

REFINEMENTS OF REVERSE TRIANGLE INEQUALITIES IN INNER PRODUCT SPACES

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Abstract

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Abstract

Refining some results of S.S. Dragomir, several new reverses of the triangle inequality in inner product spaces are obtained.

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1. Introduction

It is interesting to know under which conditions the triangle inequality reverses in a normed space X ; in other words, we would like to know if there is a constant c with the property that $c \sum_{k=1}^n \|x_k\| \leq \|\sum_{k=1}^n x_k\|$ for some finite set $x_1, \dots, x_n \in X$. M. Nakai and T. Tada [7] proved that the normed spaces with this property for any finite set $x_1, \dots, x_n \in X$ are only those of finite dimension.

The first authors to investigate the reverse of the triangle inequality in inner product spaces were J. B. Diaz and F. T. Metcalf [2]. They did so by establishing the following result as an extension of an inequality given by M. Petrovich [8] for complex numbers:

Theorem 1.1 (Diaz-Metcalf Theorem). *Let a be a unit vector in the inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose the vectors $x_k \in H$, $k \in \{1, \dots, n\}$ satisfy*

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_k, a \rangle}{\|x_k\|}, \quad k \in \{1, \dots, n\}.$$

Then

$$r \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|,$$

where equality holds if and only if

$$\sum_{k=1}^n x_k = r \sum_{k=1}^n \|x_k\| a.$$



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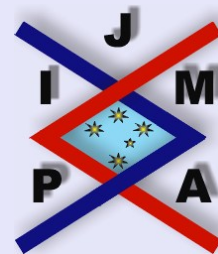
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Inequalities related to the triangle inequality are of special interest; cf. Chapter XVII of [6] and may be applied to obtain inequalities in complex numbers or to study vector-valued integral inequalities [3], [4].

Using several ideas and the notation of [3], [4] we modify or refine some results of S.S. Dragomir to procure some new reverses of the triangle inequality (see also [1]).

We use repeatedly the Cauchy-Schwarz inequality without mentioning it. The reader is referred to [9], [5] for the terminology of inner product spaces.



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2. Main Results

The following theorem is an improvement of Theorem 2.1 of [4] in which the real numbers r_1, r_2 are not necessarily nonnegative. The proof seems to be different as well.

Theorem 2.1. *Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H, k \in \{1, \dots, n\}$ satisfy*

$$(2.1) \quad 0 \leq r_1^2 \|x_k\| \leq \operatorname{Re} \langle x_k, r_1 a \rangle, \quad 0 \leq r_2^2 \|x_k\| \leq \operatorname{Im} \langle x_k, r_2 a \rangle$$

for some $r_1, r_2 \in [-1, 1]$. Then we have the inequality

$$(2.2) \quad (r_1^2 + r_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|.$$

The equality holds in (2.2) if and only if

$$(2.3) \quad \sum_{k=1}^n x_k = (r_1 + ir_2) \sum_{k=1}^n \|x_k\| a.$$

Proof. If $r_1^2 + r_2^2 = 0$, the theorem is trivial. Assume that $r_1^2 + r_2^2 \neq 0$. Summing inequalities (2.1) over k from 1 to n , we have

$$(r_1^2 + r_2^2) \sum_{k=1}^n \|x_k\| \leq \operatorname{Re} \left\langle \sum_{k=1}^n x_k, r_1 a \right\rangle + \operatorname{Im} \left\langle \sum_{k=1}^n x_k, r_2 a \right\rangle$$



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$$\begin{aligned}
&= \operatorname{Re} \left\langle \sum_{k=1}^n x_k, (r_1 + ir_2)a \right\rangle \\
&\leq \left| \left\langle \sum_{k=1}^n x_k, (r_1 + ir_2)a \right\rangle \right| \\
&\leq \left\| \sum_{k=1}^n x_k \right\| \|(r_1 + ir_2)a\| \\
&= (r_1^2 + r_2^2)^{\frac{1}{2}} \left\| \sum_{k=1}^n x_k \right\|.
\end{aligned}$$

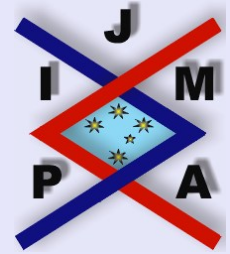
Hence (2.2) holds.

If (2.3) holds, then

$$\left\| \sum_{k=1}^n x_k \right\| = \left\| (r_1 + ir_2) \sum_{k=1}^n \|x_k\| a \right\| = (r_1^2 + r_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\|.$$

Conversely, if the equality holds in (2.2), we have

$$\begin{aligned}
(r_1^2 + r_2^2)^{\frac{1}{2}} \left\| \sum_{k=1}^n x_k \right\| &= (r_1^2 + r_2^2) \sum_{k=1}^n \|x_k\| \\
&\leq \operatorname{Re} \left\langle \sum_{k=1}^n x_k, (r_1 + ir_2)a \right\rangle \\
&\leq \left| \left\langle \sum_{k=1}^n x_k, (r_1 + ir_2)a \right\rangle \right| \leq (r_1^2 + r_2^2)^{\frac{1}{2}} \left\| \sum_{k=1}^n x_k \right\|.
\end{aligned}$$



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From this we deduce

$$\left| \left\langle \sum_{k=1}^n x_k, (r_1 + ir_2)a \right\rangle \right| = \left\| \sum_{k=1}^n x_k \right\| \|(r_1 + ir_2)a\|.$$

Consequently there exists $\eta \geq 0$ such that

$$\sum_{k=1}^n x_k = \eta(r_1 + ir_2)a.$$

From this we have

$$(r_1^2 + r_2^2)^{\frac{1}{2}}\eta = \|\eta(r_1 + ir_2)a\| = \left\| \sum_{k=1}^n x_k \right\| = (r_1^2 + r_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\|.$$

Hence

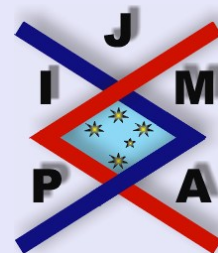
$$\eta = \sum_{k=1}^n \|x_k\|.$$

□

The next theorem is a refinement of Corollary 1 of [4] since, in the notation of Theorem 2.1, $\sqrt{2 - p_1^2 - p_2^2} \leq \sqrt{\alpha_1^2 + \alpha_2^2}$.

Theorem 2.2. *Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose the vectors $x_k \in H - \{0\}$, $k \in \{1, \dots, n\}$, are such that*

$$(2.4) \quad \|x_k - a\| \leq p_1, \quad \|x_k - ia\| \leq p_2, \quad p_1, p_2 \in (0, \sqrt{\alpha^2 + 1}),$$



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where $\alpha = \min_{1 \leq k \leq n} \|x_k\|$. Let

$$\alpha_1 = \min \left\{ \frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\},$$

$$\alpha_2 = \min \left\{ \frac{\|x_k\|^2 - p_2^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\}.$$

Then we have the inequality

$$(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|,$$

where the equality holds if and only if

$$\sum_{k=1}^n x_k = (\alpha_1 + i\alpha_2) \sum_{k=1}^n \|x_k\| a.$$

Proof. From the first inequality in (2.4) we have

$$\langle x_k - a, x_k - a \rangle \leq p_1^2,$$

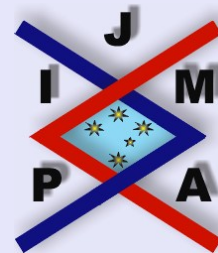
$$\|x_k\|^2 + 1 - p_1^2 \leq 2 \operatorname{Re} \langle x_k, a \rangle, \quad k = 1, \dots, n,$$

and

$$\frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} \|x_k\| \leq \operatorname{Re} \langle x_k, a \rangle.$$

Consequently,

$$\alpha_1 \|x_k\| \leq \operatorname{Re} \langle x_k, a \rangle.$$



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Similarly from the second inequality we obtain

$$\alpha_2 \|x_k\| \leq \operatorname{Re}\langle x_k, ia \rangle = \operatorname{Im}\langle x_k, a \rangle.$$

Now apply Theorem 2.1 for $r_1 = \alpha_1, r_2 = \alpha_2$. □

Corollary 2.3. *Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H - \{0\}, k \in \{1, \dots, n\}$ such that*

$$\|x_k - a\| \leq 1, \quad \|x_k - ia\| \leq 1.$$

Then

$$\frac{\alpha}{\sqrt{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|,$$

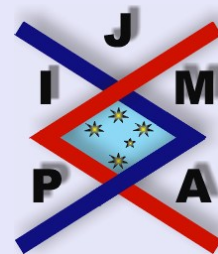
in which $\alpha = \min_{1 \leq k \leq n} \|x_k\|$. The equality holds if and only if

$$\sum_{k=1}^n x_k = \alpha \frac{(1+i)}{2} \sum_{k=1}^n \|x_k\| a.$$

Proof. Apply Theorem 2.2 for $\alpha_1 = \frac{\alpha}{2} = \alpha_2$. □

Theorem 2.4. *Let a be a unit vector in the inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field. Suppose that the vectors $x_k \in H - \{0\}, k \in \{1, \dots, n\}$ satisfy*

$$\|x_k - a\| \leq p, \quad p \in \left(0, \sqrt{\alpha^2 + 1}\right), \quad \alpha = \min_{1 \leq k \leq n} \|x_k\|.$$



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Then we have the inequality

$$\alpha_1 \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|,$$

where

$$\alpha_1 = \min \left\{ \frac{\|x_k\|^2 - p^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\}.$$

The equality holds if and only if

$$\sum_{k=1}^n x_k = \alpha_1 \sum_{k=1}^n \|x_k\| a.$$

Proof. The proof is similar to Theorem 2.2 in which we use Theorem 2.1 with $r_2 = 0$. \square

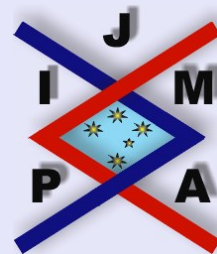
The next theorem is a generalization of Theorem 2.1. It is a modification of Theorem 3 of [4], however our proof is apparently different.

Theorem 2.5. Let a_1, \dots, a_m be orthonormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that for $1 \leq t \leq m$, $r_t, \rho_t \in \mathbb{R}$ and that the vectors $x_k \in H$, $k \in \{1, \dots, n\}$ satisfy

$$(2.5) \quad \begin{aligned} 0 &\leq r_t^2 \|x_k\| \leq \operatorname{Re} \langle x_k, r_t a_t \rangle, \\ 0 &\leq \rho_t^2 \|x_k\| \leq \operatorname{Im} \langle x_k, \rho_t a_t \rangle, \quad t \in \{1, \dots, m\}. \end{aligned}$$

Then we have the inequality

$$(2.6) \quad \left(\sum_{t=1}^m r_t^2 + \rho_t^2 \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|.$$



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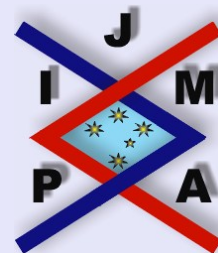
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The equality holds in (2.7) if and only if

$$(2.7) \quad \sum_{k=1}^n x_k = \sum_{k=1}^n \|x_k\| \sum_{t=1}^m (r_t + i\rho_t)a_t.$$

Proof. If $\sum_{t=1}^m (r_t^2 + \rho_t^2) = 0$, the theorem is trivial. Assume that $\sum_{t=1}^m (r_t^2 + \rho_t^2) \neq 0$. Summing inequalities (2.6) over k from 1 to n and again over t from 1 to m , we get

$$\begin{aligned} \sum_{t=1}^m (r_t^2 + \rho_t^2) \sum_{k=1}^n \|x_k\| &\leq \operatorname{Re} \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m r_t a_t \right\rangle + \operatorname{Im} \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m \rho_t a_t \right\rangle \\ &= \operatorname{Re} \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m r_t a_t \right\rangle + \operatorname{Re} \left\langle \sum_{k=1}^n x_k, i \sum_{t=1}^m \rho_t a_t \right\rangle \\ &= \operatorname{Re} \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m (r_t + i\rho_t) a_t \right\rangle \\ &\leq \left| \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m (r_t + i\rho_t) a_t \right\rangle \right| \\ &\leq \left\| \sum_{k=1}^n x_k \right\| \left\| \sum_{t=1}^m (r_t + i\rho_t) a_t \right\| \\ &= \left\| \sum_{k=1}^n x_k \right\| \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}}. \end{aligned}$$



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Then

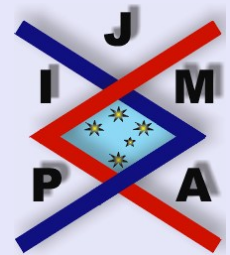
$$(2.8) \quad \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|.$$

If (2.8) holds, then

$$\left\| \sum_{k=1}^n x_k \right\| = \left\| \sum_{k=1}^n \|x_k\| \sum_{t=1}^m (r_t + i\rho_t) a_t \right\| = \sum_{k=1}^n \|x_k\| \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}}.$$

Conversely, if the equality holds in (2.7), we obtain from (2.6) that

$$\begin{aligned} \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}} \left\| \sum_{k=1}^n x_k \right\| &= \sum_{t=1}^m (r_t^2 + \rho_t^2) \sum_{k=1}^n \|x_k\| \\ &\leq \operatorname{Re} \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m (r_t + i\rho_t) a_t \right\rangle \\ &\leq \left| \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m (r_t + i\rho_t) a_t \right\rangle \right| \\ &\leq \left\| \sum_{k=1}^n x_k \right\| \left\| \sum_{t=1}^m (r_t + i\rho_t) a_t \right\| \\ &= \left\| \sum_{k=1}^n x_k \right\| \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}}. \end{aligned}$$



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Thus we have

$$\left| \left\langle \sum_{k=1}^n x_k, \sum_{t=1}^m (r_t + i\rho_t) a_t \right\rangle \right| = \left\| \sum_{k=1}^n x_k \right\| \left\| \sum_{t=1}^m (r_t + i\rho_t) a_t \right\|.$$

Consequently there exists $\eta \geq 0$ such that

$$\sum_{k=1}^n x_k = \eta \sum_{t=1}^m (r_t + i\rho_t) a_t$$

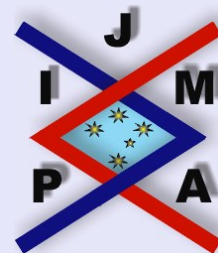
from which we have

$$\begin{aligned} \eta \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}} &= \left\| \eta \sum_{t=1}^m (r_t + i\rho_t) a_t \right\| \\ &= \left\| \sum_{k=1}^n x_k \right\| \\ &= \sum_{k=1}^n \|x_k\| \left(\sum_{t=1}^m (r_t^2 + \rho_t^2) \right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\eta = \sum_{k=1}^n \|x_k\|.$$

□



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Corollary 2.6. Let a_1, \dots, a_m be orthonormal vectors in the inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field. Suppose for $1 \leq t \leq m$ that the vectors $x_k \in H, k \in \{1, \dots, n\}$ satisfy

$$0 \leq r_t^2 \|x_k\| \leq \operatorname{Re} \langle x_k, r_t a_t \rangle.$$

Then we have the inequality

$$\left(\sum_{t=1}^m r_t^2 \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|.$$

The equality holds if and only if

$$\sum_{k=1}^n x_k = \sum_{k=1}^n \|x_k\| \sum_{t=1}^m r_t a_t.$$

Proof. Apply Theorem 2.5 for $\rho_t = 0$. □

Theorem 2.7. Let a_1, \dots, a_m be orthonormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H - \{0\}, k \in \{1, \dots, n\}$ satisfy

$$\|x_k - a_t\| \leq p_t, \quad \|x_k - ia_t\| \leq q_t, \quad p_t, q_t \in (0, \sqrt{\alpha^2 + 1}), \quad 1 \leq t \leq m,$$

where $\alpha = \min_{1 \leq k \leq n} \|x_k\|$. Let

$$\alpha_t = \min \left\{ \frac{\|x_k\|^2 - p_t^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\},$$

$$\beta_t = \min \left\{ \frac{\|x_k\|^2 - q_t^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\}.$$



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Then we have the inequality

$$\left(\sum_{t=1}^m \alpha_t^2 + \beta_t^2 \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|,$$

where equality holds if and only if

$$\sum_{k=1}^n x_k = \sum_{k=1}^n \|x_k\| \sum_{t=1}^m (\alpha_t + i\beta_t) a_t.$$

Proof. For $1 \leq t \leq m$, $1 \leq k \leq n$ it follows from $\|x_k - a_t\| \leq p_t$ that

$$\langle x_k - a_t, x_k - a_t \rangle \leq p_t^2,$$

$$\frac{\|x_k\|^2 - p_t^2 + 1}{2\|x_k\|} \|x_k\| \leq \operatorname{Re}\langle x_k, a_t \rangle,$$

$$\alpha_t \|x_k\| \leq \operatorname{Re}\langle x_k, a_t \rangle,$$

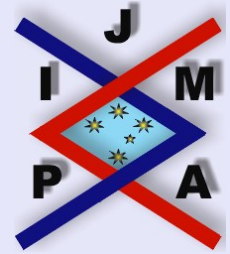
and similarly

$$\beta_t \|x_k\| \leq \operatorname{Re}\langle x_k, ia_t \rangle = \operatorname{Im}\langle x_k, a_t \rangle.$$

Now applying Theorem 2.4 with $r_t = \alpha_t$, $\rho_t = \beta_t$ we deduce the desired inequality. \square

Corollary 2.8. Let a_1, \dots, a_m be orthonormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H$, $k \in \{1, \dots, n\}$ satisfy

$$\|x_k - a_t\| \leq 1, \quad \|x_k - ia_t\| \leq 1, \quad 1 \leq t \leq m.$$



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Then

$$\frac{\alpha}{\sqrt{2}} \sqrt{m} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|.$$

The equality holds if and only if

$$\sum_{k=1}^n x_k = \alpha \frac{(1+i)}{2} \sum_{k=1}^n \|x_k\| \sum_{t=1}^m a_t.$$

Proof. Apply Theorem 2.7 for $\alpha_t = \frac{\alpha}{2} = \beta_t$. □

Remark 1. It is interesting to note that

$$\frac{\alpha}{\sqrt{2}} \sqrt{m} \leq \frac{\left\| \sum_{k=1}^n x_k \right\|}{\sum_{k=1}^n \|x_k\|} \leq 1,$$

where

$$\alpha \leq \sqrt{\frac{2}{m}}.$$

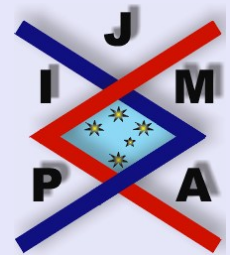
Corollary 2.9. Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H - \{0\}$, $k \in \{1, \dots, n\}$ satisfy

$$\|x_k - a\| \leq p_1, \quad \|x_k - ia\| \leq p_2, \quad p_1, p_2 \in (0, 1].$$

Let

$$\alpha_1 = \min \left\{ \frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\},$$

$$\alpha_2 = \min \left\{ \frac{\|x_k\|^2 - p_2^2 + 1}{2\|x_k\|} : 1 \leq k \leq n \right\}.$$



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If $\alpha_1 \neq (1 - p_1^2)^{\frac{1}{2}}$, or $\alpha_2 \neq (1 - p_2^2)^{\frac{1}{2}}$, then we have the following strictly inequality

$$(2 - p_1^2 - p_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| < \left\| \sum_{k=1}^n x_k \right\|.$$

Proof. If equality holds, then by Theorem 2.2 we have

$$(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\| = (2 - p_1^2 - p_2^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\|$$

and so

$$(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}} \leq (2 - p_1^2 - p_2^2)^{\frac{1}{2}}.$$

On the other hand for $1 \leq k \leq n$,

$$\frac{\|x_k\|^2 - p_1^2 + 1}{2\|x_k\|} \geq (1 - p_1^2)^{\frac{1}{2}}$$

and so

$$\alpha_1 \geq (1 - p_1^2)^{\frac{1}{2}}.$$

Similarly

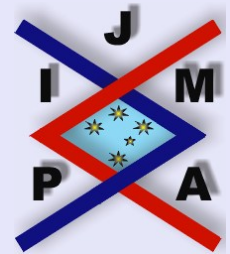
$$\alpha_2 \geq (1 - p_2^2)^{\frac{1}{2}}.$$

Hence

$$(2 - p_1^2 - p_2^2)^{\frac{1}{2}} \leq (\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}}.$$

Thus

$$\sqrt{\alpha_1^2 + \alpha_2^2} = (2 - p_1^2 - p_2^2)^{\frac{1}{2}}.$$



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Therefore

$$\alpha_1 = (1 - p_1^2)^{\frac{1}{2}} \quad \text{and} \quad \alpha_2 = (1 - p_2^2)^{\frac{1}{2}},$$

a contradiction. □

The following result looks like Corollary 2 of [4].

Theorem 2.10. *Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$, $M \geq m > 0$, $L \geq \ell > 0$ and $x_k \in H - \{0\}$, $k \in \{1, \dots, n\}$ such that*

$$\operatorname{Re}\langle Ma - x_k, x_k - ma \rangle \geq 0, \quad \operatorname{Re}\langle Lia - x_k, x_k - lia \rangle \geq 0,$$

or equivalently,

$$\|x_k - \frac{m+M}{2}a\| \leq \frac{M-m}{2}, \quad \|x_k - \frac{L+\ell}{2}ia\| \leq \frac{L-\ell}{2}.$$

Let

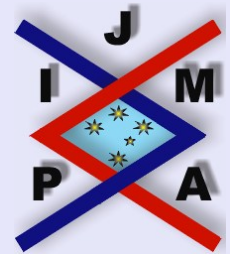
$$\alpha_{m,M} = \min \left\{ \frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} : 1 \leq k \leq n \right\}$$

and

$$\alpha_{\ell,L} = \min \left\{ \frac{\|x_k\|^2 + \ell L}{(\ell+L)\|x_k\|} : 1 \leq k \leq n \right\}.$$

Then we have the inequality

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| \leq \left\| \sum_{k=1}^n x_k \right\|.$$



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The equality holds if and only if

$$\sum_{k=1}^n x_k = (\alpha_{m,M} + i\alpha_{\ell,L}) \sum_{k=1}^n \|x_k\| a.$$

Proof. For each $1 \leq k \leq n$, it follows from

$$\|x_k - \frac{m+M}{2}a\| \leq \frac{M-m}{2}$$

that

$$\left\langle x_k - \frac{m+M}{2}a, x_k - \frac{m+M}{2}a \right\rangle \leq \left(\frac{M-m}{2} \right)^2.$$

Hence

$$\|x_k\|^2 + mM \leq (m+M) \operatorname{Re}\langle x_k, a \rangle.$$

Then

$$\frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} \|x_k\| \leq \operatorname{Re}\langle x_k, a \rangle,$$

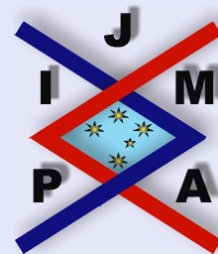
and consequently

$$\alpha_{m,M} \|x_k\| \leq \operatorname{Re}\langle x_k, a \rangle.$$

Similarly from the second inequality we deduce

$$\alpha_{\ell,L} \|x_k\| \leq \operatorname{Im}\langle x_k, a \rangle.$$

Applying Theorem 2.1 for $r_1 = \alpha_{m,M}$, $r_2 = \alpha_{\ell,L}$, we infer the desired inequality. \square



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Theorem 2.11. Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$, $M \geq m > 0$, $L \geq \ell > 0$ and $x_k \in H - \{0\}$, $k \in \{1, \dots, n\}$ such that

$$\operatorname{Re}\langle Ma - x_k, x_k - ma \rangle \geq 0, \quad \operatorname{Re}\langle Lia - x_k, x_k - lia \rangle \geq 0,$$

or equivalently

$$\left\| x_k - \frac{m+M}{2}a \right\| \leq \frac{M-m}{2}, \quad \left\| x_k - \frac{L+\ell}{2}ia \right\| \leq \frac{L-\ell}{2}.$$

Let

$$\alpha_{m,M} = \min \left\{ \frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} : 1 \leq k \leq n \right\}$$

and

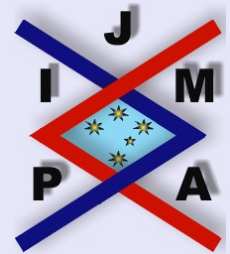
$$\alpha_{\ell,L} = \min \left\{ \frac{\|x_k\|^2 + \ell L}{(\ell+L)\|x_k\|} : 1 \leq k \leq n \right\}.$$

If $\alpha_{m,M} \neq 2\frac{\sqrt{mM}}{m+M}$, or $\alpha_{\ell,L} \neq 2\frac{\sqrt{\ell L}}{\ell+L}$, then we have

$$2 \left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2} \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| < \left\| \sum_{k=1}^n x_k \right\|.$$

Proof. If

$$2 \left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2} \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| = \left\| \sum_{k=1}^n x_k \right\|$$



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then by Theorem 2.10 we have

$$\begin{aligned}
 (\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\| &\leq \left\| \sum_{k=1}^n x_k \right\| \\
 &= 2 \left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2} \right)^{\frac{1}{2}} \sum_{k=1}^n \|x_k\|.
 \end{aligned}$$

Consequently

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \leq 2 \left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2} \right)^{\frac{1}{2}}.$$

On the other hand for $1 \leq k \leq n$,

$$\frac{\|x_k\|^2 + mM}{(m+M)\|x_k\|} \geq 2 \frac{\sqrt{mM}}{m+M}, \quad \text{and} \quad \frac{\|x_k\|^2 + \ell L}{(\ell+L)\|x_k\|} \geq 2 \frac{\sqrt{\ell L}}{\ell+L},$$

so

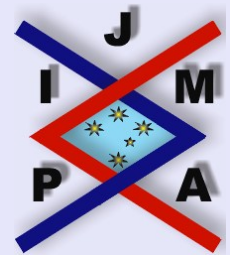
$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} \geq 2 \left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2} \right)^{\frac{1}{2}}.$$

Then

$$(\alpha_{m,M}^2 + \alpha_{\ell,L}^2)^{\frac{1}{2}} = 2 \left(\frac{mM}{(m+M)^2} + \frac{\ell L}{(\ell+L)^2} \right)^{\frac{1}{2}}.$$

Hence

$$\alpha_{m,M} = 2 \frac{\sqrt{mM}}{m+M}$$



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and

$$\alpha_{\ell,L} = 2 \frac{\sqrt{\ell L}}{\ell + L}$$

a contradiction. □

Finally we mention two applications of our results to complex numbers.

Corollary 2.12. *Let $a \in \mathbb{C}$ with $|a| = 1$. Suppose that $z_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ such that*

$$|z_k - a| \leq p_1, \quad |z_k - ia| \leq p_2, \quad p_1, p_2 \in (0, \sqrt{\alpha^2 + 1}),$$

where

$$\alpha = \min\{|z_k| : 1 \leq k \leq n\}.$$

Let

$$\alpha_1 = \min \left\{ \frac{|z_k|^2 - p_1^2 + 1}{2|z_k|} : 1 \leq k \leq n \right\},$$

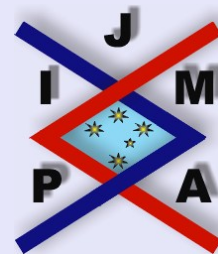
$$\alpha_2 = \min \left\{ \frac{|z_k|^2 - p_2^2 + 1}{2|z_k|} : 1 \leq k \leq n \right\}.$$

Then we have the inequality

$$\sqrt{\alpha_1^2 + \alpha_2^2} \sum_{k=1}^n |z_k| \leq \left| \sum_{k=1}^n z_k \right|.$$

The equality holds if and only if

$$\sum_{k=1}^n z_k = (\alpha_1 + i\alpha_2) \left(\sum_{k=1}^n |z_k| \right) a.$$



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Proof. Apply Theorem 2.2 for $H = \mathbb{C}$. □

Corollary 2.13. Let $a \in \mathbb{C}$ with $|a| = 1$. Suppose that $z_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ such that

$$|z_k - a| \leq 1, \quad |z_k - ia| \leq 1.$$

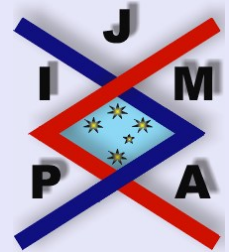
If $\alpha = \min\{|z_k| : 1 \leq k \leq n\}$. Then we have the inequality

$$\frac{\alpha}{\sqrt{2}} \sum_{k=1}^n |z_k| \leq \left| \sum_{k=1}^n z_k \right|$$

the equality holds if and only if

$$\sum_{k=1}^n z_k = \alpha \frac{(1+i)}{2} \left(\sum_{k=1}^n |z_k| \right) a.$$

Proof. Apply Corollary 2.3 for $H = \mathbb{C}$. □



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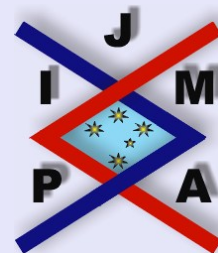
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