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ON THE PRODUCT OF DIVISORS OF n AND OF $\sigma(n)$

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Let $n \geq 1$ be a positive integer. In [7], Sándor introduced the function $T(n) := \prod_{d|n} d$ as the multiplicative analog of $\sigma(n)$, which is the sum of all the positive divisors of n, and studied some of its properties. In particular, he proved several results pertaining to *multiplicative perfect numbers*, which, by analogy, are numbers n for which the relation $T(n) = n^k$ holds with some positive integer k.

In this paper, we compare $T(n)$ with $T(\sigma(n))$. Our first result is:

Theorem 1. *The inequality $T(\sigma(n)) > T(n)$ holds for almost all positive integers n.*

In light of Theorem 1, one can ask whether or not there exist infinitely many n for which $T(\sigma(n)) \leq T(n)$ holds. The fact that this is indeed so is contained in the following more precise statement.

Theorem 2. *Each one of the divisibility relations $T(n) \mid T(\sigma(n))$ and $T(\sigma(n)) \mid T(n)$ holds for an infinite set of positive integers n.*

Finally, we ask whether there exist positive integers $n > 1$ so that $T(n) = T(\sigma(n))$. The answer is no.

Theorem 3. *The equation $T(n) = T(\sigma(n))$ has no positive integer solution $n > 1$.*

Throughout this paper, for a positive real number x and a positive integer k we write $\log_k x$ for the recursively defined function given by $\log_k x := \max\{\log \log_{k-1} x, 1\}$, where $\log$ stands for the natural logarithm function. When $k = 1$, we simply write $\log x$, and we understand that this number is always greater than or equal to 1. For a positive real number x we use $\lfloor x \rfloor$ for the integer part of x, i.e., the largest integer k so that $k \le x$. We use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with their regular meanings. For a positive integer n, we write $\tau(n)$, and $\omega(n)$ for the number of divisors of n, and the number of distinct prime divisors of n, respectively.

Proof of Theorem 1. Let x be a large positive real number, and let n be a positive integer in the interval $I := (x/\log x,\ x)$. Since

$$\frac{1}{x} \cdot \sum_{n<x} \tau(n) = O(\log x),$$

it follows that the inequality

$$\tau(n) < \log^2 x \tag{1}$$

holds for all $n \in I$, except for a subset of such n of cardinality $O(x/\log x) = o(x)$.

A straightforward adaptation of the arguments from [4, p. 349] show that the inequality

$$\omega(\sigma(n)) > \frac{1}{3} \cdot \log_2^2 x \tag{2}$$

holds for all $n \in I$, except, eventually, for a subset of such n of cardinality $o(x)$. So, we can say that for most $n \in I$ both inequalities (1) and (2) hold. For such n, we have

$$T(n) = n^{\frac{\tau(n)}{2}} = \exp\left(\frac{\tau(n)\log n}{2}\right) < \exp\left(\frac{\log^3 x}{2}\right), \tag{3}$$

while

$$T(\sigma(n)) = (\sigma(n))^{\frac{\tau(\sigma(n))}{2}} \tag{4}$$
$$> n^{\frac{\tau(\sigma(n))}{2}}$$
$$> \exp\left(\frac{\tau(\sigma(n))\log n}{2}\right)$$
$$> \exp\left(\frac{2^{\omega(\sigma(n))}\log n}{2}\right)$$
$$> \exp\left(\frac{2^{\frac{\log_2^2 x}{3}}}{2}\cdot\log\left(\frac{x}{\log x}\right)\right),$$

and it is easy to see that for large values of x the function appearing in the right hand side of (4) is larger than the function appearing on the right hand side of (3). This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. We first construct infinitely many n such that $T(n) \mid T(\sigma(n))$. Let λ be an odd number to be chosen later and put $n := 2^\lambda \cdot 3$. Then, $\tau(n) = 2(\lambda+1)$, therefore

$$T(n) = (2^\lambda \cdot 3)^{\frac{\tau(n)}{2}} \mid 6^{(\lambda+1)^2}. \tag{5}$$

Now $\sigma(n) = 4\cdot(2^{\lambda+1}-1)$ is a multiple of 6 because $\lambda+1$ is even, and so $2^{\lambda+1}-1$ is a multiple of 3. Thus, $T(\sigma(n))$ is a multiple of

$$6^{\lfloor\frac{\tau(4(2^{\lambda+1}-1))}{2}\rfloor} = 6^{\lfloor\frac{3\tau(2^{\lambda+1}-1)}{2}\rfloor},$$

and since the inequality $\lfloor 3k/2 \rfloor \geq k$ holds for all positive integers k, it follows that $T(\sigma(n))$ is a multiple of $6^{\tau(2^{\lambda+1}-1)}$.

It suffices therefore to see that we can choose infinitely many such odd λ so that $\tau(2^{\lambda+1}-1) > (\lambda+1)^2$. Since $\tau(2^{\lambda+1}-1) \geq 2^{\omega(2^{\lambda+1}-1)}$, it follows that it suffices to show that we can choose infinitely many odd λ so that

$$2^{\omega(2^{\lambda+1}-1)} > (\lambda+1)^2,$$

which is equivalent to

$$\omega(2^{\lambda+1}-1) > \frac{2}{\log 2} \cdot \log(\lambda+1).$$

Since $2/\log 2 < 3$, it suffices to show that the inequality

$$\omega(2^{\lambda+1}-1) > 3\log(\lambda+1) \tag{6}$$

holds for infinitely many odd positive integers λ.

Let $(u_k)_{k\geq 1}$ be the *Lucas sequence* of general term $u_k := 2^k - 1$ for $k = 1,\ 2,\ \dots$. The primitive divisor theorem (see [1], [2]), says that for all $d \mid k$, $d \neq 1,\ 6$, there exists a prime number $p \mid u_d$ (hence, $p|u_k$ as well), so that $p \nmid u_m$ for any $1 \leq m < d$. In particular, the inequality $\omega(2^k - 1) \geq \tau(k) - 2$ holds for all positive integers k. Thus, in order to prove that (6) holds for infinitely many odd positive integers λ, it suffices to show that the inequality

$$\tau(\lambda+1) \geq 2 + 3\log(\lambda+1)$$

holds for infinitely many odd positive integers λ.

Choose a large real number y and put

$$\lambda + 1 := \prod_{p<y} p. \tag{7}$$

Clearly, $\lambda + 1$ is even, therefore λ is odd. With the prime number theorem, we have that

$$\lambda + 1 = \exp(1 + o(1))y)$$

holds for large y, and therefore the inequality

$$\lambda + 1 < \exp(2y)$$

holds for large values of y. In particular,

$$2 + 3\log(\lambda + 1) < 2 + 6y \tag{8}$$

holds for large y. However,

$$\tau(\lambda + 1) \geq 2^{\omega(\lambda+1)} = 2^{\pi(y)},$$

where we write $\pi(y)$ for the number of prime numbers $p < y$. Since $\pi(y) \geq y/\log y$ holds for all $y > 17$ (see [6]), it follows that for y sufficiently large we have

$$\tau(\lambda + 1) \geq 2^{\frac{y}{\log y}}. \tag{9}$$

It is now clear that the right hand side of (9) is larger than the right hand side of (8) for sufficiently large values of y, and therefore the numbers λ shown at (7) do fulfill inequality (6) for large values of y.

We now construct infinitely many n such that $T(\sigma(n)) \mid T(n)$. For coprime integers a and d with d positive and a large positive real number x let $\pi(x; d, a)$ be the number of primes $p < x$ with $p \equiv a \pmod{d}$. For positive real numbers $y < x$ let $\pi(x; y)$ stand for the number of primes $p < x$ so that $p + 1$ is free of primes $q \geq y$. Let $\mathcal{E}$ denote the set of all real numbers E in the range $0 < E < 1$ so that there exists a positive constant $\gamma(E)$ and a real number $x_1(E)$ such that the inequality

$$\pi(x; x^{1-E}) > \gamma(E)\pi(x) \tag{10}$$

holds for all $x > x_1(E)$. Thus, $\mathcal{E}$ is the set of all real numbers E in the interval $0 < E < 1$ such that for large x a positive proportion (depending on E) of all the prime numbers p up to x have $p + 1$ free of primes $q \geq x^{1-E}$. Erdős (see [3]) showed that $\mathcal{E}$ is nonempty. In fact, he did not exactly treat this question, but the analogous question for the primes $p < x$ such that $p - 1$ is free of primes larger than x^{1-E}, but his argument can be adapted to the situation in which $p - 1$ is replaced by $p + 1$, which is our instance. The best result known about $\mathcal{E}$ is due to Friedlander [5], who showed that every positive number E smaller than $1 - (2\sqrt{e})^{-1}$ belongs to $\mathcal{E}$. Erdős has conjectured that $\mathcal{E}$ is the full interval $(0, 1)$.

Let E be some number in $\mathcal{E}$. Let $x > x_1(E)$ be a large real number. Let $\mathcal{P}_E(x)$ be the set of all the primes $p < x$ counted by $\pi(x; x^{1-E})$. Note that all the primes $p < x^{1-E}$ are already in $\mathcal{P}_E(x)$. Put

$$n := \prod_{p \in \mathcal{P}_E(x)} p. \tag{11}$$

Clearly,

$$T(n) = n^{\frac{\tau(n)}{2}}, \tag{12}$$

and

$$\frac{\tau(n)}{2} = 2^{\#\mathcal{P}_E(x)-1} = 2^{\pi(x;x^{1-E})-1} > 2^{c\pi(x)} > 2^{\frac{cx}{\log x}}, \tag{13}$$

where one can take $c := \gamma(E)/2$, and inequality (13) holds for sufficiently large values of x. In particular, $T(n)$ is divisible by all primes $q < x^{1-E}$, and each one of them appears at the power at least $2^{\frac{cx}{\log x}}$.

We now look at $T(\sigma(n))$. We have

$$T(\sigma(n)) = \left(\prod_{p\in\mathcal{P}_E(x)} (p+1)\right)^{\frac{\tau(\sigma(n))}{2}}. \tag{14}$$

From the definition of $\mathcal{P}_E(x)$, we know that the only primes than can divide $T(\sigma(n))$ are the primes $q < x^{1-E}$. Thus, to conclude, it suffices to show that the exponent at which each one of these primes $q < x^{1-E}$ appears in the prime factorization of $T(\sigma(n))$ is smaller than $2^{\frac{cx}{\log x}}$. Let q be such a prime, and let α_q be so that $q^{\alpha_q}||\sigma(n)$. It is easy to see that

$$\alpha_q \le \pi(x,q,-1) + \pi(x,q^2,-1) + \cdots + \pi(x,q^j,-1) + \cdots. \tag{15}$$

Let $j \ge 1$. Then $\pi(x;q^j,-1)$ is the number of primes $p < x$ such that $q^j \mid p+1$. In particular, $\pi(x;q^j,-1)$ is at most the number of numbers $m < x+1$ which are multiples of q^j, and this number is $\left\lfloor \frac{x+1}{q^j} \right\rfloor \le \frac{x+1}{q^j}$. Thus,

$$\alpha_q < (x+1)\sum_{j\ge 1} \frac{1}{q^j} = \frac{x+1}{q-1} \le x+1.$$

Thus,

$$\alpha_q + 1 \leq x + 2 < 2x$$

holds for all $q < x^{1-E}$, and therefore

$$\tau(\sigma(n)) < (2x)^{\pi(x^{1-E})} = \exp\bigl(\pi(x^{1-E}) \cdot \log(2x)\bigr).$$

By the prime number theorem,

$$\pi(x^{1-E}) = (1 + o(1)) \cdot \frac{x^{1-E}}{\log(x^{1-E})},$$

and therefore the inequality

$$\pi(x^{1-E}) < \frac{2x^{1-E}}{\log(x^{1-E})} = \frac{2}{1-E} \cdot \frac{x^{1-E}}{\log x} \tag{16}$$

holds for large values of x. Thus,

$$\tau(\sigma(n)) < \exp\left(\frac{2}{1-E} \cdot \frac{x^{1-E}}{\log x} \cdot \log(2x)\right) < \exp\left(\frac{3x^{1-E}}{1-E}\right), \tag{17}$$

holds for large values of x. In particular, the exponent at which a prime number $q < x^{1-E}$ can appear in the prime factorization of $T(\sigma(n))$ is at most

$$\alpha_q \cdot \frac{\tau(\sigma(n))}{2} < \tau(\sigma(n))^2 < \exp\left(\frac{6x^{1-E}}{1-E}\right). \tag{18}$$

Comparing (13) with (18), it follows that it suffices to show that the inequality

$$\exp\left(\frac{6x^{1-E}}{1-E}\right) < 2^{\frac{cx}{\log x}} \tag{19}$$

holds for large values of x, and taking logarithms in (19), we see that (19) is equivalent to

$$c_1 \log x < x^E, \tag{20}$$

where $c_1 := \frac{6}{c(1-E)\log 2}$, and it is clear that (20) holds for large values of x. Theorem 2 is therefore proved. $\square$

Proof of Theorem 3. Assume that $n > 1$ satisfies $T(n) = T(\sigma(n))$. Write $t := \omega(n)$. It is clear that $t > 1$, for otherwise the number n will be of the form $n = q^\alpha$ for some prime number q and some positive integer α, and the contradiction comes from the fact that $\sigma(q^\alpha)$ is coprime to q. We now note that it is not possible that the prime factors of n are in $\{2, 3\}$. Indeed, if this were so, then $n = 2^{\alpha_1} \cdot 3^{\alpha_2}$, and $\sigma(n) = (2^{\alpha_1+1} - 1)(3^{\alpha_2+1} - 1)$. Since the prime factors of $\sigma(n)$ are also in the set $\{2, 3\}$, we get the diophantine equations $2^{\alpha_1+1} - 1 = 3^{\beta_1}$ and $3^{\alpha_2+1} - 1 = 2^{\beta_2}$, and it is wellknown and very easy to prove that the only positive integer solution $(\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2)$ of the above equations is $(1,\ 1,\ 1,\ 3)$. Thus, $n = 6$, and the contradiction comes from the fact that this number does not satisfy the equation $T(n) = T(\sigma(n))$.

Write

$$n := q_1^{\alpha_1} \cdot \cdots \cdot q_t^{\alpha_t} \tag{21}$$

where $q_1 < q_2 < \cdots < q_t$ are prime numbers and α_i are positive integers for $i = 1, \ldots, t$. We claim that

$$q_1 \cdot \cdots \cdot q_t > e^t. \tag{22}$$

This is clearly so if $t = 2$, because in this case $q_1 q_2 \geq 2 \cdot 5 > e^2$. For $t \geq 3$, one proves by induction that the inequality

$$p_1 \cdot \dots \cdot p_t > e^t$$

holds, where p_i is the ith prime number. This takes care of (22).

We now claim that

$$\frac{\sigma(n)}{n} < \exp(1 + \log t). \tag{23}$$

Indeed,

$$\begin{aligned} \frac{\sigma(n)}{n} &= \prod_{i=1}^{t} \left(1 + \frac{1}{q_i} + \dots + \frac{1}{q_i^{\alpha_i}}\right) \\ &< \exp\left(\sum_{i=1}^{t} \sum_{\beta \geq 1} \frac{1}{p_i^{\beta}}\right) \\ &< \exp\left(\sum_{i=1}^{t} \frac{1}{p_i - 1}\right), \end{aligned} \tag{24}$$

and so, in order to prove (23), it suffices, via (24), to show that

$$\sum_{i=1}^{t} \frac{1}{p_i - 1} \leq 1 + \log t. \tag{25}$$

One checks that (25) holds at $t := 1$ and $t := 2$. Assume now that $t \geq 3$ and that (25) holds for $t-1$. Then,

$$\sum_{i=1}^{t} \frac{1}{p_i - 1} = \frac{1}{p_t - 1} + \sum_{i=1}^{t-1} \frac{1}{p_i - 1} < 1 + \frac{1}{p_t - 1} + \log(t-1) < 1 + \log t, \tag{26}$$

where the last inequality in (26) above holds because it is equivalent to

$$\left(1 + \frac{1}{t-1}\right)^{p_t - 1} > e,$$

which in turn holds because $p_t \geq t+1$ holds for $t \geq 3$, and

$$\left(1 + \frac{1}{t-1}\right)^{t} > e$$

holds for all positive integers $t > 1$.

After these preliminaries, we complete the proof of Theorem 3. Write the relation $T(n) = T(\sigma(n))$ as

$$\sigma(n) = n^{\frac{\tau(n)}{\tau(\sigma(n))}} = n \cdot n^{\frac{\tau(n)-\tau(\sigma(n))}{\tau(\sigma(n))}}. \tag{27}$$

Since $\sigma(n) > n$, we get that $\tau(n) > \tau(\sigma(n))$. We now use (23) to say that

$$n^{\frac{\tau(n)-\tau(\sigma(n))}{\tau(\sigma(n))}} = \frac{\sigma(n)}{n} < \exp(1 + \log t),$$

therefore

$$\frac{\tau(n) - \tau(\sigma(n))}{\tau(\sigma(n))} < \frac{1 + \log t}{\log n}. \tag{28}$$

Let $d := \gcd(\tau(n), \tau(\sigma(n))) = \gcd(\tau(n) - \tau(\sigma(n)), \tau(\sigma(n)))$. From (28), we get that

$$d < \left(\frac{1 + \log t}{\log n}\right) \cdot \tau(\sigma(n)).$$

Write

$$\frac{\tau(n) - \tau(\sigma(n))}{\tau(\sigma(n))} = \frac{\beta}{\gamma}, \tag{29}$$

where β and γ are coprime positive integers. We have

$$\gamma = \frac{\tau(\sigma(n))}{d} > \frac{\log n}{1 + \log t}. \tag{30}$$

The number $n^{\frac{\beta}{\gamma}} = \sigma(n)/n$ is both a rational number and an algebraic integer, and is therefore an integer. Since β and γ are coprime, it follows, by unique factorization, that α_i is a multiple of γ for all $i = 1, \ldots, t$. Thus, $\alpha_i \geq \gamma$ holds for $i = 1, \ldots, t$, therefore

$$n \geq (q_1 \cdot \cdots \cdot q_t)^{\gamma} > e^{t\gamma} = \exp(t\gamma) > \exp\left(\frac{t \log n}{1 + \log t}\right) = n^{\frac{t}{1+\log t}}, \tag{31}$$

and now (31) implies that

$$1 + \log t > t,$$

which is impossible. Theorem 3 is therefore proved. □

Remark 0.1. *We close by noting that if n is a multiply perfect number, then $T(n) \mid T(\sigma(n))$. Recall that a multiply perfect number n is a number so that $n \mid \sigma(n)$. If n has this property, then $\tau(\sigma(n)) > \tau(n)$, and now it is easy to see that $T(\sigma(n)) = \sigma(n)^{\frac{\tau(\sigma(n))}{2}}$ is a multiple of $n^{\frac{\tau(n)}{2}} = T(n)$. Unfortunately, we still do not know if the set of multiply perfect numbers is infinite.*

References

[1] Y. BILU, G. HANROT AND P.M. VOUTIER, Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte, *J. Reine Angew. Math.*, **539** (2001), 75–122.

[2] R.D. CARMICHAEL, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, *Ann. Math.*, **15**(2) (1913), 30–70.

[3] P. ERDŐS, On the normal number of prime factors of $p-1$ and some other related problems concerning Euler's ϕ-function, *Quart. J. of Math.* (Oxford Ser.), **6** (1935), 205–213.

[4] P. ERDŐS AND C. POMERANCE, On the normal number of prime factors of $\phi(n)$, *Rocky Mtn. J. of Math.*, **15** (1985), 343–352.

[5] J.B. FRIEDLANDER, Shifted primes without large prime factors, in *Number Theory and Applications*, (Ed. R.A. Mollin), (Kluwer, NATO ASI, 1989), 393–401.

[6] J.B. ROSSER AND L. SHOENFELD, Approximate formulas for some functions of prime numbers, *Illinois J. of Math.*, **6** (1962), 64–94.

[7] J. SÁNDOR, On multiplicatively perfect numbers, *J. Inequal. Pure Appl. Math.*, **2**(1) (2001), Article 3. [ONLINE: http://jipam.vu.edu.au/v2n1/019_99.html]