journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 10 (2009), Issue 2, Article 85, 8 pp.


# SUPERSTABILITY FOR GENERALIZED MODULE LEFT DERIVATIONS AND GENERALIZED MODULE DERIVATIONS ON A BANACH MODULE (II)

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*Received 12 January, 2009; accepted 12 May, 2009*
*Communicated by S.S. Dragomir*

ABSTRACT. In this paper, we introduce and discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

*Key words and phrases:* Superstability, Generalized module left derivation, Generalized module derivation, Module left derivation, Module derivation, Banach module.

2000 *Mathematics Subject Classification.* Primary 39B52; Secondary 39B82.

## 1. INTRODUCTION

The study of stability problems was formulated by Ulam in [28] during a talk in 1940: "Under what conditions does there exist a homomorphism near an approximate homomorphism?" In the following year 1941, Hyers in [12] answered the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \to Y$ is a map with a normed space $X$ and a Banach space $Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \tag{1.1}$$

for all $x, y$ in $X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon, \tag{1.2}$$

for all $x$ in $X$. In addition, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x$ in $X$, then the mapping $T$ is real linear. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x + y) = f(x) + f(y)$. A generalized version

This subject is supported by the NNSFs of China (No. 10571113, 10871224).
013-09

of the theorem of Hyers for approximately additive mappings was given by Aoki in [1] and for approximate linear mappings was presented by Th. M. Rassias in [26] by considering the case when the left hand side of the inequality (1.1) is controlled by a sum of powers of norms [25]. The stability of approximate ring homomorphisms and additive mappings were discussed in [6, 7, 8, 10, 11, 13, 14, 21].

The stability result concerning derivations between operator algebras was first obtained by P. Semrl in [27]. Badora [5] and Moslehian [17, 18] discussed the Hyers-Ulam stability and the superstability of derivations. C. Baak and M. S. Moslehian [4] discussed the stability of $J^*$-homomorphisms. Miura et al. proved the Hyers-Ulam-Rassias stability and Bourgin-type superstability of derivations on Banach algebras in [16]. Various stability results on derivations and left derivations can be found in [3, 19, 20, 2, 9]. More results on stability and superstability of homomorphisms, special functionals and equations can be found in J. M. Rassias' papers [22, 23, 24].

Recently, S.-Y. Kang and I.-S. Chang in [15] discussed the superstability of generalized left derivations and generalized derivations. In the present paper, we will discuss the superstability of generalized module left derivations and generalized module derivations on a Banach module.

To give our results, let us give some notations. Let $\mathscr{A}$ be an algebra over the real or complex field $\mathbb{F}$ and $X$ be an $\mathscr{A}$-bimodule.

**Definition 1.1.** A mapping $d : \mathscr{A} \to \mathscr{A}$ is said to be *module-$X$ additive* if

$$xd(a+b) = xd(a) + xd(b) \quad (a, b \in \mathscr{A}, x \in X). \tag{1.3}$$

A module-$X$ additive mapping $d : \mathscr{A} \to \mathscr{A}$ is said to be a *module-$X$ left derivation* (resp., *module-$X$ derivation*) if the functional equation

$$xd(ab) = axd(b) + bxd(a) \quad (a, b \in \mathscr{A}, x \in X) \tag{1.4}$$

(resp.,

$$xd(ab) = axd(b) + d(a)xb \quad (a, b \in \mathscr{A}, x \in X)) \tag{1.5}$$

holds.

**Definition 1.2.** A mapping $f : X \to X$ is said to be *module-$\mathscr{A}$ additive* if

$$af(x_1 + x_2) = af(x_1) + af(x_2) \quad (x_1, x_2 \in X, a \in \mathscr{A}). \tag{1.6}$$

A module-$\mathscr{A}$ additive mapping $f : X \to X$ is called a *generalized module-$\mathscr{A}$ left derivation* (resp., *generalized module-$\mathscr{A}$ derivation*) if there exists a module-$X$ left derivation (resp., module-$X$ derivation) $\delta : \mathscr{A} \to \mathscr{A}$ such that

$$af(bx) = abf(x) + ax\delta(b) \quad (x \in X, a, b \in \mathscr{A}) \tag{1.7}$$

(resp.,

$$af(bx) = abf(x) + a\delta(b)x \quad (x \in X, a, b \in \mathscr{A})). \tag{1.8}$$

In addition, if the mappings $f$ and $\delta$ are all linear, then the mapping $f$ is called a *linear generalized module-$\mathscr{A}$ left derivation* (resp., *linear generalized module-$\mathscr{A}$ derivation*).

**Remark 1.** Let $\mathscr{A} = X$ and $\mathscr{A}$ be one of the following cases:

(a) a unital algebra;
(b) a Banach algebra with an approximate unit.

Then module-$\mathscr{A}$ left derivations, module-$\mathscr{A}$ derivations, generalized module-$\mathscr{A}$ left derivations and generalized module-$\mathscr{A}$ derivations on $\mathscr{A}$ become left derivations, derivations, generalized left derivations and generalized derivations on $\mathscr{A}$ as discussed in [15].

## 2. MAIN RESULTS

**Theorem 2.1.** *Let $\mathscr{A}$ be a Banach algebra, $X$ a Banach $\mathscr{A}$-bimodule, $k$ and $l$ be integers greater than $1$, and $\varphi : X \times X \times \mathscr{A} \times X \to [0, \infty)$ satisfy the following conditions:*

(a) $\lim_{n\to\infty} k^{-n}[\varphi(k^n x, k^n y, 0, 0) + \varphi(0, 0, k^n z, w)] = 0 \ (x, y, w \in X, z \in \mathscr{A})$.

(b) $\lim_{n\to\infty} k^{-2n}\varphi(0, 0, k^n z, k^n w) = 0 \ (z \in \mathscr{A}, w \in X)$.

(c) $\tilde{\varphi}(x) := \sum_{n=0}^{\infty} k^{-n+1}\varphi(k^n x, 0, 0, 0) < \infty \ (x \in X)$.

*Suppose that $f : X \to X$ and $g : \mathscr{A} \to \mathscr{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$ and*

$$\left\|\Delta^1_{f,g}(x, y, z, w)\right\| \le \varphi(x, y, z, w) \tag{2.1}$$

*for all $x, y, w \in X$ and $z \in \mathscr{A}$ where*

$$\Delta^1_{f,g}(x, y, z, w) = f\left(\frac{x}{k} + \frac{y}{l} + zw\right) + f\left(\frac{x}{k} - \frac{y}{l} + zw\right) - \frac{2f(x)}{k} - 2zf(w) - 2wg(z).$$

*Then $f$ is a generalized module-$\mathscr{A}$ left derivation and $g$ is a module-$X$ left derivation.*

*Proof.* By taking $w = z = 0$, we see from (2.1) that

$$\left\|f\left(\frac{x}{k} + \frac{y}{l}\right) + f\left(\frac{x}{k} - \frac{y}{l}\right) - \frac{2f(x)}{k}\right\| \le \varphi(x, y, 0, 0) \tag{2.2}$$

for all $x, y \in X$. Letting $y = 0$ and replacing $x$ by $kx$ in (2.2), we get

$$\left\|f(x) - \frac{f(kx)}{k}\right\| \le \frac{1}{2}\varphi(kx, 0, 0, 0) \tag{2.3}$$

for all $x \in X$. Hence, for all $x \in X$, we have from (2.3) that

$$\begin{aligned}\left\|f(x) - \frac{f(k^2x)}{k^2}\right\| &\le \left\|f(x) - \frac{f(kx)}{k}\right\| + \left\|\frac{f(kx)}{k} - \frac{f(k^2x)}{k^2}\right\| \\ &\le \frac{1}{2}\varphi(kx, 0, 0, 0) + \frac{1}{2}k^{-1}\varphi(k^2x, 0, 0, 0).\end{aligned}$$

By induction, one can check that

$$\left\|f(x) - \frac{f(k^n x)}{k^n}\right\| \le \frac{1}{2}\sum_{j=1}^{n} k^{-j+1}\varphi(k^j x, 0, 0, 0) \tag{2.4}$$

for all $x$ in $X$ and $n = 1, 2, \ldots$. Let $x \in X$ and $n > m$. Then by (2.4) and condition (c), we obtain that

$$\begin{aligned}\left\|\frac{f(k^n x)}{k^n} - \frac{f(k^m x)}{k^m}\right\| &= \frac{1}{k^m}\left\|\frac{f(k^{n-m} \cdot k^m x)}{k^{n-m}} - f(k^m x)\right\| \\ &\le \frac{1}{k^m} \cdot \frac{1}{2}\sum_{j=1}^{n-m} k^{-j+1}\varphi(k^j \cdot k^m x, 0, 0, 0) \\ &\le \frac{1}{2}\sum_{s=m}^{\infty} k^{-s+1}\varphi(k^s x, 0, 0, 0) \\ &\to 0 \ (m \to \infty).\end{aligned}$$

This shows that the sequence $\left\{\frac{f(k^n x)}{k^n}\right\}$ is a Cauchy sequence in the Banach $\mathscr{A}$-module $X$ and therefore converges for all $x \in X$. Put $d(x) = \lim_{n\to\infty} \frac{f(k^n x)}{k^n}$ for every $x \in X$ and $f(0) = d(0) = 0$. By (2.4), we get

$$\|f(x) - d(x)\| \leq \frac{1}{2}\tilde{\varphi}(x) \quad (x \in X). \tag{2.5}$$

Next, we show that the mapping $d$ is additive. To do this, let us replace $x, y$ by $k^n x, k^n y$ in (2.2), respectively. Then

$$\left\| \frac{1}{k^n} f\left(\frac{k^n x}{k} + \frac{k^n y}{l}\right) + \frac{1}{k^n} f\left(\frac{k^n x}{k} - \frac{k^n y}{l}\right) - \frac{1}{k} \cdot \frac{2f(k^n x)}{k^n} \right\| \leq k^{-n}\varphi(k^n x, k^n y, 0, 0)$$

for all $x, y \in X$. If we let $n \to \infty$ in the above inequality, then the condition (a) yields that

$$d\left(\frac{x}{k} + \frac{y}{l}\right) + d\left(\frac{x}{k} - \frac{y}{l}\right) = \frac{2}{k} d(x) \tag{2.6}$$

for all $x, y \in X$. Since $d(0) = 0$, taking $y = 0$ and $y = \frac{l}{k}x$, respectively, we see that $d\left(\frac{x}{k}\right) = \frac{d(x)}{k}$ and $d(2x) = 2d(x)$ for all $x \in X$, and then we obtain that $d(x+y) + d(x-y) = 2d(x)$ for all $x, y \in X$. Now, for all $u, v \in X$, put $x = \frac{k}{2}(u+v)$, $y = \frac{l}{2}(u-v)$. Then by (2.6), we get that

$$\begin{aligned} d(u) + d(v) &= d\left(\frac{x}{k} + \frac{y}{l}\right) + d\left(\frac{x}{k} - \frac{y}{l}\right) \\ &= \frac{2}{k} d(x) = \frac{2}{k} d\left(\frac{k}{2}(u+v)\right) = d(u+v). \end{aligned}$$

This shows that $d$ is additive.

Now, we are going to prove that $f$ is a generalized module-$\mathscr{A}$ left derivation. Letting $x = y = 0$ in (2.1), we get

$$\|f(zw) + f(zw) - 2zf(w) - 2wg(z)\| \leq \varphi(0, 0, z, w),$$

that is

$$\|f(zw) - zf(w) - wg(z)\| \leq \frac{1}{2}\varphi(0, 0, z, w) \tag{2.7}$$

for all $z \in \mathscr{A}$ and $w \in X$. By replacing $z, w$ with $k^n z, k^n w$ in (2.7) respectively, we deduce that

$$\left\| \frac{1}{k^{2n}} f\left(k^{2n} zw\right) - z\frac{1}{k^n} f(k^n w) - w\frac{1}{k^n} g(k^n z) \right\| \leq \frac{1}{2} k^{-2n}\varphi(0, 0, k^n z, k^n w) \tag{2.8}$$

for all $z \in \mathscr{A}$ and $w \in X$. Letting $n \to \infty$, condition (b) yields that

$$d(zw) = zd(w) + w\delta(z) \tag{2.9}$$

for all $z \in \mathscr{A}$ and $w \in X$. Since $d$ is additive, $\delta$ is module-$X$ additive. Put $\Delta(z, w) = f(zw) - zf(w) - wg(z)$. Then by (2.7) we see from condition (a) that

$$k^{-n}\|\Delta(k^n z, w)\| \leq \frac{1}{2} k^{-n}\varphi(0, 0, k^n z, w) \to 0 \quad (n \to \infty)$$

for all $z \in \mathscr{A}$ and $w \in X$. Hence

$$\begin{aligned} d(zw) &= \lim_{n\to\infty} \frac{f(k^n z \cdot w)}{k^n} \\ &= \lim_{n\to\infty} \left( \frac{k^n z f(w) + wg(k^n z) + \Delta(k^n z, w)}{k^n} \right) \\ &= zf(w) + w\delta(z) \end{aligned}$$

for all $z \in \mathscr{A}$ and $w \in X$. It follows from (2.9) that $zf(w) = zd(w)$ for all $z \in \mathscr{A}$ and $w \in X$, and then $d(w) = f(w)$ for all $w \in X$. Since $d$ is additive, $f$ is module-$\mathscr{A}$ additive. So, for all $a, b \in \mathscr{A}$ and $x \in X$ by (2.9),

$$af(bx) = ad(bx) = abf(x) + ax\delta(b)$$

and

$$\begin{aligned} x\delta(ab) &= d(abx) - abf(x) \\ &= af(bx) + bx\delta(a) - abf(x) \\ &= a(d(bx) - bf(x)) + bx\delta(a) \\ &= ax\delta(b) + bx\delta(a). \end{aligned}$$

This shows that if $\delta$ is a module-$X$ left derivation on $\mathscr{A}$, then $f$ is a generalized module-$\mathscr{A}$ left derivation on $X$.

Lastly, we prove that $g$ is a module-$X$ left derivation on $\mathscr{A}$. To do this, we compute from (2.7) that

$$\left\| \frac{f(k^n zw)}{k^n} - z\frac{f(k^n w)}{k^n} - wg(z) \right\| \le \frac{1}{2}k^{-n}\varphi(0, 0, z, k^n w)$$

for all $z \in \mathscr{A}$ and all $w \in X$. By letting $n \to \infty$, we get from condition (a) that

$$d(zw) = zd(w) + wg(z)$$

for all $z \in \mathscr{A}$ and all $w \in X$. Now, (2.9) implies that $wg(z) = w\delta(z)$ for all $z \in \mathscr{A}$ and all $w \in X$. Hence, $g$ is a module-$X$ left derivation on $\mathscr{A}$. This completes the proof. □

**Corollary 2.2.** *Let $\mathscr{A}$ be a Banach algebra, $X$ a Banach $\mathscr{A}$-bimodule, $\varepsilon \ge 0$, $p, q, s, t \in [0, 1)$ and $k$ and $l$ be integers greater than $1$. Suppose that $f : X \to X$ and $g : \mathscr{A} \to \mathscr{A}$ are mappings such that $f(0) = 0$, $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$ and*

$$\left\|\Delta^1_{f,g}(x, y, z, w)\right\| \le \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t) \tag{2.10}$$

*for all $x, y, w \in X$ and all $z \in \mathscr{A}$ $(0^0 := 1)$. Then $f$ is a generalized module-$\mathscr{A}$ left derivation and $g$ is a module-$X$ left derivation.*

*Proof.* It is easy to check that the function

$$\varphi(x, y, z, w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$$

satisfies conditions (a), (b) and (c) of Theorem 2.1. □

**Corollary 2.3.** *Let $\mathscr{A}$ be a Banach algebra with unit $e$, $\varepsilon \ge 0$, and $k$ and $l$ be integers greater than $1$. Suppose that $f, g : \mathscr{A} \to \mathscr{A}$ are mappings with $f(0) = 0$ such that*

$$\left\|\Delta^1_{f,g}(x, y, z, w)\right\| \le \varepsilon$$

*for all $x, y, w, z \in \mathscr{A}$. Then $f$ is a generalized left derivation and $g$ is a left derivation.*

*Proof.* By taking $w = e$ in (2.8), we see that the limit $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$. It follows from Corollary 2.2 and Remark 1 that $f$ is a generalized left derivation and $g$ is a left derivation. This completes the proof. □

**Lemma 2.4.** *Let $X, Y$ be complex vector spaces. Then a mapping $f : X \to Y$ is linear if and only if*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

*for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.*

*Proof.* It suffices to prove the sufficiency. Suppose that $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$ for all $x,y\in X$ and all $\alpha,\beta\in\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$. Then $f$ is additive and $f(\alpha x)=\alpha f(x)$ for all $x\in X$ and all $\alpha\in\mathbb{T}$. Let $\alpha$ be any nonzero complex number. Take a positive integer $n$ such that $|\alpha/n|<2$. Take a real number $\theta$ such that $0\le a:=e^{-i\theta}\alpha/n<2$. Put $\beta=\arccos\frac{a}{2}$. Then $\alpha=n(e^{i(\beta+\theta)}+e^{-i(\beta-\theta)})$ and therefore

$$\begin{aligned}f(\alpha x)&=nf(e^{i(\beta+\theta)}x)+nf(e^{-i(\beta-\theta)}x)\\&=ne^{i(\beta+\theta)}f(x)+ne^{-i(\beta-\theta)}f(x)=\alpha f(x)\end{aligned}$$

for all $x\in X$. This shows that $f$ is linear. The proof is completed. □

**Theorem 2.5.** *Let $\mathscr{A}$ be a Banach algebra, $X$ a Banach $\mathscr{A}$-bimodule, $k$ and $l$ be integers greater than 1, and $\varphi:X\times X\times\mathscr{A}\times X\to[0,\infty)$ satisfy the following conditions:*

(a) $\lim_{n\to\infty}k^{-n}[\varphi(k^nx,k^ny,0,0)+\varphi(0,0,k^nz,w)]=0\quad(x,y,w\in X,z\in\mathscr{A})$.

(b) $\lim_{n\to\infty}k^{-2n}\varphi(0,0,k^nz,k^nw)=0\quad(z\in\mathscr{A},w\in X)$.

(c) $\tilde{\varphi}(x):=\sum_{n=0}^{\infty}k^{-n+1}\varphi(k^nx,0,0,0)<\infty\quad(x\in X)$.

*Suppose that $f:X\to X$ and $g:\mathscr{A}\to\mathscr{A}$ are mappings such that $f(0)=0$, $\delta(z):=\lim_{n\to\infty}\frac{1}{k^n}g(k^nz)$ exists for all $z\in\mathscr{A}$ and*

$$\left\|\Delta^3_{f,g}(x,y,z,w,\alpha,\beta)\right\|\le\varphi(x,y,z,w)\tag{2.11}$$

*for all $x,y,w\in X$, $z\in\mathscr{A}$ and all $\alpha,\beta\in\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$, where $\Delta^3_{f,g}(x,y,z,w,\alpha,\beta)$ stands for*

$$f\left(\frac{\alpha x}{k}+\frac{\beta y}{l}+zw\right)+f\left(\frac{\alpha x}{k}-\frac{\beta y}{l}+zw\right)-\frac{2\alpha f(x)}{k}-2zf(w)-2wg(z).$$

*Then $f$ is a linear generalized module-$\mathscr{A}$ left derivation and $g$ is a linear module-$X$ left derivation.*

*Proof.* Clearly, the inequality (2.1) is satisfied. Hence, Theorem 2.1 and its proof show that $f$ is a generalized left derivation and $g$ is a left derivation on $\mathscr{A}$ with

$$f(x)=\lim_{n\to\infty}\frac{f(k^nx)}{k^n},\qquad g(x)=f(x)-xf(e)\tag{2.12}$$

for every $x\in X$. Taking $z=w=0$ in (2.11) yields that

$$\left\|f\left(\frac{\alpha x}{k}+\frac{\beta y}{l}\right)+f\left(\frac{\alpha x}{k}-\frac{\beta y}{l}\right)-\frac{2\alpha f(x)}{k}\right\|\le\varphi(x,y,0,0)\tag{2.13}$$

for all $x,y\in X$ and all $\alpha,\beta\in\mathbb{T}$. If we replace $x$ and $y$ with $k^nx$ and $k^ny$ in (2.13) respectively, then we see that

$$\begin{aligned}&\left\|\frac{1}{k^n}f\left(\frac{\alpha k^nx}{k}+\frac{\beta k^ny}{l}\right)+\frac{1}{k^n}f\left(\frac{\alpha k^nx}{k}-\frac{\beta k^ny}{l}\right)-\frac{1}{k^n}\frac{2\alpha f(k^nx)}{k}\right\|\\&\le k^{-n}\varphi(k^nx,k^ny,0,0)\\&\to 0\end{aligned}$$

as $n\to\infty$ for all $x,y\in X$ and all $\alpha,\beta\in\mathbb{T}$. Hence,

$$f\left(\frac{\alpha x}{k}+\frac{\beta y}{l}\right)+f\left(\frac{\alpha x}{k}-\frac{\beta y}{l}\right)=\frac{2\alpha f(x)}{k}\tag{2.14}$$

for all $x,y\in X$ and all $\alpha,\beta\in\mathbb{T}$. Since $f$ is additive, taking $y=0$ in (2.14) implies that

$$f(\alpha x)=\alpha f(x)\tag{2.15}$$

for all $x \in X$ and all $\alpha \in \mathbb{T}$. Lemma 2.4 yields that $f$ is linear and so is $g$. Next, similar to the proof of Theorem 2.3 in [15], one can show that $g(\mathscr{A}) \subset \mathrm{Z}(\mathscr{A}) \cap \mathrm{rad}(\mathscr{A})$. This completes the proof. □

**Corollary 2.6.** *Let $\mathscr{A}$ be a complex semi-prime Banach algebra with unit $e$, $\varepsilon \geq 0$, $p, q, s, t \in [0, 1)$ and $k$ and $l$ be integers greater than $1$. Suppose that $f, g : \mathscr{A} \to \mathscr{A}$ are mappings with $f(0) = 0$ and satisfy following inequality:*

$$\left\|\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t) \tag{2.16}$$

*for all $x, y, z, w \in \mathscr{A}$ and all $\alpha, \beta \in \mathbb{T}$ $(0^0 := 1)$. Then $f$ is a linear generalized left derivation and $g$ is a linear left derivation which maps $\mathscr{A}$ into the intersection of the center $\mathrm{Z}(\mathscr{A})$ and the Jacobson radical $\mathrm{rad}(\mathscr{A})$ of $\mathscr{A}$.*

*Proof.* Since $\mathscr{A}$ has a unit $e$, letting $w = e$ in (2.8) shows that the limit $\delta(z) := \lim_{n\to\infty} \frac{1}{k^n} g(k^n z)$ exists for all $z \in \mathscr{A}$. Thus, using Theorem 2.5 for $\varphi(x, y, z, w) = \varepsilon(\|x\|^p + \|y\|^q + \|z\|^s\|w\|^t)$ yields that $f$ is a linear generalized left derivation and $g$ is a linear left derivation since $\mathscr{A}$ has a unit. Similar to the proof of Theorem 2.3 in [15], one can check that the mapping $g$ maps $\mathscr{A}$ into the intersection of the center $\mathrm{Z}(\mathscr{A})$ and the Jacobson radical $\mathrm{rad}(\mathscr{A})$ of $\mathscr{A}$. This completes the proof. □

**Corollary 2.7.** *Let $\mathscr{A}$ be a complex semiprime Banach algebra with unit $e$, $\varepsilon \geq 0$, $k$ and $l$ be integers greater than $1$. Suppose that $f, g : \mathscr{A} \to \mathscr{A}$ are mappings with $f(0) = 0$ and satisfy the following inequality:*

$$\left\|\Delta^3_{f,g}(x, y, z, w, \alpha, \beta)\right\| \leq \varepsilon$$

*for all $x, y, z, w \in \mathscr{A}$ and all $\alpha, \beta \in \mathbb{T}$. Then $f$ is a linear generalized left derivation and $g$ is a linear left derivation which maps $\mathscr{A}$ into the intersection of the center $\mathrm{Z}(\mathscr{A})$ and the Jacobson radical $\mathrm{rad}(\mathscr{A})$ of $\mathscr{A}$.*

**Remark 2.** Inequalities (2.10) and (2.16) are controlled by their right-hand sides by the "mixed sum-product of powers of norms", introduced by J. M. Rassias (in 2007) and applied afterwards by K. Ravi et al. (2007-2008). Moreover, it is easy to check that the function

$$\varphi(x, y, z, w) = P\|x\|^p + Q\|y\|^q + S\|z\|^s + T\|w\|^t$$

satisfies conditions (a), (b) and (c) of Theorem 2.1 and Theorem 2.5, where $P, Q, T, S \in [0, \infty)$ and $p, q, s, t \in [0, 1)$ are all constants.

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