inequalities (1.1).

Corollary 2.3. Let $p, q, r, s, t \in \mathbb{R}_+$. Then the inequalities

$$(2.7) \quad L_p \leq H_r \leq A_s \leq I_t$$

hold true if and only if $p \leq 2r \leq 3s \leq 2t$.

Proof. Letting $q = 0$, $s := 2s$, and $t = s$ in (2.1) and (2.2) we obtain the first two inequalities in (2.7). It is easy to see, using the Comparison Theorem, that the inequality $\mathcal{D}_{2s,s} \leq \mathcal{D}_{t,t}$ is valid if and only if $3s \leq 2t$. This completes the proof of the third inequality in (2.7) because of (1.3). \square

It is worth mentioning that (2.7) contains two known results: $H \leq I$ (see [7]) and $\sqrt{AL} \leq A_{2/3} \leq I$ (see [5]). Indeed, letting $p = 2$, $r = 1$, $s = \frac{3}{2}$ and $t = 1$ in Corollary 2.3 we obtain

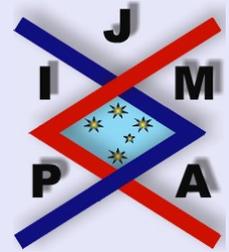
$$(2.8) \quad \sqrt{AL} \leq H \leq A_{2/3} \leq I.$$

Here we have used the formula $L_2 = \sqrt{AL}$.

The celebrated Gauss' arithmetic-geometric mean $AGM \equiv AGM(x, y)$ of $x > 0$ and $y > 0$ is the common limit of two sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$, i.e.,

$$AGM = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n,$$

where $x_0 = x$, $y_0 = y$, $x_{n+1} = (x_n + y_n)/2$, $y_{n+1} = \sqrt{x_n y_n}$ ($n \geq 0$). This important mean is used for numerical evaluation of the complete elliptic integral



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of the first kind [2]

$$R_K(x^2, y^2) = \frac{2}{\pi} \int_0^{\pi/2} (x^2 \cos^2 \phi + y^2 \sin^2 \phi)^{-1/2} d\phi.$$

Gauss' famous result states that $R_K(x^2, y^2) = 1/AGM(x, y)$.

Corollary 2.4. *Let $x > 0$ and $y > 0$. Then*

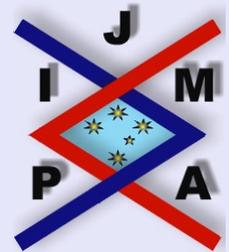
$$(2.9) \quad AGM \leq H_{3/4}.$$

Proof. J. Borwein and P. Borwein [1, Prop. 2.7] have proven that $AGM \leq L_{3/2}$. On the other hand, using the first inequality in (2.7) with $p = 3/2$ and $r = 3/4$ we obtain $L_{3/2} \leq H_{3/4}$. Hence (2.9) follows. \square

Some results of this note can be used to obtain inequalities involving hyperbolic functions. For instance, using (2.7), (1.3), and (1.2), with $x = e$ and $y = e^{-1}$, we obtain

$$\left(\frac{\sinh p}{p}\right)^{\frac{1}{p}} \leq \left(\frac{2 \cosh r + 1}{3}\right)^{\frac{1}{r}} \leq (\cosh s)^{\frac{1}{s}} \leq \exp\left(-\frac{1}{t} + \coth t\right)$$

$$(0 < p \leq 2r \leq 3s \leq 2t).$$



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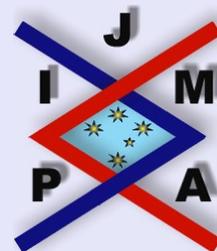
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