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# ON EQUIVALENCE OF COEFFICIENT CONDITIONS

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ABSTRACT. Two equivalence theorems and two corollaries are proved pertaining to the equiconvergence of numerical series.

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## 1. INTRODUCTION

Some papers, see [1], [2] and [5], have dealt with the equivalence of coefficient conditions, e.g. in [3] it was proved that two conditions which guarantee that a factor-sequence should be a Weyl multiplier for a certain property of a given orthogonal series is equivalent to one assumption. An example of these results is the following general theorem proved in [2].

**Theorem 1.1.** *Let $0 < p < q$, $\{\lambda_n\}$ and $\{c_n\}$ be sequences of nonnegative numbers, furthermore let $\Lambda_n := \sum_{k=1}^{n} \lambda_k$. The inequality*

$$S_{11} := \sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=n}^{\infty} c_k^q \right)^{\frac{p}{q}} < \infty \tag{1.1}$$

*holds if and only if there exists a nondecreasing sequence $\{\mu_n\}$ of positive numbers satisfying the following conditions*

$$S_{12} := \sum_{n=1}^{\infty} c_n^q \mu_n < \infty \tag{1.2}$$

*and*

$$S_{13} := \sum_{n=1}^{\infty} \lambda_n \left( \frac{\Lambda_n}{\mu_n} \right)^{\frac{p}{q-p}} < \infty. \tag{1.3}$$

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In the special case $p = 1,\ q = 2$, with $a_n$ and $\lambda_n^{-1}$ in the place of $c_n$ and $\lambda_n$, the author in [3] showed that if

$$\mu_n := \Lambda_n\, C_n^{-1}, \quad \text{where} \quad C_n := \left(\sum_{k=n}^{\infty} c_k^2\right)^{\frac{1}{2}}, \tag{1.4}$$

then (1.1) implies (1.2) and (1.3).

In this special case it is easy to see that if $\varepsilon_n \to 0$, then with $\mu_n^* := \varepsilon_n \mu_n$ in place of $\mu_n$, the condition (1.2) is also satisfied, but it can be that

$$\sum_{n=1}^{\infty} \lambda_n \frac{\Lambda_n}{\mu_n^*} = \infty \tag{1.5}$$

will occur.

This raises the question: Do (1.2) and (1.5) with the sequence $\{\mu_n^*\}$ also imply (1.1) for arbitrary $\{c_n\}$? In [3] we showed that the answer is negative. In other words, this verified the necessity of condition (1.3).

This shows that condition (1.2), in general, does not imply the inequality (1.3).

Thus, we can ask, promptly in connection with the general case considered in Theorem 1.1: What is the "optimal sequence $\{\mu_n\}$" when (1.2) implies (1.3) and conversely?

We shall show that the optimal sequence is

$$\mu_n := \Lambda_n\, C_n^{p-q}, \quad \text{where} \quad C_n := \left(\sum_{k=n}^{\infty} c_k^q\right)^{\frac{1}{q}}, \tag{1.6}$$

and with this $\{\mu_n\}$ (1.2) holds if and only if (1.3) also holds, that is, the assumptions (1.2) and (1.3) are equivalent.

Since the following symmetrical analogue of Theorem 1.1 was also verified in [2], therefore we shall set the same question pertaining to the series appearing in it.

**Theorem 1.2.** *If $p, q, \{\lambda_n\}$ and $\{c_n\}$ are as in Theorem 1.1, furthermore $\tilde{\Lambda}_n := \sum_{k=n}^{\infty} \lambda_k$, then*

$$S_{17} := \sum_{m=1}^{\infty} \lambda_m \left(\sum_{k=1}^{m} c_k^q\right)^{\frac{p}{q}} < \infty \tag{1.7}$$

*holds if and only if there exists a nondecreasing sequence $\{\mu_n\}$ of positive numbers satisfying conditions (1.2) and*

$$S_{18} := \sum_{n=1}^{\infty} \lambda_n \left(\frac{\tilde{\Lambda}_n}{\mu_n}\right)^{\frac{p}{q-p}} < \infty. \tag{1.8}$$

In order to verify our assertions made above, first we shall prove two theorems regarding the equiconvergence of two special series.

## 2. RESULTS

We prove the following assertions.

**Theorem 2.1.** *Let $0 < \alpha < 1,\ \{a_n\}$ and $\{\lambda_n\}$ be sequences of nonnegative numbers, furthermore let $\Lambda_n := \sum_{k=1}^{n} \lambda_k,\ A_n := \sum_{k=n}^{\infty} a_k$ and $\mu_n := \Lambda_n\, A_n^{\alpha-1}$. Then the sum*

$$S_{21} := \sum_{n=1}^{\infty} a_n \mu_n < \infty \tag{2.1}$$

*if and only if*

$$S_{22} := \sum_{n=1}^{\infty} \lambda_n A_n^{\alpha} < \infty. \tag{2.2}$$

**Theorem 2.2.** *If* $\alpha,\ \{a_n\}$ *and* $\{\lambda_n\}$ *are as in Theorem 2.1, furthermore* $\tilde{\Lambda}_n := \sum_{k=n}^{\infty} \lambda_k,\ \tilde{A}_n := \sum_{k=1}^{n} a_k,\ \tilde{A}_0 := 0,$ *and* $\tilde{\mu}_n := \tilde{\Lambda}_n \tilde{A}_n^{\alpha-1}$, *then*

$$S_{23} := \sum_{n=1}^{\infty} a_n \tilde{\mu}_n < \infty \tag{2.3}$$

*if and only if*

$$S_{24} := \sum_{n=1}^{\infty} \lambda_n \tilde{A}_n^{\alpha} < \infty. \tag{2.4}$$

**Corollary 2.3.** *If* $p, q, \{\lambda_n\}, \{c_n\}$ *and* $\Lambda_n$ *are as in Theorem 1.1, and* $\mu_n$ *is defined in (1.6), then the sums in (1.1), (1.2) and (1.3) are equiconvergent.*

**Corollary 2.4.** *If* $p, q, \{\lambda_n\}, \{c_n\}$ *and* $\tilde{\Lambda}_n$ *are as in Theorem 1.2, furthermore*

$$\mu_n := \tilde{\Lambda}_n \tilde{C}_n^{p-q}, \quad \textit{where} \quad \tilde{C}_n := \left( \sum_{k=1}^{n} c_k^q \right)^{\frac{1}{q}},$$

*then the sums in (1.2), (1.7) and (1.8) are equiconvergent.*

**Remark 2.5.** Corollary 2.3 shows that if (1.1) implies a certain property of a fixed orthogonal series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$, then there is no exact universal Weyl multiplier concerning this property, namely the multiplier sequence $\{\mu_n\}$ depends on $\{c_n\}$.

**Remark 2.6.** The interested reader can check that the proofs of the implications (1.1)⇒(1.2) and (1.7)⇒(1.2) given by our corollaries are shorter than those in [2].

**Remark 2.7.** As far as we know, Y. Okuyama and T. Tsuchikara [4]were the first to study conditions of the type (1.7).

## 3. PROOFS

*Proof of Theorem 2.1.* First we show that (2.1) implies (2.2). Since $A_n \to 0$, then

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n A_n^{\alpha} &= \sum_{n=1}^{\infty} \lambda_n \sum_{m=n}^{\infty} (A_m^{\alpha} - A_{m+1}^{\alpha}) \\ &= \sum_{m=1}^{\infty} (A_m^{\alpha} - A_{m+1}^{\alpha}) \sum_{n=1}^{m} \lambda_n. \end{aligned} \tag{3.1}$$

An easy consideration yields that if $0 \le a < b,\ 0 < \alpha < 1$ and

$$\frac{b^{\alpha} - a^{\alpha}}{b - a} = \alpha\, \xi^{\alpha-1}, \tag{3.2}$$

then

$$\xi \ge \alpha^{1/(1-\alpha)} b =: \xi_0, \tag{3.3}$$

namely if $a = 0$ then $\xi = \xi_0$.

Using the relations (3.2) and (3.3) we obtain that

$$A_m^\alpha - A_{m+1}^\alpha = a_m \alpha \xi^{\alpha-1} \le a_m A_m^{\alpha-1}. \tag{3.4}$$

This and (3.1) yield that

$$\sum_{n=1}^\infty \lambda_n\, A_n^\alpha \le \sum_{m=1}^\infty a_m\, A_m^{\alpha-1}\Lambda_m = \sum_{m=1}^\infty a_m \mu_m.$$

Herewith the implication (2.1)⇒(2.2) is proved.

The proof of (2.2)⇒(2.1) is similar. We use the first part of (3.4), $\alpha < 1$ and $\xi < A_m$. Thus

$$\begin{aligned}\sum_{m=1}^\infty a_m \mu_m &= \sum_{m=1}^\infty a_m A_m^{\alpha-1}\Lambda_m \\ &= \sum_{m=1}^\infty (A_m - A_{m+1}) A_m^{\alpha-1}\Lambda_m \\ &\le \alpha^{-1} \sum_{m=1}^\infty (A_m^\alpha - A_{m+1}^\alpha) \sum_{n=1}^m \lambda_n \\ &= \alpha^{-1} \sum_{n=1}^\infty \lambda_n \sum_{m=n}^\infty (A_m^\alpha - A_{m+1}^\alpha) \\ &= \alpha^{-1} \sum_{n=1}^\infty \lambda_n A_n^\alpha,\end{aligned}$$

that is, (2.2)⇒(2.1) is verified.

The proof of Theorem 2.1 is complete. □

*Proof of Theorem 2.2.* The proof is almost the same as that of Theorem 2.1. By (3.2) and (3.3) we get that

$$\tilde{A}_m^\alpha - \tilde{A}_{m-1}^\alpha = a_m \alpha \xi^{\alpha-1} \le a_m \tilde{A}_m^{\alpha-1}. \tag{3.5}$$

Utilizing this at the final step we have

$$\begin{aligned}\sum_{n=1}^\infty \lambda_n \tilde{A}_n^\alpha &= \sum_{n=1}^\infty \lambda_n \sum_{k=1}^n (\tilde{A}_k^\alpha - \tilde{A}_{k-1}^\alpha) \\ &= \sum_{k=1}^\infty (\tilde{A}_k^\alpha - \tilde{A}_{k-1}^\alpha) \sum_{n=k}^\infty \lambda_n \\ &\le \sum_{k=1}^\infty a_k \tilde{A}_k^{\alpha-1} \tilde{\Lambda}_k \equiv \sum_{k=1}^\infty a_k \tilde{\mu}_k;\end{aligned}$$

this proves the implication (2.3)⇒(2.4).

To verify (2.4)⇒(2.3) we use

$$\alpha^{-1}(\tilde{A}_m^\alpha - \tilde{A}_{m-1}^\alpha) \ge a_m \tilde{A}_m^{\alpha-1},$$

which follows from the first part of (3.5) by $\alpha < 1$ and $\xi < \tilde{A}_m$.

Thus we get that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n\tilde{\mu}_n &= \sum_{n=1}^{\infty} a_n\tilde{A}_n^{\alpha-1}\tilde{\Lambda}_n\\ &\le \alpha^{-1}\sum_{n=1}^{\infty}(\tilde{A}_n^{\alpha}-\tilde{A}_{n-1}^{\alpha})\sum_{k=n}^{\infty}\lambda_k\\ &= \alpha^{-1}\sum_{k=1}^{\infty}\lambda_k\sum_{n=1}^{k}(\tilde{A}_n^{\alpha}-\tilde{A}_{n-1}^{\alpha})\\ &= \alpha^{-1}\sum_{k=1}^{\infty}\lambda_k\tilde{A}_k^{\alpha}.\end{aligned}$$

Summing up, the proof of Theorem 2.2 is complete. □

*Proof of Corollary 2.3.* We shall use the results of Theorem 2.1 with $\alpha = \frac{p}{q}$ and $a_n = c_n^q$. Then $\mu_n = \Lambda_n \left(\sum_{k=n}^{\infty} c_k^q\right)^{\frac{p-q}{q}}$, and

$$S_{22} \equiv S_{11} \equiv S_{13} \quad \text{as well as} \quad S_{21} \equiv S_{12},$$

moreover, by Theorem 2.1, $S_{21}$ and $S_{22}$ are equiconvergent, herewith Corollary 2.3 is proved. □

*Proof of Corollary 2.4.* Now we utilize Theorem 2.2 with $\alpha = \frac{p}{q}$ and $a_n = c_n^q$. Then $\tilde{\mu}_n = \tilde{\Lambda}_n \left(\sum_{k=1}^{n} c_k^q\right)^{\frac{p-q}{q}}$, furthermore

$$S_{24} \equiv S_{17} \equiv S_{18} \quad \text{and} \quad S_{23} \equiv S_{12},$$

hold. Since, by Theorem 2.2, $S_{23}$ and $S_{24}$ are equiconvergent, thus Corollary 2.4 is verified. □

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