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IMPROVED INTEGRAL INEQUALITIES FOR PRODUCTS OF CONVEX FUNCTIONS

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ABSTRACT. The inequalities, which Pachpatte has derived from the well known Hadamard's inequality for convex functions, are improved, obtaining new integral inequalities for products of convex functions. These inequalities are sharp for linear functions, while the initial Pachpatte's ones do not have this property.

Key words and phrases: Convex function, Hadamard's inequality, Pachpatte's inequalities.

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1. PRELIMINARIES

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality (see [3] or [2]) which has generated a wide range of directions for extension and a rich mathematical literature. Below, we recall this inequality, together with its framework.

A function $f : [a, b] \to \mathbb{R}$, with $[a, b] \subset \mathbb{R}$, is said to be convex if whenever $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{1.1}$$

This definition has its origins in Jensen's results from [4] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them. They were known before the analytical foundation of convexity theory, due to the deep geometrical significance and many geometrical applications related to the convex shapes (see, for example, [1], [5], [7]). One of these results, known as

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Hadamard's inequality, which was first published in [3], states that a convex function f satisfies

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Recent inequalities derived from Hadamard's inequality can be found in Pachpatte's paper [6] and we recall two of them in the following theorem, because we intend to improve them. Let us suppose that the interval $[a,b]$ has the property that $b-a \leq 1$. Then the following result holds.

Theorem 1.1. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\begin{aligned} &\frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx \\ &\qquad\leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{8}\left[\frac{M(a,b)+N(a,b)}{(b-a)^2}\right] \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx \\ &\qquad\leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{4}\cdot\frac{1+b-a}{b-a}\left[M(a,b)+N(a,b)\right], \end{aligned} \tag{1.4}$$

where

$$M(a,b) = f(a)\,g(a) + f(b)\,g(b) \tag{1.5}$$

and

$$N(a,b) = f(a)\,g(b) + f(b)\,g(a). \tag{1.6}$$

Remark 1.2. The inequalities (1.3) and (1.4) are valid when the length of the interval $[a,b]$ does not exceed 1. Unfortunately, this condition is accidentally omitted in [6], but it is implicitly used in the proof of Theorem 1.1.

Of course, there are cases when at least one of the two inequalities from the previous theorem is satisfied for $b-a>1$, but it is easy to find counterexamples in this case, as follows.

Example 1.1. Let us take $[a,b]=[0,2]$. The functions $f:[0,2]\to\mathbb{R}$ and $g:[0,2]\to\mathbb{R}$ are defined by $f(x)=x$ and $g(x)=x$. Then it is obvious that $M(a,b)=4$, $N(a,b)=0$. Then, the direct calculus of both sides of (1.3) leads to

$$\frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx = \frac{11}{6},$$

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{8}\left[\frac{M(a,b)+N(a,b)}{(b-a)^2}\right] = \frac{35}{24}$$

and, obviously, inequality (1.3) is false.

Remark 1.3. Inequality (1.3) is sharp for linear functions defined on $[0,1]$, while inequality (1.4) does not have the same property.

In this paper we improve the previous theorem, such that the condition

$$b-a\leq 1$$

is eliminated and the derived inequalities are sharp for the whole class of linear functions.

2. IMPROVED INEQUALITIES

Let us consider an interval $[a,b] \subset \mathbb{R}$ and a function $f:[a,b] \to \mathbb{R}$. The following classical result is very useful in this section.

Lemma 2.1. *The following statements are equivalent:*

(i) *Function f is convex on $[a,b]$,*

(ii) *For every $x,y \in [a,b]$, the function $\gamma:[0,1] \to \mathbb{R}$, defined by*

$$\gamma(t) = f(tx + (1-t)y) \tag{2.1}$$

is convex on $[0,1]$.

The main result of this section is in the following theorem.

Theorem 2.2. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\begin{aligned}\frac{3}{2}\cdot\frac{1}{(b-a)^2}\int_a^b\int_a^b\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dtdydx \\ \leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{8}\left[M(a,b)+N(a,b)\right]\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\frac{3}{b-a}\int_a^b\int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx \\ \leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{2}\left[M(a,b)+N(a,b)\right],\end{aligned} \tag{2.3}$$

where the numbers $M(a,b)$ and $N(a,b)$ are defined by (1.5) and (1.6).

Proof. The proof follows, mainly, the same procedure as that of the previous theorem from [6]. But, at a certain point, it becomes more refined. Since both functions f and g are convex, for every two points $x,y \in [a,b]$ and $t \in [0,1]$, the following inequalities are valid

$$f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$$

and

$$g(tx+(1-t)y) \leq tg(x) + (1-t)g(y).$$

Multiplying the above mentioned inequalities, we find the following one

$$\begin{aligned}f(tx+(1-t)y)\,g(tx+(1-t)y) \\ \leq t^2 f(x)\,g(x) + (1-t)^2 f(y)\,g(y) + t(1-t)\left[f(x)\,g(y)+f(y)\,g(x)\right].\end{aligned}$$

Both sides of this inequality are integrable with respect to t on $[0,1]$, due to Lemma 2.1 together with the known properties of the convex functions. Then, integrating them over $[0,1]$, one gets

$$\begin{aligned}\int_0^1 f(tx+(1-t)y)\,g(tx+(1-t)y)\,dt \\ \leq \frac{2(b-a)}{3}\int_a^b f(x)\,g(x)\,dx + \frac{1}{3}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(y)\,dy\right).\end{aligned} \tag{2.4}$$

The convexity properties of f and g allows us to use the right side of the inequality (1.1), written as below:

$$\int_a^b f(x)\,dx \leq (b-a)\frac{f(a)+f(b)}{2}, \tag{2.5}$$

$$\int_a^b g(y)\,dy \le (b-a)\frac{g(a)+g(b)}{2}. \tag{2.6}$$

Replacing (2.5) and (2.6) in the last two integrals from the right side of (2.4), one obtains the following inequality

$$\begin{aligned}\int_0^1 & f(tx+(1-t)y)\,g(tx+(1-t)y)\,dt \\ &\le \frac{2(b-a)}{3}\int_a^b f(x)\,g(x)\,dx + \frac{(b-a)^2}{12}\left[f(b)+f(a)\right]\left[g(b)+g(a)\right].\end{aligned} \tag{2.7}$$

Direct calculus shows that

$$\left[f(b)+f(a)\right]\left[g(b)+g(a)\right] = M(a,b)+N(a,b), \tag{2.8}$$

according to (1.5) and (1.6). Replacing (2.8) in (2.7) and multiplying both sides of (2.7) by $\frac{3}{2(b-a)^2}$ one completes the proof of (2.2).

Inequality (2.3) has a similar starting point for its proof. Again the convexity of the two functions f and g gives us the starting inequalities

$$f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) \le tf(x)+(1-t)\,f\left(\frac{a+b}{2}\right)$$

and

$$g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) \le tg(x)+(1-t)\,g\left(\frac{a+b}{2}\right).$$

As above, we multiply the previous inequalities, obtaining

$$\begin{aligned} f&\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) \\ &\le t^2 f(x)\,g(x) + (1-t)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ &\quad + t(1-t)\left[f(x)\,g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) g(x)\right].\end{aligned} \tag{2.9}$$

As in the proof of the first inequality, we integrate both sides of (2.9) over $[0,1]$, deriving the following relation

$$\begin{aligned}\int_0^1 & f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dt \\ &\le \frac{1}{3}\left[f(x)\,g(x) + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\right] \\ &\quad + \frac{1}{6}\left[f(x)\,g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) g(x)\right].\end{aligned} \tag{2.10}$$

Now, using the convexity of f and g and integrating both sides of (2.10) over $[a,b]$, we get

$$\begin{aligned}(2.11)\quad &\int_a^b \int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx \\ &\leq \frac{1}{3}\int_a^b f(x)\,g(x)\,dx + \frac{b-a}{12}\left[f(a)+f(b)\right]\left[g(a)+g(b)\right] \\ &\quad + \frac{g(a)+g(b)}{12}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{12}\int_a^b g(x)\,dx.\end{aligned}$$

Once again we replace (2.5) and (2.6) in the last two integrals of the right side of the last inequality and, using (2.8), we obtain

$$\begin{aligned}(2.12)\quad &\int_a^b \int_0^1 f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) g\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) dtdx \\ &\leq \frac{1}{3}\int_a^b f(x)\,g(x)\,dx + \frac{b-a}{6}\left[M(a,b)+N(a,b)\right].\end{aligned}$$

The multiplication of both sides of this inequality by $\frac{3}{b-a}$ completes the proof of (2.3). □

Remark 2.3. Both inequalities (2.2) and (2.3) are sharp for the whole class of linear functions defined on an arbitrary closed real interval $[a,b]$.

Indeed, let us suppose that the functions $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are defined by $f(x)=mx+n$ and $g(x)=px+q$. Then, direct calculus shows that

$$M(a,b) = mp\left(a^2+b^2\right)+(mq+np)(a+b)+nq$$

and

$$N(a,b) = 2mpab+(mq+np)(a+b)+nq.$$

Both sides of the inequality (2.2) become, in this case, equal to

$$\frac{mp}{24}\left(11a^2+14ab+11b^2\right)+\frac{3}{4}(mq+np)(a+b)+\frac{3}{2}nq,$$

while both sides of (2.3) are

$$\frac{mp}{6}\left(5a^2+8ab+5b^2\right)+\frac{3}{2}(mq+np)(a+b)+3nq.$$

Therefore, both (2.2) and (2.3) are sharp if the two functions f and g are linear. It is an expected result, taking into account the fact that all the inequalities used during their proofs (the definition of convex functions, the inequality of Hadamard) are sharp for the linear functions.

It would be of interest to study various inequalities for functions that have generalized convexity properties corresponding to non-connected domains with generalized convexity properties, as discussed in [1] and [7].

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