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THE EXTENSION OF A CYCLIC INEQUALITY TO THE SYMMETRIC FORM

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ABSTRACT. Let n be a natural number such that $n \geq 2$, and let $a_1, \dots\ a_n$ be positive numbers. Considering the notations

$$S_{i_1 \dots i_k} = a_{i_1} + \cdots + a_{i_k},$$
$$S = a_1 + \cdots + a_n,$$

we prove certain inequalities connected to conjugate sums of the form:

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \dots i_k}}{S - S_{i_1 \dots i_k}}$$

Then provided that $1 \leq k \leq n-1$ we give certain lower estimates for expressions of the above form, that extend some cyclic inequalities of Mitrinovic and others.

We also give certain inequalities that are more or less direct applications of the previous mentioned results.

Key words and phrases: Symmetric inequalities, Cyclic inequalities, Inequalities for sums.

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1. INTRODUCTION

Let k and n be natural numbers such that $1 \leq k \leq n-1$ and let $a_1, \dots, a_n$ be positive numbers.

In this paper we first prove the inequality

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \cdots i_k}}{S - S_{i_1 \cdots i_k}} \leq \frac{k^2}{(n-k)^2} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S - S_{i_1 \cdots i_k}}{S_{i_1 \cdots i_k}}, \tag{1.1}$$

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where $k \leq \left[\frac{n}{2}\right]$. We then present a result which states that if $\mathcal{I} = \{\{i_1, \ldots, i_k\} | \ 1 \leq i_1 < \cdots < i_k \leq n\}$, then the following inequality holds:

$$\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} \geq \frac{k}{n-k} \binom{n}{k}. \tag{1.2}$$

This is a result that extends the next cyclic inequalities to their symmetric form:

For $k = 1$ and $n = 3$ we obtain the result of Nesbit [6] (see e.g. [2], [3]),

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}. \tag{1.3}$$

For $k = 1$, we obtain the result of Peixoto [7] (see e.g [5]),

$$\frac{a_1}{S - a_1} + \cdots + \frac{a_n}{S - a_n} \geq \frac{n}{n-1}. \tag{1.4}$$

For arbitrary naturals n, k provided that $1 \leq k \leq n-1$, we get the result of Mitrinović [4] (see e.g [5]),

$$\frac{a_1 + a_2 + \cdots + a_k}{a_{k+1} + \cdots + a_n} + \frac{a_2 + a_3 + \cdots + a_{k+1}}{a_{k+2} + \cdots + a_n + a_1} + \cdots + \frac{a_n + a_1 + \cdots + a_{k-1}}{a_k + \cdots + a_{n-1}} \geq \frac{nk}{n-k}. \tag{1.5}$$

As a remark, we note that this is a cyclic summation.

By considering $n = 3$ and $k = 1$ in Theorem 2.3, we obtain the following result of J. Nesbitt (see e.g. [2, pp.87])

$$\frac{a_1 + a_2}{a_3} + \frac{a_2 + a_3}{a_1} + \frac{a_3 + a_1}{a_2} \geq \frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_1} + \frac{a_3}{a_1 + a_2} + \frac{9}{2}. \tag{1.6}$$

2. MAIN RESULTS

In this section we are going to present the results that we have mentioned in Introduction.

Theorem 2.1. *Let n and k be natural numbers such that $n \geq 2$ and $k \leq \left[\frac{n}{2}\right]$. Then for all positive numbers $a_1, \ldots, a_n$ the following inequality holds:*

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S_{i_1 \cdots i_k}}{S - S_{i_1 \cdots i_k}} \leq \frac{k^2}{(n-k)^2} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{S - S_{i_1 \cdots i_k}}{S_{i_1 \cdots i_k}}, \tag{2.1}$$

where

$$S_{i_1 \cdots i_k} = a_{i_1} + \cdots + a_{i_k},$$
$$S = a_1 + \cdots + a_n.$$

Theorem 2.2. *Let n and k be natural numbers, such that $n \geq 2$ and $1 \leq k \leq n-1$. Then for all positive numbers $a_1, \ldots, a_n$ and $\mathcal{I} = \{\{i_1, \ldots, i_k\} | \ 1 \leq i_1 < \cdots < i_k \leq n\}$, the next inequality holds:*

$$\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} \geq \frac{k}{n-k} \binom{n}{k}. \tag{2.2}$$

We have considered that $S_I = a_{i_1} + \cdots + a_{i_k}$, for $I = \{i_1, \ldots, i_k\}$.

In what follows we are going to refer to the expressions $\frac{S_{i_1 \cdots i_k}}{S - S_{i_1 \cdots i_k}}$ and $\frac{S - S_{i_1 \cdots i_k}}{S_{i_1 \cdots i_k}}$ as complementary.

Using Theorem 2.1 and Theorem 2.2, we obtain the following result which gives lower estimates for the difference of two complementary symmetric sums.

Theorem 2.3. *Let n and k be natural numbers such that $n \geq 2$ and $k \leq \left[\frac{n}{2}\right]$. Then for all positive numbers $a_1, \dots, a_n$ we have*

$$\sum_{1\leq i_1<\cdots<i_k\leq n} \frac{S - S_{i_1\cdots i_k}}{S_{i_1\cdots i_k}} - \sum_{1\leq i_1<\cdots<i_k\leq n} \frac{S_{i_1\cdots i_k}}{S - S_{i_1\cdots i_k}} \geq \frac{(n-2k)\,n}{(n-k)\,k}\binom{n}{k}. \tag{2.3}$$

Using the previous results we also find a lower estimate for the sum of two complementary symmetric sums.

Theorem 2.4. *Let n and k be natural numbers, such that $1 \leq k \leq n-1$, and $a_1, \dots, a_n$ positive numbers. Then the next inequality holds:*

$$\sum_{I\in\mathcal{I}} \frac{S_I}{S - S_I} + \sum_{I\in\mathcal{I}} \frac{S - S_I}{S_I} \geq \frac{(n-k)^2 + k^2}{k(n-k)}\binom{n}{k}. \tag{2.4}$$

3. PROOFS

Proof of Theorem 2.1. Using the notations introduced before, inequality (2.1) becomes

$$\sum_{I\in\mathcal{I}} \frac{S_I}{S - S_I} \leq \frac{k^2}{(n-k)^2} \sum_{I\in\mathcal{I}} \frac{S - S_I}{S_I}. \tag{3.1}$$

Denote by

$$E = \sum_{I\in\mathcal{I}} \frac{S - S_I}{S_I} = \sum_{I\in\mathcal{I}} \frac{\sum_{j\notin I} a_j}{S_I}, \tag{3.2}$$

and note that $|\{j \in \{1, \dots, n\} |\, j \notin I\}| = n-k \geq k$. We write the sum $\sum_{j\notin I} a_j$ as a symmetric sum containing all possible sums of k distinct terms, which do not contain indices in I. Each such sum of k terms appears once. In the case $n = 5$, $k = 2$ we have:

$$a_1 + a_2 + a_3 = \frac{(a_1 + a_2) + (a_1 + a_3) + (a_2 + a_3)}{2}.$$

In the general case we write, for example, the sum of the first $n-k$ terms:

$$a_1 + \cdots + a_{n-k} = \frac{(a_1 + \cdots + a_k) + \cdots + (a_{n-2k+1} + \cdots + a_{n-k})}{\alpha}. \tag{3.3}$$

Clearly in the right member, a_1 appears for $\binom{n-k-1}{k-1}$ times, so $\alpha = \binom{n-k-1}{k-1}$. It is now easy to see that we may write

$$\sum_{j\notin I} a_j = \frac{\sum_{J\in\mathcal{I}} S_J}{\binom{n-k-1}{k-1}}, \tag{3.4}$$

where $J = \{j_1, \dots, j_k\}$, with $I \cap J = \emptyset$.

With our notations, (3.4) is equivalent to

$$S - S_I = \sum_{\substack{J\in\mathcal{I}\\ J\cap I=\emptyset}} \frac{S_J}{\binom{n-k-1}{k-1}}.$$

We obtain

$$E = \sum_{I\in\mathcal{I}} \frac{1}{\binom{n-k-1}{k-1}} \sum_{\substack{J\in\mathcal{I}\\ J\cap I=\emptyset}} \frac{S_J}{S_I},$$

that is

$$E = \frac{1}{\binom{n-k-1}{k-1}} \sum_{J \in \mathcal{I}} S_J \sum_{\substack{I \in \mathcal{I} \\ I \cap J = \emptyset}} \frac{1}{S_I}.$$

Interchanging now I and J we obtain:

$$E = \frac{1}{\binom{n-k-1}{k-1}} \sum_{I \in \mathcal{I}} S_I \sum_{\substack{J \in \mathcal{I} \\ I \cap J = \emptyset}} \frac{1}{S_J}.$$

Denote

$$E_I = \frac{S_I}{\binom{n-k-1}{k-1}} \sum_{\substack{J \in \mathcal{I} \\ I \cap \mathcal{J} = \emptyset}} \frac{1}{S_J}.$$

We prove the following relation:

$$\frac{S_I}{S - S_I} \leq \frac{\beta}{\binom{n-k-1}{k-1}} S_I \sum_{\substack{J \in \mathcal{I} \\ I \cap J = \emptyset}} \frac{1}{S_J} = \beta \cdot E_I. \tag{3.5}$$

It is easy to see that summing (3.5) after $I \in \mathcal{I}$ we get (3.1) and β will be determined later. We have that (3.5) is equivalent to

$$\frac{1}{S - S_I} \leq \frac{\beta}{\binom{n-k-1}{k-1}} \sum_{\substack{J \in \mathcal{I} \\ I \cap J = \emptyset}} \frac{1}{S_J}, \tag{3.6}$$

which is also equivalent to:

$$1 \leq \frac{\beta}{\binom{n-k-1}{k-1}^2} \left(\sum_{\substack{J \in \mathcal{I} \\ I \cap J = \emptyset}} S_J \right) \left(\sum_{\substack{J \in \mathcal{I} \\ I \cap J = \emptyset}} \frac{1}{S_J} \right).$$

Each of the sums in the right-hand side has exactly $\binom{n-k}{k}$ terms, and by Cauchy's inequality we obtain that:

$$\binom{n-k}{k}^2 \leq \left(\sum_{\substack{J \in \mathcal{I} \\ J \cap I = \emptyset}} S_J \right) \left(\sum_{\substack{J \in \mathcal{I} \\ J \cap I = \emptyset}} \frac{1}{S_J} \right).$$

Finally, we get the required β which is:

$$\beta := \frac{\binom{n-k-1}{k-1}^2}{\binom{n-k}{k}^2} = \left[\frac{(n-k-1)!}{(k-1)!(n-2k)!} \cdot \frac{k!(n-2k)!}{(n-k)!} \right]^2 = \left[\frac{k}{n-k} \right]^2.$$

Hence in view of (3.5) we have obtained that:

$$\frac{S_I}{S - S_I} \leq \left(\frac{k}{n-k} \right)^2 E_I$$

By summing we finally get (3.1). □

Proof of Theorem 2.2. By the Cauchy inequality we have that:

$$\left(\sum_{I \in \mathcal{I}} \frac{S_I}{S - S_I} \right) \left(\sum_{I \in \mathcal{I}} S_I (S - S_I) \right) \geq \left(\sum_{I \in \mathcal{I}} S_I \right)^2. \tag{3.7}$$

In order to prove (2.2) it is enough to show that:

$$\left(\sum_{I\in\mathcal{I}} S_I\right)^2 \geq \frac{k}{n-k}\binom{n}{k}\sum_{I\in\mathcal{I}} S_I(S-S_I) \tag{3.8}$$

and by (3.7) and (3.8) we obtain (2.2) by making the product.

Let us prove (3.8). We begin with the next lemma.

Lemma 3.1. $\sum_{I\in\mathcal{I}} S_I = \binom{n-1}{k-1} S.$

Proof of Lemma 3.1. We have to find the multiplicity of a_1 in $\sum_{I\in\mathcal{I}} S_I$. If a_1 appears in the first position, the other $k-1$ position from I may be chosen in $\binom{n-1}{k-1}$ ways and because the sum is symmetric it follows the conclusion. □

Using the lemma we obtain

$$\sum S_I \cdot S = \binom{n-1}{k-1} S^2$$

and (3.8) becomes:

$$\binom{n-1}{k-1}^2 \cdot S^2 \geq \frac{k}{n-k}\binom{n}{k}\left[\binom{n-1}{k-1}S^2 - \sum_{I\in\mathcal{I}} S_I^2\right] \tag{3.9}$$

which is

$$\frac{k}{n-k}\binom{n}{k}\left(\sum_{I\in\mathcal{I}} S_I^2\right) \geq \binom{n-1}{k-1}S^2\left[\frac{k}{n-k}\binom{n}{k} - \binom{n-1}{k-1}\right]. \tag{3.10}$$

Using the identity:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1},$$

it follows that (3.10) is equivalent to:

$$\frac{k}{n-k}\binom{n}{k}\left(\sum_{I\in\mathcal{I}} S_I^2\right) \geq \binom{n-1}{k-1}S^2\left[\frac{k}{n-k}\binom{n-1}{k-1}\right],$$

which is also equivalent to:

$$\left(\sum_{I\in\mathcal{I}} S_I^2\right)\binom{n}{k} \geq \binom{n-1}{k-1}^2 S^2.$$

By the Cauchy inequality and using Lemma 3.1, we have that

$$\left(\sum_{I\in\mathcal{I}} S_I^2\right)\binom{n}{k} \geq \left(\sum_{I\in\mathcal{I}} S_I\right)^2.$$

(Clearly, both sums have $\binom{n}{k}$ terms). So, (3.8) holds.

Note that the equality holds if and only if $S_I = S_J$ for $I, J \in \mathcal{I}$, which gives that $a_1 = \cdots = a_n$. □

Remark 3.2. In [1] a shorter proof for this theorem is given by using Jensen's inequality for some convex function.

Proof of Theorem 2.3. Using Theorem 2.1, we find that our sum is in fact greater or equal to

$$\left(\frac{(n-k)^2}{k^2}-1\right)\sum_{1\le i_1<\cdots<i_k\le n}\frac{S_{i_1\cdots i_k}}{S-S_{i_1\cdots i_k}}.$$

By Theorem 2.2, this is greater than

$$\frac{(n-2k)\,n}{k^2}\cdot\frac{k}{n-k}\cdot\binom{n}{k}.$$

This ends the proof of Theorem 2.3. □

Proof of Theorem 2.4. Using Theorem 2.1 and Theorem 2.2 together with the notations $\mathcal{I}=\{\{i_1,\dots,i_k\}|\ 1\le i_1<\cdots<i_k\le n\}$ and $\mathcal{J}=\{\{j_1,\dots,j_{n-k}\}|\ 1\le j_1<\cdots<j_{n-k}\le n\}$, we obtain:

$$\sum_{I\in\mathcal{I}}\frac{S-S_I}{S_I}=\sum_{J\in\mathcal{I}}\frac{S_J}{S-S_J}\ge\frac{n-k}{k}\binom{n}{k}.$$

It is clear that

$$\sum_{I\in\mathcal{I}}\frac{S_I}{S-S_I}+\sum_{I\in\mathcal{I}}\frac{S-S_I}{S_I}\ge\left(\frac{n-k}{k}+\frac{k}{n-k}\right)\cdot\binom{n}{k},$$

and this is exactly the required inequality. □

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