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POSITIVE SOLUTIONS FOR SECOND-ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE ON A HALF-LINE

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ABSTRACT. This paper deals with the second order boundary value problem with integral boundary conditions on a half-line:

$$(p(t)x'(t))' + g(t)f(t,x(t)) = 0, \text{ a.e. in } (0,\infty),$$
$$x(0) = \int_0^\infty x(s)g(s)ds, \qquad \lim_{t\to\infty} p(t)x'(t) = p(0)x'(0).$$

A new result on the existence of positive solutions is obtained. The interesting points are: firstly, the boundary value problem involved in the integral boundary condition on unbounded domains; secondly, we employ a new tool – the recent Leggett-Williams norm-type theorem for coincidences and obtain positive solutions. Also, an example is constructed to illustrate that our result here is valid.

Key words and phrases: Boundary value problem; Resonance; Cone; Positive solution; Coincidence.

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1. INTRODUCTION

In this paper, we study the existence of positive solutions to the following boundary value problem at resonance:

$$(p(t)x'(t))' + g(t)f(t,x(t)) = 0, \quad \text{a.e.in } (0,\infty), \tag{1.1}$$

$$x(0) = \int_0^\infty x(s)g(s)ds, \quad \lim_{t\to\infty} p(t)x'(t) = p(0)x'(0), \tag{1.2}$$

where $g \in L^1[0,\infty)$ with $g(t) > 0$ on $[0,\infty)$ and $\int_0^\infty g(s)ds = 1$, $p \in C[0,\infty) \cap C^1(0,\infty)$, $\frac{1}{p} \in L^1[0,\infty)$, $\int_0^\infty \frac{1}{p(t)}dt \leq 1$ and $p(t) > 0$ on $[0,\infty)$.

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Second-order boundary value problems (in short: BVPs) on infinite intervals, arising from the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium [10], have received much attention, to identify a few, we refer the readers to [9] – [11] and references therein. For example, in [9], Lian and Ge studied the following second-order BVPs on a half-line

$$x''(t) = f(t, x(t), x'(t)), \qquad 0 < t < \infty, \tag{1.3}$$

$$x(0) = x(\eta), \quad \lim_{t\to\infty} x'(t) = 0 \tag{1.4}$$

and

$$x''(t) = f(t, x(t), x'(t)) + e(t), \qquad 0 < t < \infty, \tag{1.5}$$

$$x(0) = x(\eta), \quad \lim_{t\to\infty} x'(t) = 0, \tag{1.6}$$

By using Mawhin's continuity theorem, they obtained the existence results.

N. Kosmanov in [11] considered the second-order nonlinear differential equation at resonance

$$(p(t)u'(t))' = f(t, u(t), u'(t)), \quad \text{a.e. in } (0,\infty) \tag{1.7}$$

with two sets of boundary conditions:

$$u'(0) = 0, \quad \sum_{i=1}^{n} \kappa_i u_i(T_i) = \lim_{t\to\infty} u(t) \tag{1.8}$$

and

$$u(0) = 0, \quad \sum_{i=1}^{n} \kappa_i u_i(T_i) = \lim_{t\to\infty} u(t). \tag{1.9}$$

The author established existence theorems by the coincidence degree theorem of Mawhin under the condition that $\sum_{i=1}^{n} \kappa_i = 1$.

Although the existing literature on solutions of BVPs is quite wide, to the best of our knowledge, only a few papers deal with the existence of positive solutions to BVPs at resonance. In particular, there has been no work done for the boundary value problems with integral boundary conditions on a half-line, such as the BVP (1.1) – (1.2). Moreover, our main approach is different from the existing ones and our main ingredient is the Leggett-Williams norm-type theorem for coincidences obtained by O'Regan and Zima [4], which is a new tool used to study the existence of positive solutions for nonlocal BVPs at resonance. An example is constructed to illustrate that our result here is valid and almost sharp.

2. RELATED LEMMAS

For the convenience of the reader, we review some standard facts on Fredholm operators and cones in Banach spaces. Let X, Y be real Banach spaces. Consider a linear mapping $L : \operatorname{dom} L \subset X \to Y$ and a nonlinear operator $N : X \to Y$. Assume that

1° L is a Fredholm operator of index zero, i.e., $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

The assumption 1° implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Moreover, since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Denote by L_p the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L$. Clearly, L_p is an isomorphism from $\operatorname{Ker} P \cap \operatorname{dom} L$ to $\operatorname{Im} L$, we denote its inverse by K_p :

$\operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$. It is known (see [3]) that the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Let C be a cone in X such that
(i) $\mu x \in C$ for all $x \in C$ and $\mu \geq 0$,
(ii) $x, -x \in C$ implies $x = \theta$.
It is well known that C induces a partial order in X by

$$x \preceq y \quad \text{if and only if} \quad y - x \in C.$$

The following property is valid for every cone in a Banach space X.

Lemma 2.1 ([7]). *Let C be a cone in X. Then for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that*

$$||x + u|| \geq \sigma(u)||x|| \quad \textit{for all} \quad x \in C.$$

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I - Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

In order to prove the existence result, we present here a definition.

Definition 2.1. $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is called a g-Carath$\acute{e}$odory function if

(A1) for each $u \in \mathbb{R}$, the mapping $t \mapsto f(t, u)$ is Lebesgue measurable on $[0, \infty)$,
(A2) for a.e. $t \in [0, \infty)$, the mapping $u \mapsto f(t, u)$ is continuous on $\mathbb{R}$,
(A3) for each $l > 0$ and $g \in L^1[0, \infty)$, there exists $\alpha_l : [0, \infty) \to [0, \infty)$ satisfying $\int_0^\infty g(s)\alpha_l(s)ds < \infty$ such that

$$|u| \leq l \quad \text{implies} \quad |f(t, u)| \leq \alpha_l(t) \quad \text{for a.e.} \quad t \in [0, \infty).$$

We make use of the following result due to O'Regan and Zima.

Theorem 2.2 ([4]). *Let C be a cone in X and let $\Omega_1, \ \Omega_2$ be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that 1° and the following conditions hold.*

2° *N is L-compact, that is, $QN : X \to Y$ is continuous and bounded and $K_p(I - Q)N : X \to X$ is compact on every bounded subset of X,*

3° *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \operatorname{Im} L$ and $\lambda \in (0, 1)$,*

4° *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*

5° *$\deg_B\{[I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0\} \neq 0$, where $\deg_B$ denotes the Brouwer degree,*

6° *there exists $u_0 \in C \setminus \{0\}$ such that $||x|| \leq \sigma(u_0)||\Psi x||$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \preceq x$ for some $\mu > 0\}$ and $\sigma(u_0)$ such that $||x + u_0|| \geq \sigma(u_0)||x||$ for every $x \in C$,*

7° *$(P + JQN)\gamma(\partial\Omega_2) \subset C$,*

8° *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For simplicity of notation, we set

$$\omega := \int_0^\infty \left(\int_0^s \frac{1}{p(\tau)} d\tau \right) g(s) ds \tag{2.1}$$

and

$$G(t,s) = \begin{cases} \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \Big[\int_s^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr d\tau \\ \qquad - \int_0^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr \int_0^\tau g(r) dr d\tau \Big] \\ \qquad +1 + \int_0^t \frac{1}{p(\tau)} \int_0^\tau g(r) dr d\tau - \int_s^t \frac{1}{p(\tau)} d\tau, & 0 \le s < t < \infty, \\ \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \Big[\int_s^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr d\tau \\ \qquad - \int_0^\infty \frac{1}{p(\tau)} \int_\tau^\infty g(r) dr \int_0^\tau g(r) dr d\tau \Big] \\ \qquad +1 + \int_0^t \frac{1}{p(\tau)} \int_0^\tau g(r) dr d\tau, & 0 \le t \le s < \infty. \end{cases}$$

Note that $G(t,s) \ge 0$ for $t,s \in [0,1]$, and set

$$0 < \kappa \le \min \left\{ 1, \; \frac{1}{\sup\limits_{t,s \in [0,\infty)} G(t,s)} \right\}. \tag{2.2}$$

3. MAIN RESULT

We work in the Banach spaces

$$X = \left\{ x \in C[0,\infty) : \lim_{t \to \infty} x(t) \text{ exists} \right\} \tag{3.1}$$

and

$$Y = \left\{ y : [0,\infty) \to \mathbb{R} : \int_0^\infty g(t) |y(t)| dt < \infty \right\} \tag{3.2}$$

with the norms $||x||_X = \sup\limits_{t \in [0,\infty)} |x(t)|$ and $||y||_Y = \int_0^\infty g(t)|y(t)|dt$, respectively.

Define the linear operator $L : \operatorname{dom} L \subset X \to Y$ and the nonlinear operator $N : X \to Y$ with

$$\begin{aligned} \operatorname{dom} L = \Big\{ x \in X : \lim_{t \to \infty} p(t) x'(t) \quad &\text{exists}, \quad x, px' \in AC[0,\infty) \\ \text{and} \quad gx, (px')' \in L^1[0,\infty), x(0) &= \int_0^\infty x(s) g(s) ds \\ &\text{and} \quad \lim_{t \to \infty} p(t) x'(t) = p(0) x'(0) \Big\} \end{aligned} \tag{3.3}$$

by $Lx(t) = -\frac{1}{g(t)} (p(t) x'(t))'$ and $Nx(t) = f(t, x(t))$, $t \in [0,\infty)$, respectively. Then

$$\operatorname{Ker} L = \{ x \in \operatorname{dom} L : x(t) \equiv c \quad \text{on } [0,\infty) \}$$

and

$$\operatorname{Im} L = \left\{ y \in Y : \int_0^\infty g(s) y(s) ds = 0 \right\}.$$

Next, define the projections $P : X \to X$ by $(Px)(t) = \int_0^\infty g(s) x(s) ds$ and $Q : Y \to Y$ by

$$(Qy)(t) = \int_0^\infty g(s) y(s) ds.$$

Clearly, $\mathrm{Im}\, P = \mathrm{Ker}\, L$ and $\mathrm{Ker}\, Q = \mathrm{Im}\, L$. So $\dim \mathrm{Ker}\, L = 1 = \dim \mathrm{Im}\, Q = \mathrm{codim}\, \mathrm{Im}\, L$. Notice that $\mathrm{Im}\, L$ is closed, L is a Fredholm operator of index zero.

Note that the inverse $K_p : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \mathrm{Ker}\, P$ of L_p is given by

$$(K_p y)(t) = \int_0^\infty k(t,s) g(s) y(s) ds,$$

where

$$k(t,s) := \begin{cases} \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau - \int_s^t \frac{1}{p(\tau)} d\tau, & 0 \le s < t < \infty, \\ \frac{1}{\omega} \int_0^t \frac{1}{p(\tau)} d\tau \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau, & 0 \le t \le s < \infty. \end{cases} \tag{3.4}$$

It is easy to see that $|k(t,s)| \le 2 \int_0^\infty \frac{1}{p(s)} ds$.

In order to apply Theorem 2.2, we have to prove that N is L-compact, that is, QN is continuous and bounded and $K_p(I-Q)N$ is compact on every bounded subset of X. Since the Arzelà-Ascoli theorem fails in the noncompact interval case, we will use the following criterion.

Theorem 3.1 ([10]). *Let $M \subset \left\{ x \in C[0,\infty) : \lim\limits_{t\to\infty} x(t) \text{ exists} \right\}$. Then M is relatively compact if the following conditions hold:*

(B1) *all functions from M are uniformly bounded,*

(B2) *all functions from M are equicontinuous on any compact interval of $[0,\infty)$,*

(B3) *all functions from M are equiconvergent at infinity, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $|f(t) - f(\infty)| < \varepsilon$ for all $t > T$ and $f \in M$.*

Lemma 3.2. *If $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a g-Carathéodory function, then N is L-compact.*

Proof. Suppose that $\Omega \subset X$ is a bounded set. Then there exists $l > 0$ such that $||x||_X \le l$ for $x \in \overline{\Omega}$. Since f is a g-Carathéodory function, there exists $\alpha_l \in L^1[0,\infty)$ satisfying $\alpha_l(t) > 0$, $t \in (0,\infty)$ and $\int_0^\infty g(s)\alpha_l(s) ds < \infty$ such that for a.e. $t \in [0,\infty)$, $|f(t,x(t))| \le \alpha_l(t)$ for $x \in \overline{\Omega}$. Then for $x \in \overline{\Omega}$,

$$||QNx||_Y = \int_0^\infty g(t) \left| \int_0^\infty g(s) f(s,x(s)) ds \right| dt \le \int_0^\infty g(s)\alpha_l(s) ds < \infty,$$

which implies that QN is bounded on $\overline{\Omega}$.

Next, we show that $K_p(I-Q)N$ is compact, i.e., $K_p(I-Q)N$ maps bounded sets into relatively compact ones. Furthermore, denote $K_{P,Q} = K_P(I-Q)N$ (see [9], [11]). For $x \in \overline{\Omega}$, one gets

$$\begin{aligned} |(K_{P,Q}x)(t)| &\le \int_0^\infty \left| k(t,s) g(s) \left[f(s,x(s)) - \int_0^\infty g(\tau) f(\tau, x(\tau)) d\tau \right] \right| ds \\ &\le 2 \int_0^\infty \frac{1}{p(\tau)} d\tau \left[\int_0^\infty g(s) |f(s,x(s))| ds \right. \\ &\qquad \left. + \int_0^\infty g(s) \int_0^\infty g(\tau) |f(\tau, x(\tau))| d\tau ds \right] \\ &\le 4 \int_0^\infty \frac{1}{p(\tau)} d\tau \int_0^\infty g(s) \alpha_l(s) ds < \infty, \end{aligned}$$

that is, $K_{P,Q}(\overline{\Omega})$ is uniformly bounded. Meanwhile, for any $t_1, t_2 \in [0, T]$ with T a positive constant,

$$\begin{aligned}
&|(K_{P,Q}x)(t_1) - (K_{P,Q}x)(t_2)| \\
&= \bigg| \frac{1}{\omega} \int_0^\infty \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau g(s) \bigg[f(s, x(s)) \\
&\quad - \int_0^\infty g(\tau) f(\tau, x(\tau)) d\tau \bigg] ds \int_{t_2}^{t_1} \frac{1}{p(\tau)} d\tau \\
&\quad - \bigg\{ \int_0^{t_1} \int_s^{t_1} \frac{1}{p(\tau)} d\tau g(s) \left[f(s, x(s)) - \int_0^\infty g(\tau) f(\tau, x(\tau)) d\tau \right] ds \\
&\quad - \int_0^{t_2} \int_s^{t_2} \frac{1}{p(\tau)} d\tau g(s) \left[f(s, x(s)) - \int_0^\infty g(\tau) f(\tau, x(\tau)) d\tau \right] ds \bigg\} \bigg| \\
&\le \frac{1}{\omega} \bigg[\int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_0^\tau g(r) |f(r, x(r))| dr d\tau ds + \int_0^\infty g(\tau) |f(\tau, x(\tau))| d\tau \\
&\quad \cdot \int_0^\infty g(s) \int_0^s \frac{1}{p(\tau)} \int_0^\tau g(r) dr d\tau ds \bigg] \cdot \left| \int_{t_2}^{t_1} \frac{1}{p(\tau)} d\tau \right| \\
&\quad + \left| \int_{t_2}^{t_1} \frac{1}{p(s)} \left[\int_0^s g(\tau) |f(\tau, x(\tau))| d\tau + \int_0^s g(\tau) \int_0^\infty g(r) |f(r, x(r))| dr d\tau \right] ds \right| \\
&\le \left[\int_0^\infty g(r) |f(r, x(r))| dr + \int_0^\infty g(\tau) |f(\tau, x(\tau))| d\tau \cdot \int_0^\infty g(r) dr \right] \left| \int_{t_2}^{t_1} \frac{1}{p(\tau)} d\tau \right| \\
&\quad + 2 \int_0^\infty g(r) |f(r, x(r))| dr \left| \int_{t_2}^{t_1} \frac{1}{p(\tau)} d\tau \right| \\
&\le 4 \int_0^\infty g(s) \alpha_l(s) ds \cdot \left| \int_{t_2}^{t_1} \frac{1}{p(\tau)} d\tau \right| \to 0, \qquad \text{uniformly as } |t_1 - t_2| \to 0,
\end{aligned}$$

which means that $K_{P,Q}(\overline{\Omega})$ is equicontinuous. In addition, we claim that $K_{P,Q}(\overline{\Omega})$ is equiconvergent at infinity. In fact,

$$\begin{aligned}
&|(K_{P,Q}x)(\infty) - (K_{P,Q}x)(t)| \\
&\le \frac{1}{\omega} \int_0^\infty \int_s^\infty \int_s^\tau \frac{1}{p(r)} dr g(\tau) d\tau g(s) \left[|f(s, x(s))| + \int_0^\infty g(\tau) |f(\tau, x(\tau))| d\tau \right] ds \cdot \int_t^\infty \frac{1}{p(\tau)} d\tau \\
&\quad + \int_t^\infty \frac{1}{p(s)} ds \left[\int_0^s g(\tau) |f(\tau, x(\tau)| d\tau + \int_0^s g(\tau) \int_0^\infty g(r) |f(r, x(r))| dr d\tau \right] ds \\
&\le 4 \int_0^\infty g(s) \alpha_l(s) ds \cdot \int_t^\infty \frac{1}{p(\tau)} d\tau \to 0, \quad \text{uniformly as } t \to \infty.
\end{aligned}$$

Hence, Theorem 3.1 implies that $K_p(I - Q)N(\overline{\Omega})$ is relatively compact. Furthermore, since f satisfies g-Carathéodory conditions, the continuity of QN and $K_p(I - Q)N$ on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem. This completes the proof. □

Now, we state our main result on the existence of positive solutions for the BVP (1.1) – (1.2).

Theorem 3.3. *Assume that*

(H1) $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *is a g-Carathéodory function,* (H2) *there exist positive constants* $b_1, b_2, b_3, c_1, c_2, B$ *with*

$$B > \frac{c_2}{c_1} + 2\left(\frac{b_2 c_2}{b_1 c_1} + \frac{b_3}{b_1}\right)\int_0^\infty \frac{1}{p(s)} ds \tag{3.5}$$

such that

$$-\kappa x \le f(t,x),$$
$$f(t,x) \le -c_1 x + c_2,$$
$$f(t,x) \le -b_1|f(t,x)| + b_2 x + b_3$$

for $t \in [0,\infty)$, $x \in [0,B]$,

(H3) *there exist* $b \in (0,B)$, $t_0 \in [0,\infty)$, $\rho \in (0,1]$ *and* $\delta \in (0,1)$. *For each* $t \in [0,\infty)$, $\frac{f(t,x)}{x^\rho}$ *is non-increasing on* $x \in (0,b]$ *with*

$$\int_0^\infty G(t_0,s)g(s)\frac{f(s,b)}{b} ds \ge \frac{1-\delta}{\delta^\rho}. \tag{3.6}$$

Then the BVP *(1.1) – (1.2) has at least one positive solution on* $[0,\infty)$.

Proof. Consider the cone

$$C = \{x \in X : x(t) \ge 0 \quad \text{on} \quad [0,\infty)\}.$$

Let

$$\Omega_1 = \{x \in X : \delta||x||_X < |x(t)| < b \quad \text{on} \quad [0,\infty)\}$$

and

$$\Omega_2 = \{x \in X : ||x||_X < B\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\overline{\Omega}_1 = \{x \in X : \delta||x||_X \le |x(t)| \le b \quad \text{on} \quad [0,\infty)\} \subset \Omega_2$$

(see [4]). Moreover, $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \ne \emptyset$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that 4° holds.

In order to prove 3°, suppose that there exist $x_0 \in \partial\Omega_2 \cap C \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$, then $(p(t)x_0'(t))' + \lambda_0 g(t) f(t, x_0(t)) = 0$ for all $t \in [0,\infty)$. In view of (H2), we have

$$-\frac{1}{\lambda_0 g(t)}(p(t)x_0'(t))' = f(t,x_0(t)) \le -b_1 \frac{1}{\lambda_0 g(t)}|(p(t)x_0'(t))'| + b_2 x_0(t) + b_3.$$

Hence,

$$-(p(t)x_0'(t))' \le -b_1|(p(t)x_0'(t))'| + \lambda_0 b_2 g(t) x_0(t) + \lambda_0 b_3 g(t). \tag{3.7}$$

Integrating both sides of (3.7) from 0 to ∞, one gets

$$\begin{aligned} 0 &= -\int_0^\infty (p(t)x_0'(t))' dt \\ &\le -b_1 \int_0^\infty |(p(t)x_0'(t))'| dt + \lambda_0 b_2 \int_0^\infty g(t)x_0(t) dt + \lambda_0 b_3 \int_0^\infty g(t) dt, \end{aligned}$$

which gives

$$\int_0^\infty |(p(t)x_0'(t))'| dt < \frac{b_2}{b_1}\int_0^\infty g(t)x_0(t) dt + \frac{b_3}{b_1}. \tag{3.8}$$

Similarly, from (H2), we also obtain

$$\int_0^\infty g(t)x_0(t) dt \le \frac{c_2}{c_1}. \tag{3.9}$$

On the other hand,

$$x_0(t) = \int_0^\infty g(t)x_0(t)dt + \int_0^\infty k(t,s)(p(s)x_0'(s))'ds \tag{3.10}$$

$$\le \int_0^\infty g(t)x_0(t)dt + \int_0^\infty |k(t,s)| \cdot |(p(s)x_0'(s))'|ds. \tag{3.11}$$

Then, (3.8)-(3.9) yield

$$B = ||x_0||_X \le \frac{c_2}{c_1} + 2\left(\frac{b_2c_2}{b_1c_1} + \frac{b_3}{b_1}\right)\int_0^\infty \frac{1}{p(s)}ds,$$

which contradicts (3.5).

To prove 5°, consider $x \in \operatorname{Ker} L \cap \overline{\Omega}_2$. Then $x(t) \equiv c$ on $[0,\infty)$. Let

$$H(c,\lambda) = c - \lambda|c| - \lambda\int_0^\infty g(s)f(s,|c|)ds$$

for $c \in [-B, B]$ and $\lambda \in [0,1]$. It is easy to show that $0 = H(c,\lambda)$ implies $c \ge 0$. Suppose $0 = H(B,\lambda)$ for some $\lambda \in (0,1]$. Then, (3.5) leads to

$$0 \le B(1-\lambda) = \lambda\int_0^\infty g(s)f(s,B)ds \le \lambda(-c_1B + c_2) < 0,$$

which is a contradiction. In addition, if $\lambda = 0$, then $B = 0$, which is impossible. Thus, $H(x,\lambda) \ne 0$ for $x \in \operatorname{Ker} L \cap \partial\Omega_2$ and $\lambda \in [0,1]$. As a result,

$$\deg_B\{H(\cdot,1), \operatorname{Ker} L \cap \Omega_2, 0\} = \deg_B\{H(\cdot,0), \operatorname{Ker} L \cap \Omega_2, 0\}.$$

However,

$$\deg_B\{H(\cdot,0), \operatorname{Ker} L \cap \Omega_2, 0\} = \deg_B\{I, \operatorname{Ker} L \cap \Omega_2, 0\} = 1.$$

Then,

$$\deg_B\{[I - (P + JQN)\gamma]_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0\} = \deg_B\{H(\cdot,1), \operatorname{Ker} L \cap \Omega_2, 0\} \ne 0.$$

Next, we prove 8°. Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,\infty)$,

$$\begin{aligned}(\Psi_\gamma x)(t) &= \int_0^\infty g(s)|x(s)|ds + \int_0^\infty g(s)f(s,|x(s)|)ds \\ &\quad + \int_0^\infty k(t,s)g(s)\left[f(s,|x(s)|) - \int_0^\infty g(\tau)f(\tau,|x(\tau)|)d\tau\right]ds \\ &= \int_0^\infty g(s)|x(s)|ds + \int_0^\infty G(t,s)g(s)f(s,|x(s)|)ds \\ &\ge \int_0^\infty (1-\kappa G(t,s))g(s)|x(s)|ds \ge 0.\end{aligned}$$

Hence, $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$, i.e. 8° holds.

Since for $x \in \partial\Omega_2$,

$$\begin{aligned}(P + JQN)\gamma x &= \int_0^\infty g(s)|x(s)|ds + \int_0^\infty g(s)f(s,|x(s)|)ds \\ &\ge \int_0^\infty (1-\kappa)g(s)|x(s)|ds \ge 0,\end{aligned}$$

then, $(P + JQN)\gamma x \subset C$ for $x \in \partial\Omega_2$, and 7° holds.

It remains to verify 6°. Let $u_0(t) \equiv 1$ on $[0,\infty)$. Then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0,\infty)\}$ and we can take $\sigma(u_0) = 1$. Let $x \in C(u_0) \cap \partial\Omega_1$. Then $x(t) > 0$ on $[0,\infty)$, $0 < ||x||_X \leq b$ and $x(t) \geq \delta ||x||_X$ on $[0,\infty)$. For every $x \in C(u_0) \cap \partial\Omega_1$, by (H3)

$$\begin{aligned}(\Psi x)(t_0) &= \int_0^\infty g(s)x(s)ds + \int_0^\infty G(t_0,s)g(s)f(s,x(s))ds \\ &\geq \delta||x||_X + \int_0^\infty G(t_0,s)g(s)\frac{f(s,x(s))}{x^\rho(s)}x^\rho(s)ds \\ &\geq \delta||x||_X + \delta^\rho||x||_X^\rho \int_0^\infty G(t_0,s)g(s)\frac{f(s,b)}{b^\rho}ds \\ &= \delta||x||_X + \delta^\rho||x||_X \frac{b^{1-\rho}}{||x||_X^{1-\rho}} \int_0^\infty G(t_0,s)g(s)\frac{f(s,b)}{b}ds \\ &\geq \delta||x||_X + \delta^\rho||x||_X \int_0^\infty G(t_0,s)g(s)\frac{f(s,b)}{b}ds \\ &\geq ||x||_X.\end{aligned}$$

Thus, $||x||_X \leq \sigma(u_0)||\Psi x||_X$ for all $x \in C(u_0) \cap \partial\Omega_1$.

In addition, 1° holds and Lemma 3.2 yields 2°. Then, by Theorem 2.2, the BVP (1.1) – (1.2) has at least one positive solution x^* on $[0,\infty)$ with $b \leq ||x^*||_X \leq B$. This completes the proof of Theorem 3.3. □

Remark 1. Note that with the projection $P(x) = x(0)$, Conditions 7° and 8° of Theorem 2.2 are no longer satisfied.

To illustrate how our main result can be used in practice, we present here an example.

Example 3.1. Consider the following BVP

$$\begin{cases} 2(e^t x'(t))' + e^{-t} f(t,x(t)) = 0, \quad \text{a.e. in } (0,\infty), \\ x'(0) = \lim\limits_{t\to\infty} e^t x'(t), \quad x(0) = \int_0^\infty e^{-s}x(s)ds. \end{cases} \tag{3.12}$$

Corresponding to the BVP (1.1) – (1.2), $p(t) = 2e^t$, $g(t) = e^{-t}$ and $f(t,x) = (t - \frac{1}{2})e^{-2t}x + e^{-t}x^2$. We can get $\omega = \frac{1}{4}$ and

$$G(t,s) = \begin{cases} \frac{13}{12} + \frac{1}{6}(e^{-t} - 3e^{-s}) + \frac{1}{4}(e^{-2t} + 2e^{-2s}) - \frac{1}{2}e^{-(t+2s)}, & 0 \leq s \leq t < \infty, \\ \frac{13}{12} - \frac{1}{3}e^{-t} + \frac{1}{4}(e^{-2t} + 2e^{-2s}) - \frac{1}{2}e^{-(t+2s)}, & 0 \leq t \leq s < \infty. \end{cases} \tag{3.13}$$

Obviously, $G(t,s) \geq 0$ for $t,s \in [0,+\infty)$. Choose $\kappa = \frac{1}{2}$, $B = 5$, $c_1 = \frac{2}{5}$, $c_2 = \frac{1}{2}e^{-\frac{3}{2}}$, $b_1 = \frac{1}{2}$, $b_2 = \frac{3}{2}$ and $b_3 = \frac{3}{2}e^{-\frac{3}{2}}$ such that (H2) holds, and take $b = \frac{5}{4}$, $t_0 = 0$, $\rho = 1$ and $\delta = \frac{4}{9}$ such that (H3) is satisfied. Then thanks to Theorem 3.3, the BVP (3.12) has a positive solution on $[0,\infty)$.

REFERENCES

[1] K. DEIMLING, *Nonlinear Functional Analysis*, New York, 1985.

[2] D. GUO AND V. LAKSHMIKANTHAM, *Nonlinear Problems in Abstract Cones*, New York, 1988.

[3] J. MAWHIN, *Topological degree methods in nonlinear boundary value problems*, in: NSFCBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1979.

[4] D. O'REGAN AND M. ZIMA, Leggett-Williams norm-type theorems for coincidences, *Arch. Math.*, **87** (2006), 233–244.

[5] W. GE, *Boundary value problems for ordinary nonlinear differential equations*, Science Press, Beijing, 2007 (in Chinese).

[6] G. INFANTE AND M. ZIMA, Positive solutions of multi-point boundary value problems at resonance, *Nonlinear Anal.*, **69** (2008), 2458–2465.

[7] W.V. PETRYSHYN, On the solvability of $x \in Tx + \lambda Fx$ in quasinormal cones with T and F k-set contractive, *Nonlinear Anal.*, **5** (1981), 585–591.

[8] R.E. GAINES AND J. SANTANILLA, A coincidence theorem in convex sets with applications to periodic solutions of ordinary differential equations, *Rocky Mountain. J. Math*., **12** (1982), 669–678.

[9] H. LIAN, H. PANG AND W. GE, Solvability for second-order three-point boundary value problems at resonance on a half-line, *J. Math. Anal. Appl.*, **337** (2008), 1171–1181.

[10] R.P. AGARWAL AND D. O'REGAN, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic, 2001.

[11] N. KOSMATOV, Multi-point boundary value problems on an unbounded domain at resonance, *Nonlinear Anal.*, **68** (2008), 2158–2171.