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ON AN INEQUALITY OF FENG QI

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ABSTRACT. Recently Feng Qi has presented a sharp inequality between the sum of squares and the exponential of the sum of a nonnegative sequence. His result has been extended to more general power sums by Huan-Nan Shi, and, independently, by Yu Miao, Li-Min Liu, and Feng Qi. In this note we generalize those inequalitites by introducing weights and permitting more general functions. Inequalities in the opposite direction are also presented.

Key words and phrases: Optimal inequality, Power sum, Subadditive, Superadditive, Power mean inequality.

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1. INTRODUCTION

The following inequality is due to Feng Qi [2].

Let $x_1, x_2, \dots, x_n$ be arbitrary nonnegative numbers. Then

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \le \exp\left(\sum_{i=1}^{n} x_i\right). \tag{1.1}$$

Equality holds if and only if all but one of $x_1, \dots, x_n$ are 0, and the missing one is equal to 2. Thus the constant $e^2/4$ is the best possible. Moreover, (1.1) is also valid for infinite sums.

In answer of an open question posed by Qi, Shi [3] extended (1.1) to more general power sums on the left-hand side, proving that

$$\frac{e^\alpha}{\alpha^\alpha}\sum_{i=1}^{n} x_i^\alpha \le \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.2}$$

for $\alpha \ge 1$, and $n \le \infty$.

After the present paper had been prepared, Yu Miao, Li-Min Liu, and Feng Qi also published Shi's result for integer values of α, see [1].

In papers [2] and [3], after taking the logarithm of both sides, the authors considered the left-hand side expression as an n-variate function, and maximized it under the condition of

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$x_1 + \cdots + x_n$ fixed. To this end Qi applied differential calculus, while Shi used Schur convexity. Both methods relied heavily on the properties of the log function.

On the other hand, [1] uses a probability theory argument, which also seems to utilize the particular choice of functions in the inequality.

In the present note we present extensions of (1.2) by permitting arbitrary positive functions on both sides and weights in the sums. Our method is simple and elementary.

Theorem 1.1. *Let $w_1, w_2, \dots, w_n$ be positive weights, f a positive function defined on $[0, \infty)$, and let $\alpha > 0$. Then for arbitrary nonnegative numbers $x_1, x_2, \dots, x_n$ the inequality*

$$C \sum_{i=1}^{n} w_i x_i^{\alpha} \le f\left(\sum_{i=1}^{n} w_i x_i\right) \tag{1.3}$$

is valid with

$$C = w_0^{\alpha-1} \inf_{x>0} x^{-\alpha} f(x), \tag{1.4}$$

where

$$w_0 = \begin{cases} \min\{w_1, \dots, w_n\} & \text{if } \alpha \ge 1, \\ w_1 + \cdots + w_n & \text{if } \alpha < 1. \end{cases} \tag{1.5}$$

This inequality is sharp in the sense that C cannot be replaced by any greater constant.

Remark 1. The necessary and sufficient condition for equality in (1.3) is the following.

Case $\alpha > 1$. There is exactly one x_i differing from zero, for which $w_i = w_0$ and $w_0 x_i$ minimizes $x^{-\alpha} f(x)$ in $(0, \infty)$.

Case $\alpha = 1$. $\sum_{i=1}^{n} w_i x_i$ minimizes $x^{-\alpha} f(x)$ in $(0, \infty)$.

Case $\alpha < 1$. $x_1 = \cdots = x_n$, and $w_0 x_1$ minimizes $x^{-\alpha} f(x)$ in $(0, \infty)$.

Remark 2. Inequality (1.3) can be extended to infinite sums. Let f and α be as in Theorem 1.1, and let $\{w_i\}_{i=1}^{\infty}$ be an infinite sequence of positive weights such that $w_0 := \inf_{1 \le i < \infty} w_i > 0$ when $\alpha \ge 1$, and $w_0 := \sum_{i=1}^{\infty} w_i < \infty$ when $\alpha < 1$. Then for an arbitrary nonnegative sequence $\{x_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} w_i x_i < \infty$ the following inequality holds.

$$C \sum_{i=1}^{\infty} w_i x_i^{\alpha} \le f\left(\sum_{i=1}^{\infty} w_i x_i\right),$$

where C is defined in (1.4).

Remark 3. By setting $\alpha \ge 1$, $f(x) = e^x$ and $w_1 = w_2 = \cdots = 1$ we get Theorems 1 and 2 of [3]. In particular, taking $\alpha = 2$ implies Theorems 1.1 and 1.2 of [2].

2. CONVERSE INEQUALITIES

Qi posed the problem of determining the optimal constant C for which

$$\exp\left(\sum_{i=1}^{n} x_i\right) \le C \sum_{i=1}^{n} x_i^{\alpha} \tag{2.1}$$

holds for arbitrary nonnegative $x_1, \dots, x_n$, with a given positive α. As Shi pointed out, such an inequality is generally untenable, because the exponential function grows faster than any power function. However, if the exponential function is replaced with a suitable one, the following inequalities, analogous to those of Theorem 1.1, have sense.

Theorem 2.1. *Let* $w_1, w_2, \dots, w_n$ *be positive weights,* f *a positive function defined on* $[0,\infty)$*, and let* $\alpha > 0$*. Suppose* $\sup_{x>0} x^{-\alpha} f(x) < \infty$*. Then for arbitrary nonnegative numbers* $x_1, x_2, \dots, x_n$ *the inequality*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le C \sum_{i=1}^{n} w_i x_i^{\alpha} \tag{2.2}$$

is valid with

$$C = w_0^{\alpha-1} \sup_{x>0} x^{-\alpha} f(x), \tag{2.3}$$

where

$$w_0 = \begin{cases} \min\{w_1, \dots, w_n\} & \text{if } \alpha \le 1, \\ w_1 + \cdots + w_n & \text{if } \alpha > 1. \end{cases} \tag{2.4}$$

This inequality is sharp in the sense that C *cannot be replaced by any smaller constant.*

Remark 4. The necessary and sufficient condition for equality in (2.2) is the following.

Case $\alpha < 1$. There is exactly one x_i differing from zero, for which $w_i = w_0$ and $w_0 x_i$ maximizes $x^{-\alpha} f(x)$ in $(0,\infty)$.

Case $\alpha = 1$. $\sum_{i=1}^{n} w_i x_i$ maximizes $x^{-\alpha} f(x)$ in $(0,\infty)$.

Case $\alpha > 1$. $x_1 = \cdots = x_n$, and $w_0 x_1$ maximizes $x^{-\alpha} f(x)$ in $(0,\infty)$.

Remark 5. Inequality (2.2) also remains valid for infinite sums. Let f and α be as in Theorem 2.1, and let $\{w_i\}_{i=1}^{\infty}$ be an infinite sequence of positive weights such that $w_0 := \inf_{1\le i<\infty} w_i > 0$ when $\alpha > 1$, and $w_0 := \sum_{i=1}^{\infty} w_i < \infty$ when $\alpha < 1$. Then for an arbitrary nonnegative sequence $\{x_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} w_i x_i < \infty$ the following inequality holds.

$$f\left(\sum_{i=1}^{\infty} w_i x_i\right) \le C \sum_{i=1}^{\infty} w_i x_i^{\alpha},$$

where C is defined in (2.3).

3. FURTHER GENERALIZATIONS

Inequalities (1.3) and (2.2) can be further generalized by replacing the power function with more general functions. Unfortunately, the inequalities thus obtained are not necessarily sharp anymore.

Let us introduce four classes of nonnegative power-like functions $g : [0,\infty) \to \mathbb{R}$ that are positive for positive x.

$$\mathcal{F}_1 = \{g : g(x) + g(y) \le g(x+y),\ g(x)g(y) \le g(xy) \text{ for } x, y \ge 0\}, \tag{3.1}$$

$$\mathcal{F}_2 = \{g : g \text{ is concave, } g(x)g(y) \le g(xy) \text{ for } x, y \ge 0\}, \tag{3.2}$$

$$\mathcal{F}_3 = \{g : g(x) + g(y) \ge g(x+y),\ g(x)g(y) \ge g(xy) \text{ for } x, y \ge 0\}, \tag{3.3}$$

$$\mathcal{F}_4 = \{g : g \text{ is convex, } g(x)g(y) \ge g(xy) \text{ for } x, y \ge 0\}. \tag{3.4}$$

Obviously, the power function $g(x) = x^{\alpha}$ belongs to $\mathcal{F}_1$ and $\mathcal{F}_4$ if $\alpha \ge 1$, and to $\mathcal{F}_2$ and $\mathcal{F}_3$ if $\alpha \le 1$. In fact, our classes are wider.

Theorem 3.1. *Let* p_1, p_2, α_1, α_2 *be positive parameters and*

$$g(x) = \begin{cases} p_1 x^{\alpha_1}, & \text{if } 0 \le x \le 1, \\ p_2 x^{\alpha_2}, & \text{if } 1 < x. \end{cases} \tag{3.5}$$

Then

$$p_1 \le p_2 \le 1,\ \ 1 \le \alpha_2 \le \alpha_1 \ \Rightarrow\ g \in \mathcal{F}_1, \tag{3.6}$$

$$p_1 = p_2 \le 1,\ \ \alpha_2 \le \alpha_1 \le 1 \ \Rightarrow\ g \in \mathcal{F}_2, \tag{3.7}$$

$$1 \le p_2 \le p_1,\ \ \alpha_1 \le \alpha_2 \le 1 \ \Rightarrow\ g \in \mathcal{F}_3, \tag{3.8}$$

$$1 \le p_2 = p_1,\ \ 1 \le \alpha_1 \le \alpha_2 \ \Rightarrow\ g \in \mathcal{F}_4. \tag{3.9}$$

It would be of independent interest to characterize these four classes.

Our last theorem generalizes Theorems 1.1 and 2.1.

Theorem 3.2. *Let $w_1, w_2, \dots, w_n$ be fixed positive weights, and $x_1, x_2, \dots, x_n$ arbitrary non-negative numbers. Let f be a positive function defined on $[0,\infty)$.*

Suppose $g \in \mathcal{F}_1$. Then

$$C \sum_{i=1}^n w_i g(x_i) \le f\left(\sum_{i=1}^n w_i x_i\right) \tag{3.10}$$

is valid with

$$C = \min_{1\le i\le n} \frac{g(w_i)}{w_i} \cdot \inf_{x>0} \frac{f(x)}{g(x)}. \tag{3.11}$$

Suppose $g \in \mathcal{F}_2$. Then (3.10) *holds with*

$$C = \frac{g(w_0)}{w_0} \cdot \inf_{x>0} \frac{f(x)}{g(x)}, \tag{3.12}$$

where $w_0 = w_1 + \dots + w_n$.

Suppose $g \in \mathcal{F}_3$, and $\sup_{x>0} \frac{f(x)}{g(x)} < \infty$. Then

$$f\left(\sum_{i=1}^n w_i x_i\right) \le C \sum_{i=1}^n w_i g(x_i) \tag{3.13}$$

is valid with

$$C = \max_{1\le i\le n} \frac{g(w_i)}{w_i} \cdot \sup_{x>0} \frac{f(x)}{g(x)}. \tag{3.14}$$

Suppose $g \in \mathcal{F}_4$, and $\sup_{x>0} \frac{f(x)}{g(x)} < \infty$. Then (3.13) *holds with*

$$C = \frac{g(w_0)}{w_0} \cdot \sup_{x>0} \frac{f(x)}{g(x)}, \tag{3.15}$$

where $w_0 = w_1 + \dots + w_n$.

4. PROOFS

Proof of Theorem 1.1. First, let $\alpha \ge 1$. Making use of the superadditive property of the α-power function we obtain

$$\begin{aligned} f\left(\sum_{i=1}^n w_i x_i\right) &\ge \inf_{x>0} x^{-\alpha} f(x) \left(\sum_{i=1}^n w_i x_i\right)^\alpha \\ &\ge \inf_{x>0} x^{-\alpha} f(x) \sum_{i=1}^n (w_i x_i)^\alpha \\ &\ge w_0^{\alpha-1} \inf_{x>0} x^{-\alpha} f(x) \cdot \sum_{i=1}^n w_i x_i^\alpha, \end{aligned} \tag{4.1}$$

which was to be proved.

Suppose (1.3) is valid for arbitrary nonnegative numbers x_i with some constant C. Let $x_j = 0$ for $j \neq i$, where i is chosen to satisfy $w_i = w_0$. Then from (1.3) we obtain that $Cw_0 x_i^\alpha \leq f(w_0 x_i)$ must hold for every $x_i > 0$. Hence $C \leq w_0^{\alpha-1} \inf x^{-\alpha} f(x)$.

The proof is similar for $\alpha < 1$. By applying the α-power mean inequality we have

$$
\begin{aligned}
f\left(\sum_{i=1}^n w_i x_i\right) &\geq \inf_{x>0} x^{-\alpha} f(x) \left(\sum_{i=1}^n w_i x_i\right)^\alpha \\
&= \inf_{x>0} x^{-\alpha} f(x)\, w_0^\alpha \left(w_0^{-1} \sum_{i=1}^n w_i x_i\right)^\alpha \\
&\geq \inf_{x>0} x^{-\alpha} f(x)\ w_0^{\alpha-1} \sum_{i=1}^n w_i x_i^\alpha,
\end{aligned} \tag{4.2}
$$

as required.

Again, if (1.3) is valid for arbitrary nonnegative numbers x_i with some constant C, let $x_1 = \cdots = x_n = x > 0$. Then it follows that $Cw_0 x^\alpha \leq f(w_0 x)$ for every $x > 0$, implying $C \leq w_0^{\alpha-1} \inf x^{-\alpha} f(x)$. □

Proof of Remark 1. Let $\alpha > 1$. In the second inequality of (4.1) equality holds if and only if there is at most one positive term in the sum. Since f is positive, for $x_1 = \cdots = x_n = 0$ (1.3) holds true with strict inequality. Let x_i be the only positive term in the sum, then the first inequality fulfils with equality if and only if $w_i x_i = \arg\min x^{-\alpha} f(x)$. The last inequality is strict if $w_i > w_0$.

Similarly, in the case of $\alpha < 1$ we need $x_1 = \cdots = x_n$ for equality in the α-power mean inequality. Then $\sum_{i=1}^n w_i x_i = w_0 x_1$, and the first inequality of (4.2) is strict if $w_0 x_1$ does not minimize $x^{-\alpha} f(x)$.

Finally, the case of $\alpha = 1$ is obvious. □

Proof of Remark 2. The proof of (1.3) is valid for infinite sums, too, because both the superadditivity of power functions with exponent $\alpha \geq 1$, and the α-power mean inequality remain true for an infinite number of terms. □

Proof of Theorem 2.1. The proof of Theorem 1.1 can be repeated with obvious alterations. Let $\alpha \leq 1$. Then, by the subadditivity of the α-power function we have

$$
\begin{aligned}
f\left(\sum_{i=1}^n w_i x_i\right) &\leq \sup_{x>0} x^{-\alpha} f(x) \left(\sum_{i=1}^n w_i x_i\right)^\alpha \\
&\leq \sup_{x>0} x^{-\alpha} f(x) \sum_{i=1}^n (w_i x_i)^\alpha \\
&\leq w_0^{\alpha-1} \sup_{x>0} x^{-\alpha} f(x) \cdot \sum_{i=1}^n w_i x_i^\alpha.
\end{aligned} \tag{4.3}
$$

If $\alpha > 1$, we have to apply the α-power mean inequality again.

$$
f\left(\sum_{i=1}^n w_i x_i\right) \leq \sup_{x>0} x^{-\alpha} f(x) \left(\sum_{i=1}^n w_i x_i\right)^\alpha \tag{4.4}
$$

$$= \sup_{x>0} x^{-\alpha} f(x)\, w_0^{\alpha} \left(w_0^{-1} \sum_{i=1}^{n} w_i x_i \right)^{\alpha}$$
$$\leq \sup_{x>0} x^{-\alpha} f(x)\ w_0^{\alpha-1} \sum_{i=1}^{n} w_i x_i^{\alpha}.$$

Suppose (2.2) is valid for arbitrary nonnegative numbers x_i with some constant C. If $\alpha \leq 1$, let $x_j = 0$ for $j \neq i$, where i is chosen to satisfy $w_i = w_0$, and let $x_i = x > 0$. In the complementary case let $x_1 = \cdots = x_n = x > 0$. In both cases from (2.2) we obtain that $f(w_0 x) \leq C w_0 x^{\alpha}$ must hold for every $x > 0$. Hence $C \geq w_0^{\alpha-1} \sup x^{-\alpha} f(x)$. □

The proofs of Remarks 4 and 5, being straightforward adaptations of what we have done in the proofs of Remarks 1 and 2, resp., are left to the reader.

Proof of Theorem 3.1. Throughout we will suppose that $x \leq y$.
Proof of (3.6). First we show that g is superadditive. It obviously holds if $x + y \leq 1$ or $x > 1$. If $y \leq 1 < x + y$, then

$$g(x) + g(y) = p_1(x^{\alpha_1} + y^{\alpha_1}) \leq p_1(x^{\alpha_2} + y^{\alpha_2}) \leq p_1(x+y)^{\alpha_2} \leq p_2(x+y)^{\alpha_2} = g(x+y).$$

Finally, if $x \leq 1 < y$, then

$$g(x) + g(y) = p_1 x^{\alpha_1} + p_2 y^{\alpha_2} \leq p_2(x^{\alpha_2} + y^{\alpha_2}) \leq p_2(x+y)^{\alpha_2} = g(x+y).$$

Let us turn to supermultiplicativity. It is valid if $y \leq 1$ or $x > 1$. Let $x \leq 1 < y$, then $g(x)g(y) = p_1 x^{\alpha_1} p_2 y^{\alpha_2} \leq p_1 (xy)^{\alpha_1}$, because $p_2 y^{\alpha_2} \leq y^{\alpha_1}$. On the other hand, $g(x)g(y) \leq p_2(xy)^{\alpha_2}$, because $p_1 x^{\alpha_1} \leq x^{\alpha_2}$. Thus $g(x)g(y) \leq g(xy)$.
Proof of (3.7). $g'(x) = p_1 \alpha_1 x^{\alpha_1 - 1}$ if $0 < x < 1$, and $g'(x) = p_1 \alpha_2 x^{\alpha_2 - 1}$ if $x > 1$. Thus $g'(x)$ is decreasing, hence g is concave. The proof of supermultiplicativity is the same as in the proof of (3.6).
Proof of (3.8). It can be done along the lines of the proof of (3.6), but with all inequality signs reversed. Let us begin with the subadditivity. It is obvious, if $x + y \leq 1$ or $x > 1$. If $y \leq 1 < x + y$, then

$$g(x) + g(y) = p_1(x^{\alpha_1} + y^{\alpha_1}) \geq p_1(x^{\alpha_2} + y^{\alpha_2}) \geq p_1(x+y)^{\alpha_2} \geq p_2(x+y)^{\alpha_2} = g(x+y).$$

If $x \leq 1 < y$, then

$$g(x) + g(y) = p_1 x^{\alpha_1} + p_2 y^{\alpha_2} \geq p_2(x^{\alpha_2} + y^{\alpha_2}) \geq p_2(x+y)^{\alpha_2} = g(x+y).$$

Concerning submultiplicativity, it obviously holds when $y \leq 1$ or $x > 1$. Let $x \leq 1 < y$. Then $g(x)g(y) = p_1 x^{\alpha_1} p_2 y^{\alpha_2}$ does not exceed $p_1(xy)^{\alpha_1}$ on the one hand, and $p_2(xy)^{\alpha_2}$ on the other hand. Hence $g(x)g(y) \geq g(xy)$.
Proof of (3.9). This time $g'(x)$ is increasing, thus g is convex. The submultiplicativity of g has already been proved above. □

Proof of Theorem 3.2. We proceed similarly to the proofs of Theorems 1.1 and 2.1.
Let $g \in \mathcal{F}_1$. Then

$$f\left(\sum_{i=1}^{n} w_i x_i \right) \geq \inf_{x>0} \frac{f(x)}{g(x)} \cdot g\left(\sum_{i=1}^{n} w_i x_i \right) \tag{4.5}$$
$$\geq \inf_{x>0} \frac{f(x)}{g(x)}\ \sum_{i=1}^{n} g(w_i x_i)$$

$$\geq \inf_{x>0} \frac{f(x)}{g(x)} \sum_{i=1}^{n} g(w_i) g(x_i)$$

$$\geq \inf_{x>0} \frac{f(x)}{g(x)} \min_{1\leq i\leq n} \frac{g(w_i)}{w_i} \sum_{i=1}^{n} w_i g(x_i).$$

For the second inequality we applied the superadditivity of g, and for the third one the supermultiplicativity.

Let $g \in \mathcal{F}_2$. Using concavity at first, then supermultiplicativity, we obtain that

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \geq \inf_{x>0} \frac{f(x)}{g(x)} \cdot g\left(\sum_{i=1}^{n} w_i x_i\right) \tag{4.6}$$

$$= \inf_{x>0} \frac{f(x)}{g(x)} \cdot g\left(\frac{1}{w_0} \sum_{i=1}^{n} w_i w_0 x_i\right)$$

$$\geq \inf_{x>0} \frac{f(x)}{g(x)} \cdot \frac{1}{w_0} \sum_{i=1}^{n} w_i g(w_0 x_i)$$

$$\geq \inf_{x>0} \frac{f(x)}{g(x)} \cdot \frac{1}{w_0} \sum_{i=1}^{n} w_i g(w_0) g(x_i),$$

as required.

The proof of (3.13) in the cases of $g \in \mathcal{F}_3$ and $g \in \mathcal{F}_4$ can be performed analogously to (4.5) and (4.6), resp., with every inequality sign reversed, and wherever $\inf$ or $\min$ appears they have to be changed to $\sup$ and $\max$, resp. □

Unfortunately, nothing can be said about the condition of equality in the sub/supermultiplicative steps. This is why inequalities (3.10) and (3.13) are not sharp in general.

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