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# HISTORY, GENERALIZATIONS AND UNIFIED TREATMENT OF TWO OSTROWSKI'S INEQUALITIES

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ABSTRACT. In this paper we present a historical review of the investigation of two Ostrowski inequalities and describe several distinct streams for their generalizations. Also we point out some new methods to obtain known results and give a number of new results related to Ostrowski's inequalities.

*Key words and phrases:* Ostrowski's inequalities, Cauchy-Buniakowski-Schwarz inequality, Gram's determinant, Unitary vector space, Superadditive function, Interpolation.

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## 1. HISTORY AND GENERALIZATIONS

In his book *Vorlesungen über Differential und Integralrechnung II,* A. Ostrowski presented the following interesting inequalities.

**Theorem 1.1.** [12, p. 289, problem 61], [10, pp. 92–93]. *The minimum of the sum* $x_1^2+\cdots+x_n^2$ *under the conditions*

$$\sum_{i=1}^n a_i x_i = 0 \ \text{ and } \ \sum_{i=1}^n b_i x_i = 1 \tag{1.1}$$

*is*

$$\frac{\sum_{i=1}^n a_i^2}{\sum_{i<j}(a_i b_j - a_j b_i)^2}, \quad \left(\sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 > 0\right). \tag{1.2}$$

**Theorem 1.2.** [12, p. 290, problem 63], [10, p. 94]. *The maximum of the sum* $\left(\sum_{i=1}^n b_i x_i\right)^2$ *under the conditions*

$$\sum_{i=1}^n a_i x_i = 0 \ \text{ and } \ \sum_{i=1}^n x_i^2 = 1 \tag{1.3}$$

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*is*

$$\frac{\sum_{i<j}(a_ib_j-a_jb_i)^2}{\sum_{i=1}^n a_i^2}, \quad \left(\sum_{i=1}^n a_i^2>0\right). \tag{1.4}$$

According to the Lagrange identity [10, p. 84], Theorem 1.1 can be rewritten in the following form.

**Theorem 1.3.** *Let $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ be two nonproportional sequences of real numbers and let $x=(x_1,\dots,x_n)$ be any real sequence which satisfies*

$$\sum_{i=1}^n a_ix_i=0 \text{ and } \sum_{i=1}^n b_ix_i=1. \tag{1.5}$$

*Then*

$$\sum_{i=1}^n x_i^2 \geq \frac{\sum_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)-(\sum_{i=1}^n a_ib_i)^2}. \tag{1.6}$$

The second Ostrowski problem can also be written in the analogue form. In the literature those forms are used more frequently than the original and have been extended, improved and generalized in different ways.

The aim of this paper is to give a brief historical review and to carry those ideas somewhat further.

K. Fan and J. Todd, [8], using Theorem 1.1, i.e. Theorem 1.3, they established the following theorem.

**Theorem 1.4.** *Let $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ $(n\geq 2)$ be two sequences of real numbers such that $a_ib_j\neq a_jb_i$ for $i\neq j$. Then*

$$\frac{\sum_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)-(\sum_{i=1}^n a_ib_i)^2} \leq \left(\frac{2}{n(n-1)}\right)^2 \sum_{i=1}^n \left(\sum_{j=1,j\neq i}^n \frac{a_j}{a_jb_i-a_ib_j}\right)^2. \tag{1.7}$$

They also generalized Theorem 1.4 using more than two vectors.

Another direction of generalization has arisen from the fact that the map $(x_1,\dots,x_n)\mapsto \sqrt{\sum_{i=1}^n x_i^2}$ is a Euclidean norm in $\mathbb{R}^n$ generated by the inner product $\langle x,y\rangle=\sum_{i=1}^n x_iy_i$. It is natural to consider an arbitrary inner product instead of the Euclidean inner product. The first generalization of that kind was done by Ž. Mitrović, [11] and after that some similar results were given in [6], [7] and [15]. Here we quote Mitrović's result.

**Theorem 1.5.** *Let $a$ and $b$ be linearly independent vectors of a unitary complex vector space $V$ and let $x$ be a vector in $V$ such that*

$$\langle x,a\rangle=\alpha \text{ and } \langle x,b\rangle=\beta. \tag{1.8}$$

*Then*

$$G(a,b)\|x\|^2\geq \|\overline{\alpha}b-\overline{\beta}a\|^2, \tag{1.9}$$

*where $G(a,b)$ is the Gram determinant of vectors $a$ and $b$. Equality holds if and only if*

$$x=\frac{1}{G(a,b)}(\langle a,\overline{\beta}a-\overline{\alpha}b\rangle b-\langle b,\overline{\beta}a-\overline{\alpha}b\rangle a). \tag{1.10}$$

Here we present a rough outline of Mitrović's proof. Let $y$ be a vector in $V$ given by

$$y = \frac{1}{G(a,b)}(\langle a, \overline{\beta}a - \overline{\alpha}b\rangle b - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle a).$$

If vector $x$ satisfies conditions (1.8), then $\langle y, y\rangle = \langle x, y\rangle = \frac{1}{G(a,b)}\|\overline{\alpha}b - \overline{\beta}a\|^2$ and $\|x-y\|^2 = \|x\|^2 - \|y\|^2$. Since $\|x-y\|^2 \geq 0$, we obtain

$$\|x\|^2 \geq \|y\|^2 = \frac{1}{G(a,b)}\|\overline{\alpha}b - \overline{\beta}a\|^2$$

and inequality (1.9) holds.

**Remark 1.6.** Now, we point out another proof of Theorem 1.5. It is well known that Gram's determinant of the vectors $x_1, x_2, x_3$ is nonnegative, i.e. inequality

$$G(x_1, x_2, x_3) \geq 0$$

holds with equality iff the vectors $x_1, x_2, x_3$ are linearly dependent. Putting $x_1 = x$, $x_2 = a$, $x_3 = b$ and using notations $\langle x, a\rangle = \alpha$ and $\langle x, b\rangle = \beta$ we have the following

$$\begin{aligned} 0 &\leq G(x,a,b) \\ &= \begin{vmatrix} \langle x,x\rangle & \langle x,a\rangle & \langle x,b\rangle \\ \langle a,x\rangle & \langle a,a\rangle & \langle a,b\rangle \\ \langle b,x\rangle & \langle b,a\rangle & \langle b,b\rangle \end{vmatrix} \\ &= G(a,b)\|x\|^2 - \langle x,a\rangle \begin{vmatrix} \langle a,x\rangle & \langle a,b\rangle \\ \langle b,x\rangle & \langle b,b\rangle \end{vmatrix} + \langle x,b\rangle \begin{vmatrix} \langle a,x\rangle & \langle a,a\rangle \\ \langle b,x\rangle & \langle b,a\rangle \end{vmatrix} \\ &= G(a,b)\|x\|^2 - \alpha(\overline{\alpha}\langle b,b\rangle - \overline{\beta}\langle a,b\rangle) + \beta(\overline{\alpha}\langle b,a\rangle - \overline{\beta}\langle a,a\rangle), \end{aligned}$$

$$G(a,b)\|x\|^2 \geq |\alpha|^2\langle b,b\rangle - \alpha\overline{\beta}\langle a,b\rangle - \beta\overline{\alpha}\langle b,a\rangle + |\beta|^2\langle a,a\rangle = \|\overline{\alpha}b - \overline{\beta}a\|^2.$$

Equality holds iff vectors $x, a$ and $b$ are linearly dependent, i.e. there exist scalars $\lambda$ and $\mu$ such that

$$x = \lambda a + \mu b.$$

Multiplying that identity by $a$ and $b$ respectively, we obtain $\alpha = \lambda\langle a,a\rangle + \mu\langle b,a\rangle$ and $\beta = \lambda\langle a,b\rangle + \mu\langle b,b\rangle$ from where we easily find that

$$\lambda = \frac{1}{G(a,b)}(\alpha\langle b,b\rangle - \beta\langle b,a\rangle), \qquad \mu = \frac{1}{G(a,b)}(\beta\langle a,a\rangle - \alpha\langle a,b\rangle).$$

So, $x$ is the vector given in (1.10).

In the same paper [11] a generalization of Fan-Todd's result is given. Furthermore, in the paper [2] P.R. Beesack noticed that inequality (1.9) and a fortiori also Ostrowski's inequality (1.6) can be regarded as a special case of the Bessel inequality for non-orthonormal vectors.

**Theorem 1.7.** [2] *Let $a_1, \ldots, a_k$, $(k \geq 1)$ be linearly independent vectors of a Hilbert space $H$ and let $\alpha_1, \ldots, \alpha_k$ be given scalars. If $x \in H$ satisfies*

$$\langle x, a_i\rangle = \alpha_i \quad 1 \leq i \leq k, \tag{1.11}$$

*then*

$$G(a_1, \ldots, a_k)^2\|x\|^2 \geq \left\|\sum_{i=1}^{k}\gamma_i^{(k)}a_i\right\|^2, \tag{1.12}$$

*where* $G(a_1,\dots,a_k)$ *is the Gram determinant of* $a_1,\dots,a_k$ *and* $\gamma_i^{(k)}$ *is the determinant obtained from* $G$ *by replacing the elements of the* $i^{\text{th}}$ *row of* $G$ *by* $(\alpha_1,\dots,\alpha_k)$. *Moreover, equality holds if and only if*

$$G(a_1,\dots,a_k)x=\sum_{i=1}^{k}\gamma_i^{(k)}a_i. \tag{1.13}$$

Finally, an analogue of Theorem 1.5 and related generalizations in 2-inner and $n$-inner spaces are given in [4] and [5].

The second stream of generalization of Ostrowski's inequality (1.6) was started by Madevski's paper [9]. He used Theorem 1.3 to obtain inequalities between certain statistical central moments. Also, he gave the following $p$-version of Ostrowski's inequality.

**Theorem 1.8.** [9] *Let* $a=(a_1,\dots,a_n)$ *and* $b=(b_1,\dots,b_n)$ *be two nonproportional sequences of real numbers and let* $x=(x_1,\dots,x_n)$ *be any real sequence which satisfies*

$$\sum_{i=1}^{n}a_ix_i=0 \ \text{ and } \ \sum_{i=1}^{n}b_ix_i=1. \tag{1.14}$$

*If* $p$ *is an integer, then*

$$\left(\sum_{i=1}^{n}x_i^2\right)^p\ge\frac{(\sum_{i=1}^{n}a_i^2)^p}{(\sum_{i=1}^{n}a_i^2)^p(\sum_{i=1}^{n}b_i^2)^p-(\sum_{i=1}^{n}a_ib_i)^{2p}}. \tag{1.15}$$

In [1] M. Alić and J. Pečarić proved that the integer $p$ can be substituted by an arbitrary real number $p\ge 1$. In the same paper a sequence of results involving moments of discrete distribution function has been given. An integral version of those results and some generalizations of known statistical inequalities given in [9], [14] and [16] are obtained in [13].

Recently, Theorems 1.1 and 1.2 have been the focus of investigation. In the papers [6] and [7] the authors have used elementary arguments and the Cauchy-Buniakowski-Schwarz inequality to obtain Ostrowski type inequalities in unitary space. Indeed, the following theorems are obtained.

**Theorem 1.9.** [7] *Let* $a$ *and* $b$ *be linearly independent vectors of a real or complex unitary vector space* $V$ *and let* $x$ *be a vector in* $V$ *such that*

$$\langle x,a\rangle=0 \ \text{ and } \ |\langle x,b\rangle|=1. \tag{1.16}$$

*Then*

$$\|x\|^2\ge\frac{\|a\|^2}{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}. \tag{1.17}$$

*Equality holds if and only if*

$$x=\mu\left(b-\frac{\langle a,b\rangle}{\|a\|^2}a\right), \tag{1.18}$$

*where* $\mu\in\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ *is such that*

$$|\mu|=\frac{\|a\|^2}{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}.$$

**Theorem 1.10.** [6] *Let* $a$ *and* $b$ *be linearly independent vectors of a real or complex unitary vector space* $V$ *and let* $x$ *be a vector in* $V$ *such that*

$$\langle x,a\rangle=0 \ \text{ and } \ \|x\|=1. \tag{1.19}$$

*Then*

$$|\langle x,b\rangle|^2 \le \frac{\|a\|^2\|b\|^2-|\langle a,b\rangle|^2}{\|a\|^2}. \tag{1.20}$$

*Equality holds if and only if*

$$x=\nu\left(b-\frac{\langle b,a\rangle}{\|a\|^2}a\right), \tag{1.21}$$

*where* $\nu\in\mathbb{K}(\mathbb{K}=\mathbb{R},\mathbb{C})$ *is so that*

$$|\nu|=\frac{\|a\|}{(\|a\|^2\|b\|^2-|\langle a,b\rangle|^2)^{\frac{1}{2}}}.$$

It is obvious that these results are special cases of Theorem 1.5 but we mentioned it because the method of proving is different from Mitrović's method and leads to another generalization which will be given in the next section. Proofs of the previous two theorems are based on the Cauchy-Buniakowski-Schwarz inequality:

$$\|u\|^2\|v\|^2\ge|\langle u,v\rangle|^2,\quad u,v\in V.$$

Applying it on vectors $u=z-\frac{\langle z,c\rangle}{\|c\|^2}c$ and $v=d-\frac{\langle d,c\rangle}{\|c\|^2}c$, where $c\neq 0$ and taking into account that

$$\left\|z-\frac{\langle z,c\rangle}{\|c\|^2}c\right\|^2=\frac{\|z\|^2\|c\|^2-|\langle z,c\rangle|^2}{\|c\|^2}, \tag{1.22}$$

$$\left\|d-\frac{\langle d,c\rangle}{\|c\|^2}c\right\|^2=\frac{\|d\|^2\|c\|^2-|\langle d,c\rangle|^2}{\|c\|^2}, \tag{1.23}$$

and

$$\left\langle z-\frac{\langle z,c\rangle}{\|c\|^2}c,\,d-\frac{\langle d,c\rangle}{\|c\|^2}c\right\rangle=\frac{\langle z,d\rangle\|c\|^2-\langle z,c\rangle\langle c,d\rangle}{\|c\|^2} \tag{1.24}$$

we have the following inequality

$$\left(\|z\|^2\|c\|^2-|\langle z,c\rangle|^2\right)\left(\|d\|^2\|c\|^2-|\langle d,c\rangle|^2\right)\ge\left|\langle z,d\rangle\|c\|^2-\langle z,c\rangle\langle c,d\rangle\right|^2. \tag{1.25}$$

Putting in inequality (1.25) $z=x$, $c=a$ and $d=b$ where $a$ and $x$ satisfy $\langle x,a\rangle=0$ and $\|x\|=1$ we get inequality (1.20), while if $a$ and $x$ satisfy $\langle x,a\rangle=0$ and $|\langle x,b\rangle|=1$ inequality (1.17) is obtained.

**Remark 1.11.** Let us mention that inequality (1.9) also can be obtained by the above-mentioned method. In fact, putting in inequality (1.25) $z=x$, $c=a$ and $d=b$, $\langle x,a\rangle=\alpha$ and $\langle x,b\rangle=\beta$ we get

$$\left(\|x\|^2\|a\|^2-|\langle x,a\rangle|^2\right)\left(\|b\|^2\|a\|^2-|\langle b,a\rangle|^2\right)\ge\left|\beta\|a\|^2-\alpha\langle a,b\rangle\right|^2,$$

$$\|x\|^2\|a\|^2G(a,b)\ge\left|\beta\|a\|^2-\alpha\langle a,b\rangle\right|^2+|\alpha|^2G(a,b)=\|a\|^2\|\overline{\alpha}b-\overline{\beta}a\|^2$$

from where inequality (1.9) occurs. Using the fact that in the Cauchy-Buniakowski-Schwarz inequality, equality holds iff vectors are proportional, we get (1.10).

**Remark 1.12.** Inequality (1.25) is a special case of the more general result related to Gram's determinant given in [10, p. 599]. That result is as follows.

**Theorem 1.13.** *For vectors $x_1, \dots, x_n$ and $y_1, \dots, y_n$ from unitary space $V$ the following inequality holds*

$$\left| \det \begin{bmatrix} \langle x_1, y_1 \rangle & \dots & \langle x_1, y_n \rangle \\ \vdots & & \vdots \\ \langle x_n, y_1 \rangle & \dots & \langle x_n, z_n \rangle \end{bmatrix} \right|^2 \le G(x_1, \dots, x_n) G(y_1, \dots, y_n),$$

*with equality iff the vectors $x_1, \dots, x_n$ span the same subspace as the vectors $y_1, \dots, y_n$.*

## 2. Further Generalizations of Ostrowski's Inequalities

In this section we extend results from papers [1], [6], [7], [13] introducing super(sub)additive function.

**Theorem 2.1.** *Let $a$ and $b$ be linearly independent vectors of a unitary complex vector space $V$ and let $x$ be a vector in $V$ such that*

$$\langle x, a \rangle = \alpha \ \text{ and } \ \langle x, b \rangle = \beta. \tag{2.1}$$

*If $\phi : [0, \infty) \to \mathbb{R}$ is a nondecreasing, superadditive function, then*

$$\phi(\|a\|^2 \|b\|^2) - \phi(|\langle a, b \rangle|^2) \ge \phi\left( \frac{\|\overline{\alpha} b - \overline{\beta} a\|^2}{\|x\|^2} \right). \tag{2.2}$$

*If $\phi$ is a nonincreasing, subadditive function then a reverse in (2.2) holds.*

*Proof.* Let us suppose that $\phi$ is a superadditive nondecreasing function. Then we have

$$\phi(u) = \phi((u - v) + v) \ge \phi(u - v) + \phi(v), \ \text{ i.e. } \phi(u) - \phi(v) \ge \phi(u - v). \tag{2.3}$$

Taking into account the nondecreasing property of $\phi$, results of Theorem 1.5 and inequality (2.3) we conclude

$$\begin{aligned} \phi(\|a\|^2 \|b\|^2) - \phi(|\langle a, b \rangle|^2) &\ge \phi(\|a\|^2 \|b\|^2 - (|\langle a, b \rangle|^2) \\ &\ge \phi\left( \frac{\|\overline{\alpha} b - \overline{\beta} a\|^2}{\|x\|^2} \right). \end{aligned}$$

The case when $\phi$ is a nonincreasing and subadditive function has been done similarly. □

In particular, inequality (2.2) holds for any nondecreasing convex function $\phi$, while its reverse holds for any nonincreasing concave function. The result of Theorem 2.1 can be improved if function $\phi$ is a power function. In that case we have the following result.

**Theorem 2.2.** *Suppose that $a$, $b$ and $x$ are as in Theorem 2.1. If $p \ge 1$, then*

$$\begin{aligned} &\|x\|^{2p} \|a\|^{2p} (\|a\|^{2p} \|b\|^{2p} - |\langle a, b \rangle|^{2p}) \\ &\quad \ge \max \left\{ \|a\|^{2p} \|\overline{\alpha} b - \overline{\beta} a\|^{2p}, |\beta \|a\|^2 - \alpha \langle a, b \rangle|^{2p} + |\alpha|^{2p} (\|a\|^{2p} \|b\|^{2p} - |\langle a, b \rangle|^{2p}) \right\}. \end{aligned} \tag{2.4}$$

*Proof.* The function $\phi(x) = x^p$, $p \ge 1$ is a nondecreasing superadditive function so, a direct consequence of the previous theorem is that for $a$, $b$ and $x$ which satisfy assumptions of Theorem 2.1 we have the following inequality

$$\|a\|^{2p} \|b\|^{2p} - |\langle a, b \rangle|^{2p} \ge \frac{\|\overline{\alpha} b - \overline{\beta} a\|^{2p}}{\|x\|^{2p}}. \tag{2.5}$$

Applying the method of proving in Theorem 2.1 on inequality (1.25) we get

$$\left( \|z\|^{2p} \|c\|^{2p} - |\langle z, c \rangle|^{2p} \right) \left( \|d\|^{2p} \|c\|^{2p} - |\langle d, c \rangle|^{2p} \right) \ge \left| \langle z, d \rangle \|c\|^2 - \langle z, c \rangle \langle c, d \rangle \right|^{2p} \tag{2.6}$$

i.e. putting $z = x$, $c = a$, $d = b$ and taking into account that $\langle x, a\rangle = \alpha$ and $\langle x, b\rangle = \beta$ we have

$$\left(\|x\|^{2p}\|a\|^{2p} - |\alpha|^{2p}\right)\left(\|b\|^{2p}\|a\|^{2p} - |\langle b, a\rangle|^{2p}\right) \geq \left|\beta\|a\|^2 - \alpha\langle a, b\rangle\right|^{2p}. \tag{2.7}$$

After simple calculations, inequalities (2.6) and (2.7) give inequality (2.4). □

**Remark 2.3.** If $p = 1$, then the two terms on the righthand side of inequality (2.4) are equal, but if $p > 1$ terms $\|a\|^{2p}\|\overline{\alpha}b - \overline{\beta}a\|^{2p}$ and $|\beta\|a\|^2 - \alpha\langle a, b\rangle|^{2p} + |\alpha|^{2p}(\|a\|^{2p}\|b\|^{2p} - |\langle a, b\rangle|^{2p})$ are not comparable. For example, if $p = 2$, $\|a\| = 1$, $\|b\| = 1$, $\langle a, b\rangle = \frac{1}{2}$ and $\alpha = 1$, $\beta \in \mathbb{R}$ the first term is equal to $(\beta^2 - \beta + 1)^2$, while the second term is equal to $(\beta^2 - \beta + \frac{1}{4})^2 + \frac{15}{16}$. If $\beta \in (0, 1)$ the first term is less than the second term and if $\beta > 1$ the opposite inequality holds.

## 3. INTERPOLATION

Some results about refinements of the original first Ostrowski's inequality are given in [3]. Here we give more general results in which we consider refinements of Ostrowski's inequalities in arbitrary unitary complex vector spaces.

**Theorem 3.1.** *Let $a$ and $b$ be linearly independent vectors in a unitary complex vector space $V$ and let $x$ be a vector in $V$ such that*

$$\langle x, a\rangle = \alpha \quad \text{and} \quad \langle x, b\rangle = \beta. \tag{3.1}$$

*Let $y$ be a vector defined by*

$$y = \frac{1}{G(a, b)}(\langle a, \overline{\beta}a - \overline{\alpha}b\rangle b - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle a). \tag{3.2}$$

*Then the vector $F(x) = \theta x + (1 - \theta)y$, $\theta \in [0, 1]$, satisfies*

$$\|x\|^2 \geq \|F(x)\|^2 \tag{3.3}$$

*and*

$$G(a, b)\|F(x)\|^2 \geq \|\overline{\alpha}b - \overline{\beta}a\|^2. \tag{3.4}$$

*Proof.* Let us note that $y$ is a vector for which equality in (1.9) holds, i.e.

$$G(a, b)\|y\|^2 = \|\overline{\alpha}b - \overline{\beta}a\|^2. \tag{3.5}$$

So, without any calculation we conclude that $\langle y, a\rangle = \alpha$ and $\langle y, b\rangle = \beta$. Now,

$$\langle F(x), a\rangle = \langle \theta x + (1 - \theta)y, a\rangle = \theta\alpha + (1 - \theta)\alpha = \alpha. \tag{3.6}$$

Similarly, we obtain

$$\langle F(x), b\rangle = \beta. \tag{3.7}$$

According to Theorem 1.5 and in view of (3.6) and (3.7) we get

$$G(a, b)\|F(x)\|^2 \geq \|\overline{\alpha}b - \overline{\beta}a\|^2.$$

Let us calculate the product $\langle y, x\rangle$.

$$\begin{aligned}
G(a, b)\langle y, x\rangle &= \langle\langle a, \overline{\beta}a - \overline{\alpha}b\rangle b - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle a, x\rangle \\
&= \langle a, \overline{\beta}a - \overline{\alpha}b\rangle\langle b, x\rangle - \langle b, \overline{\beta}a - \overline{\alpha}b\rangle\langle a, x\rangle \\
&= \overline{\beta}(\beta\langle a, a\rangle - \alpha\langle a, b\rangle) - \overline{\alpha}(\beta\langle b, a\rangle - \alpha\langle b, b\rangle) \\
&= |\beta\|^2\|a\|^2 - \alpha\overline{\beta}\langle a, b\rangle - \overline{\alpha}\beta\langle b, a\rangle + |\alpha|^2\|b\|^2 \\
&= \|\overline{\alpha}b - \overline{\beta}a\|^2.
\end{aligned}$$

Comparing this result with (3.5) we have $\langle y, x\rangle = \langle y, y\rangle = \langle x, y\rangle$. Using these equalities we obtain

$$\begin{aligned}\|F(x)\|^2 &= \langle F(x), F(x)\rangle \\ &= \theta^2\|x\|^2 + \theta(1-\theta)\langle x, y\rangle + (1-\theta)\theta\langle y, x\rangle + (1-\theta)^2\langle y, y\rangle \\ &= \theta^2\|x\|^2 + (1-\theta^2)\|y\|^2.\end{aligned}$$

$$\|x\|^2 - \|F(x)\|^2 = (1-\theta^2)(\|x\|^2 - \|y\|^2) = (1-\theta^2)(\|x-y\|^2) \geq 0$$

and inequality (3.3) has been established. □

Thus we obtain a sequence of succesive approximations

$$x, F(x), F^2(x), \dots, F^n(x), \dots$$

converging to $y$ for $\theta < 1$ which interpolate inequality (1.9)

$$\|x\|^2 \geq \|F(x)\|^2 \geq \|F^2(x)\|^2 \geq \cdots \geq \|F^n(x)\|^2 \geq \cdots \geq \|y\|^2 = \frac{\|\overline{\alpha}b - \overline{\beta}a\|^2}{G(a,b)}.$$

If $\alpha = 0$, $\beta = 1$, $\theta = \frac{1}{2}$ and $\|x\|^2 = \sum_{i=1}^n x_i^2$, then we get a result of M. Bjelica, [3].

## REFERENCES


[1] M. ALIĆ AND J. PEČARIĆ, On some inequalities of Ž. Madevski and A.M. Ostrowski, *Rad HAZU, Matematičke znanosti,* **472** (1997), 77–82.

[2] P.R. BEESACK, On Bessel's inequality and Ostrowski's, *Univ. Beograd Publ.Elektrotehn. Fak. Ser. Mat. Fiz.,* No **498-541** (1975), 69–71.

[3] M. BJELICA, Refinements of Ostrowski's and Fan-Todd's inequalities, in *Recent Progress in Inequalities*, G.V. Milovanović (Ed.), Kluwer Academic Publishers, Dordrecht, 1998, 445–448.

[4] Y.J. CHO, M. MATIĆ AND J. PEČARIĆ, On Gram's determinant in 2-inner product spaces, *J. Korean Math. Soc.,* **38**(6) (2001), 1125–1156.

[5] Y.J. CHO, M. MATIĆ AND J. PEČARIĆ, On Gram's determinant in n-inner product spaces, *Bull. Korean Math. Soc.,*to appear.

[6] S.S. DRAGOMIR AND A.C. GOŞA, A generalization of an Ostrowski inequality in inner product spaces, *RGMIA Research Report Collection,* **6**(2) (2003), Article 20. [ONLINE] `http://rgmia.vu.edu.au/v6n2.html`.

[7] S.S. DRAGOMIR, Ostrowski's inequality in complex inner product spaces, *RGMIA Research Report Collection* **6**(Supp.) (2003), Article 6. [ONLINE] `http://rgmia.vu.edu.au/v6(E).html`.

[8] K. FAN AND J. TODD, A determinantal inequality, *J. London Math. Soc.,* **30** (1955), 58–64.

[9] Ž. MADEVSKI, Quelques conséquences d'une inégalité d'Ostrowski, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.,* No **412-460** (1973), 168–170.

[10] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities,* Kluwer Academic Publishers, Dordrecht, 1993.

[11] Ž. MITROVIĆ, On a generalization of Fan-Todd's inequality, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No **412-460** (1973), 151–154.

[12] A. OSTROWSKI, *Vorlesungen über Differential und Integralrechnung II*, Verlag Birkhüser Basel, 1951.

[13] C.E.M. PEARCE, J. PEČARIĆ AND S. VAROŠANEC, An integral analogue of the Ostrowski inequality, *J. Ineq. Appl.*, **2** (1998), 275–283.

[14] K. PEARSON, Mathematical contributions to the theory of evolution. XIX: second supplement to a memoir on skew variation, *Phil.Trans.Roy. Soc. A*, **216** (1916), 432.

[15] H. ŠIKIĆ AND T. ŠIKIĆ, A note on Ostrowski's inequality, *Math. Ineq. Appl.*, **4**(2) (2001), 297–299.

[16] J.E. WILKINS, A note on skewness and kurtosis, *Ann. Math. Statist.*, **15** (1944), 333–335.