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ON THE B-ANGLE AND g-ANGLE IN NORMED SPACES

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ABSTRACT. It is known that in a strictly convex normed space, the $B-$orthogonality (Birkhoff orthogonality) has the property, “$B-$orthogonality is unique to the left“. Using this property, we introduce the definition of the so-called $B-$angle between two vectors, in a smooth and uniformly convex space. Also, we define the so-called $g-$angle between two vectors. It is demonstrated that the $g-$angle in a unilateral triangle, in a quasi-inner product space, is $\pi/3$. The $g-$angle between a side and a diagonal, in a so-called $g-$quandrangle, is $\pi/4$.

Key words and phrases: Smooth normed spaces, quasi-inner product spaces, oriented (non-oriented) $B-$angle between two vectors, oriented (non-oriented) $g-$angle between two vectors.

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Let X be a real smooth normed space of dimension greater than 1. It is well known that the functional

(1) $$g(x,y) := \|x\| \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \qquad (x,y \in X)$$

always exists (see [5]).

This functional is linear in the second argument and it has the following properties:

(2) $$g(\alpha x, y) = \alpha g(x,y)\,(\alpha \in \mathbb{R}), \qquad g(x,x) = \|x\|^2, \qquad |g(x,y)| \le \|x\|\,\|y\|\,.$$

Definition 1 ([10]). A normed space X is a quasi-inner product space (q.i.p. space) if the equality

(3) $$\|x+y\|^4 - \|x-y\|^4 = 8\left[\|x\|^2 g(x,y) + \|y\|^2 g(y,x)\right]$$

holds for all $x, y \in X$.

The space of sequences l^4 is a $q.i.p.$ space, but l^1 is not a $q.i.p.$ space.

It is proved in [10] and [11] that a $q.i.p.$ space X is very smooth, uniformly smooth, strictly convex and, in the case of Banach spaces, reflexive.

The orthogonality of the vector $x \neq 0$ to the vector $y \neq 0$ in a normed space X may be defined in several ways. We mention some kinds of orthogonality and their notations:

- $x \perp_B y \Leftrightarrow (\forall \lambda \in \mathbb{R})\, \|x\| \le \|x + \lambda y\|$ (Birkhoff orthogonality),

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- $x \perp_J y \Leftrightarrow \|x-y\| = \|x+y\|$ (James orthogonality),
- $x \perp_S y \Leftrightarrow \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|$ (Singer orthogonality).

In the papers [8], [6] and [9], by using the functional g, the following orthogonal relations were introduced:

$$x \perp_g y \Leftrightarrow g(x,y) = 0,$$
$$x \overset{g}{\perp} y \Leftrightarrow g(x,y) + g(y,x) = 0,$$
$$x \underset{g}{\perp} y \Leftrightarrow \|x\|^2 g(x,y) + \|y\|^2 g(y,x) = 0.$$

In [6, Theorem 2] the following assertion is proved: If X is smooth, then $x \perp_g y \Leftrightarrow x \perp_B y$.

In [11] we have proved the following assertion: If X is a $q.i.p.$ space, then

$$x \underset{g}{\perp} y \Leftrightarrow x \perp_J y \quad \text{and} \quad x \overset{g}{\perp} y \Leftrightarrow x \perp_S y.$$

If there exists an inner product $\langle \cdot, \cdot \rangle$ in X, ($i.p.$), then it is easy to see that $x \rho y \Leftrightarrow \langle x, y \rangle = 0$ holds for every

$$\rho \in \left\{ \perp_B, \perp_J, \perp_S, \perp_g, \overset{g}{\perp}, \underset{g}{\perp} \right\}.$$

For more on $B-$orthogonality and $g-$orthogonality, see the papers [1], [2], [13] and [14]. Some additional properties of this orthogonality are quoted below. Denote by $P_{[x]}y$ the set of the best approximations of y with vectors from $[x]$.

Theorem 1. *Let X be a smooth and uniformly convex normed space, and let $x, y \in X - \{0\}$ be fixed linearly independent vectors. The following assertions are valid.*

(1) *There exists a unique $a \in \mathbb{R}$ such that*

$$P_{[x]}y = ax \Leftrightarrow g(y - ax, x) = 0 \Leftrightarrow \|y - ax\|^2 = g(y - ax, y),$$
$$\operatorname{sgn}\, a = \operatorname{sgn}\, g(y,x).$$

(2) *If $z \in \operatorname{span}\{x,y\}$ and $y \perp_B x \wedge z \perp_B x$, then there exists $\lambda \in \mathbb{R}$ such that $z = \lambda y$.*

(3) *If $x \perp_B y - \alpha x \wedge x \perp_B y - \beta x$ then $\alpha = \beta$.*

Proof.

(1) The proof can be found in [14].

(2) Since X is smooth, the equivalence

$$y \perp_B x \wedge z \perp_B x \Leftrightarrow g(y,x) = 0 \wedge g(z,x) = 0$$

holds.

Hence

$$x = \alpha y + \beta z \;\Rightarrow g(y, \alpha x + \beta z) = 0 \wedge g(z, \alpha x + \beta z) = 0.$$

We get the system of equations

$$\alpha \|y\|^2 + \beta g(y,z) = 0$$
$$\alpha g(z,x) + \beta \|z\|^2 = 0.$$

This system has a non-trivial solution for α and β iff

$$g(y,z)g(z,y) = \|y\|^2 \|z\|^2 \Leftrightarrow |g(y,z)|\, |g(z,y)| = \|y\|^2 \|z\|^2 .$$

The last equation is not correct if $|g(y,z)| < \|y\| \, \|z\|$. So, $|g(y,z)| = \|y\| \, \|z\|$.Then by Lemma 5 of [3], there exists $\lambda \in \mathbb{R}$ such that $z = \lambda y$.

(3) In accordance with 1) we have

$$g(x, y - \alpha x) = 0 \wedge g(x, y - \beta x) = 0 \Leftrightarrow g(x, y) - \alpha \|x\|^2 = 0 \wedge g(x, y) - \beta \|x\|^2 = 0 \Rightarrow \alpha = \beta.$$

□

From now on we assume that points $0, x, y$ are the vertices of the triangle $(0, x, y)$ and points $0, x, y, x+y$ are the vertices of the parallelogram $(0, x, y, x+y)$. The numbers $\|x - y\|$, $\|x + y\|$ are the lengths of diagonal of this parallelogram. If $\|x\| = \|y\|$, we say that this parallelogram is a rhomb, and if $x \perp_\rho y$, we say that this parallelogram is a ρ-rectangle, $\rho \in \left\{ \perp_B, \perp_J, \perp_S, \underset{g}{\perp} \right\}$.

From the next theorem, we see the similarity of $q.i.p.$ spaces to inner-product spaces ($i.p.$ spaces).

Theorem 2. *Let X be a $q.i.p.$ space. The following assertions are valid.*

(1) *The lengths of the diagonals in parallelogram $(0, x, y, x+y)$ are equal if and only if the parallelogram is a $g-$rectangle, i.e., $x \underset{g}{\perp} y$.*

(2) *The diagonals of the rhomb $(0, x, y, x + y)$ are $g-$orthogonal, i.e., $(x - y) \underset{g}{\perp} (x + y)$.*

(3) *The parallelogram $(0, x, y, x + y)$ is a $g-$quadrangle if and only if the lengths of its diagonals are equal and the diagonals are $g-$orthogonal.*

The proof of Theorem 2 can be found in [11].

The angle between two vectors x and y in a real normed space was introduced in [7] as

$$\angle(x, y) := \arccos \frac{g(x, y) + g(y, x)}{2 \|x\| \|y\|} \quad (x, y \in X - \{0\}).$$

So, $x \overset{g}{\perp} y \Leftrightarrow \cos \angle(x, y) = 0$.

In this paper we introduce several definitions of angles in a smooth normed space X.

Let us begin with the following observations. By (2), it is easily seen that we have

$$-1 \leq \frac{\|x\|^2 g(x, y) + \|y\|^2 g(y, x)}{\|x\| \|y\| (\|x\|^2 + \|y\|^2)} \leq 1 \quad (x, y \in X - \{0\}). \tag{4}$$

Hence we define new angle between the vectors x and y ,denoted as $\underset{g}{\angle}(x, y)$.

Definition 2. The number

$$\underset{g}{\angle}(x, y) := \arccos \frac{\|x\|^2 g(x, y) + \|y\|^2 g(y, x)}{\|x\| \ \|y\| \ (\|x\|^2 + \|y\|^2)}$$

is called the $g-$angle between the vector x and the vector y.

It is very easy to see that :

$$\underset{g}{\angle}(x, y) = \underset{g}{\angle}(y, x), \quad \underset{g}{\angle}(\lambda x, \lambda y) = \underset{g}{\angle}(x, y), \quad x \underset{g}{\perp} y \Leftrightarrow \cos \underset{g}{\angle}(x, y) = 0.$$

Theorem 3. *Let X be a $q.i.p.$ space. Then the following assertions hold.*

(1) *The $g-$angle over the diameter of a circle is $g-$right, i.e., if c is the circle in $\operatorname{span}\{x, y\}$, centered at $\frac{x+y}{2}$ of radius $\frac{\|x-y\|}{2}$, then $z \in c \Rightarrow (x - z) \underset{g}{\perp} (y - z)$, Figure 1.*

(2) *The centre of the circumscribed circumference about the $g-$right triangle is the centre of the $g-$hypotenuse.*

Proof.

(1) If $z \in c$, then $\left\|z - \frac{x+y}{2}\right\| = \frac{\|x-y\|}{2}$, i.e. $\|2z - (x+y)\| = \|x-y\|$. Hence

$$(x-z)\perp_J(y-z) \Leftrightarrow (x-z)\underset{g}{\perp}(y-z),$$

because X is a $q.i.p.$ space.

(2) Let c be the circle defined by the equation $\left\|z - \frac{x+y}{2}\right\| = \left\|\frac{x-y}{2}\right\|$, where $x \underset{g}{\perp} y$ i.e. $\|x-y\| = \|x+y\|$. Then $0 \in c$.

□

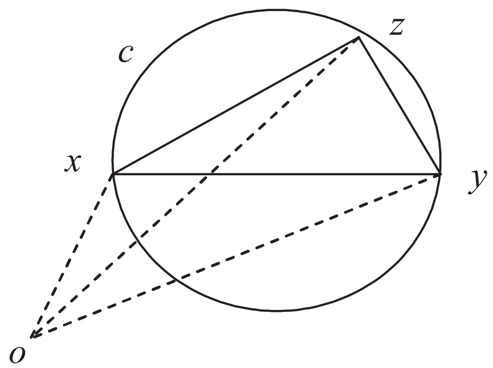

Figure 1:

In accordance with $B-$orthogonality, now we define the oriented $B-$angle between vectors x and y.

Firstly, we have the following observation. Let $P_{[x]}y = ax, \quad (a = a(x,y))$.If $\|ax\| \le \|y\|$ for every $x, y \in X - \{0\}$, then X is an $i.p.$ space (see (18.1) in [4]). So, in a normed (non trivial) space, a $B-$catheti may be greater than the hypotenuse.

Lemma 4. *Let X be a smooth and uniformly convex space and $x, y \in X - \{0\}$ linearly independent. Then there exists a unique $\tau = \tau(x,y)$ such that $\|y\| = \|y - \tau x\|$. If X is a q.i.p. space and y is not $B-$orthogonal to x, then there exist unique $p \in \mathbb{R}$ such that $(y - px) \underset{g}{\perp} px$.*

Proof. We consider the function

$$f(t) = \|y - tx\| \quad (x, y \in X - \{0\}, \quad t \in \mathbb{R}).$$

Since X is smooth and uniformly convex, there exists a unique $a = a(x,y) \in \mathbb{R}$ such that

$$\min_{t\in\mathbb{R}} f(t) = f(a) = \|y - ax\|, \quad g(y - ax, x) = 0, \quad \text{sgn}\ a = \text{sgn}\, g(y,x). \tag{5}$$

(The vector ax is the best approximation of vector y with vectors of $[x]$, i.e., $P_{[x]}y = ax$ (see [14]).

On the other hand, the function f is continuous and convex on $\mathbb{R}$ and therefore there exists a unique $\tau = \tau(x,y) \in \mathbb{R}$ (see Figure 2) such that

$$f(a) < \|y\| = \|y - \tau x\|.$$

If X is a $q.i.p.$ space, we get $p = \frac{\tau}{2}$. In this case, we have $\|y\| = \|y - 2px\|$, hence

$$\|(y - px) + px\| = \|(y - px) - px\|,$$

i.e.

$$(y - px)\perp_J px \Leftrightarrow (y - px) \underset{g}{\perp} px.$$

In this case we shall write $P_x^g y = px$. Clearly $\|y\| = \|y - 2px\| \Rightarrow \|px\| \leq \|y\|$.

In (5) we have:

$$0 < a < \tau \Leftrightarrow g(y, x) > 0, \quad \tau < a < 0 \Leftrightarrow g(y, x) < 0 \quad \text{(Figure 2)}.$$

Hence, by $\|y\| = \|y - \tau x\|$ we get $\|\tau x\| - \|y\| \leq \|y\|$, i.e.

$$\frac{\|\tau x\|}{2} \leq \|y\|. \tag{6}$$

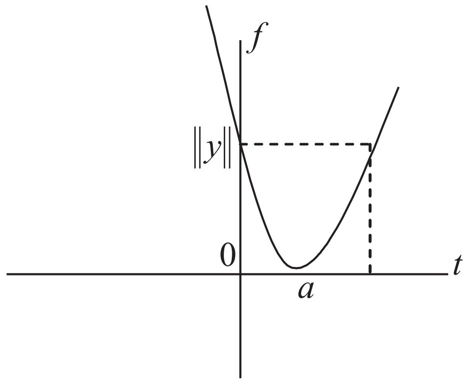

Figure 2:

Assume that $g(y, x) > 0$. If $a < \frac{\tau}{2}$, then by (5) we have $\|ax\| \leq \frac{\|\tau x\|}{2} \leq \|y\|$. If $a \geq \frac{\tau}{2}$, then $\tau - a \leq \frac{\tau}{2}$ and we have $\|(\tau - a)x\| \leq \frac{\|\tau x\|}{2} \leq \|y\|$. Hence we get $\min\{a, \tau - a\} \leq \frac{\tau}{2}$.

Of course, if $g(y, x) < 0$, we get $\min\{|a|, |\tau - a|\} \leq \frac{|\tau|}{2}$. Thus, we conclude that

$$-1 \leq \frac{\|kx\|}{\|y\|} \operatorname{sgn} g(y, x) \leq 1 \quad (x, y \in X - \{0\}), \tag{7}$$

where $k = \min\{|a|, |\tau - a|\} \quad (k = k(x, y))$. □

Keeping in mind (7) and the characteristics of $B-$ orthogonality, we introduce the following definitions of the oriented $B-$angle between the vector x and the vector y.

Definition 3. Let X be smooth and uniformly convex. The number

$$\cos_B(\overrightarrow{x, y}) := \frac{\|kx\|}{\|y\|} \operatorname{sgn} g(y, x), \tag{8}$$

$$k = \min\{|a|, |\tau - a|\}, \quad (x, y \in X - \{0\})$$

is called the $B-$cosine of the oriented angle between x and y.

The number

$$\angle_B(\overrightarrow{x, y}) := \arccos_B(\overrightarrow{x, y})$$

is the oriented $B-$angle between the vector x and the vector y.

Definition 4.

$$\cos_B(x, y) := \sqrt{|\cos_B(\overrightarrow{x, y}) \cos_B(\overrightarrow{y, x})|} \operatorname{sgn} g(x, y) \operatorname{sgn} g(y, x).$$

The number $\angle_B(x, y) := \arccos_B(x, y)$ is called the $B-$angle between the vector x and the vector y.

If X is an $i.p.$ space with $i.p.$ $\langle \cdot, \cdot \rangle$, we have $a = \frac{g(x,y)}{\|x\|^2} = \frac{\langle x,y \rangle}{\|x\|^2} = \frac{g(y,x)}{\|x\|^2}$ (see [14]). So, in this case $\cos_B(\overrightarrow{x,y}) = \frac{\langle x,y\rangle}{\|x\|\|y\|}$. Observe that $\cos_B(\overrightarrow{x,y})$ is not symmetric in x and y, so, in the triangle $(0,x,y)$ we have 6 oriented $B-$angles.

Since inequalities $-1 \leq \frac{|g(x,y)|}{\|x\|\,\|y\|} \leq 1$ are valid for every $x, y \in X - \{0\}$ and $y \perp_B x \Leftrightarrow g(y,x) = 0$ in a smooth space, we may ask whether $\cos_B(\overrightarrow{x,y}) = \frac{g(y,x)}{\|x\|\,\|y\|}$ for every $x, y \in X$. The answer is no. Namely, in this case we have $a(x,y) = \frac{g(y,x)}{\|x\|^2}$ and hence, for every $x, y \in X - \{0\}$, we get $\|ax\| = \frac{|g(y,x)|}{\|x\|} \leq \|y\|$. It follows from 18.1 of [4] that X is an $i.p.$ space.

Theorem 5. *Let X be a smooth and strictly convex space. Then,*

(1) $\cos_B(\overrightarrow{\lambda x, y}) = \cos_B(\overrightarrow{x,y}) \operatorname{sgn} \lambda \quad (\lambda \in \mathbb{R} - \{0\}),$
(2) $\cos_B(\overrightarrow{x, \lambda y}) = \cos_B(\overrightarrow{x,y}) \operatorname{sgn} \lambda \quad (\lambda \in \mathbb{R} - \{0\}).$

Proof.

(1) Assume that $P_{[x]}y = ax, \quad k = \{|a|, |\tau - a|\}, \|y\| = \|y - \tau x\|, \quad P_{[\lambda x]}y = b\lambda x$. Then $b\lambda = a$ and $\min\{|b\lambda|, |\tau - b\lambda|\} = \min\{|a|, |\tau - a|\} = k$. Hence, by Definition 3, we have

$$\begin{aligned} \cos_B(\overrightarrow{\lambda x, y}) &= \frac{\min\{|\lambda b|, |\tau - \lambda b|\}\,\|x\|}{\|y\|} \operatorname{sgn} g(y, \lambda x) \\ &= \frac{\|kx\|}{\|y\|} \operatorname{sgn} \lambda g(y,x) = \cos_B(\overrightarrow{x,y}) \operatorname{sgn} \lambda. \end{aligned}$$

(2) Let be $P_{[x]}y = ax \quad \|y\| = \|y - \tau x\|$ and $\|\lambda y\| = \|\lambda y - \tau_\lambda x\|$. Then $P_{[x]}\lambda y = \lambda a x$ and by $\|\lambda y\| = \|\lambda y - \lambda \tau x\|$ we get $\tau_\lambda = \lambda \tau$ and $k_\lambda = \min\{|\lambda a|, |\lambda\tau - \lambda a|\} = |\lambda|\, k$. Thus

$$\begin{aligned} \cos_B(\overrightarrow{x, \lambda y}) &= \frac{\|k_\lambda x\|}{\|\lambda y\|} \operatorname{sgn} g(\lambda y, x) \\ &= \frac{\|kx\|}{\|y\|} \operatorname{sgn} \lambda g(y,x) \\ &= \cos_B(\overrightarrow{x,y}) \operatorname{sgn} \lambda. \end{aligned}$$

□

Theorem 6. *Let X be smooth, $x, y \in X - \{0\}$ linearly independent, $\|y - x\| = \|y\|$. Then $\left(\angle_B \overline{(x,y)}\right) = \angle_B \overline{(-x, y-x)}$, (Figure 3).*

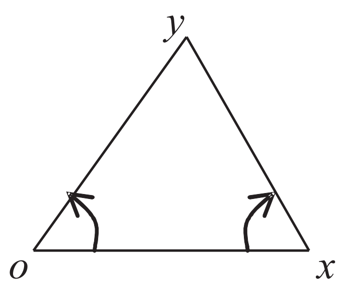

Figure 3:

Proof. In a smooth space X (see [12]), for $x, y \in X$, we have

(9) $$\|x\| \ (\|x\| - \|x-y\|) \le g(x,y) \le \|x\| \ (\|x+y\| - \|x\|).$$

Since $\|y-x\| = \|y\|$, we get $g(y,x) > 0$ and $g(y-x,-x) > 0$. Let $P_{[x]}y = ax$ and $P_{[x]}(y-x) = b$. Then: $a > 0,\ b > 0$ (see [14]), $g(y-ax,x) = 0$ and

$$g(y-x-bx,x) = 0 \Leftrightarrow g(y-(1+b)x,x) = 0.$$

By virtue of 2) in Theorem 1, we get $1+b = a$ such that $P_{[x]}(y-x) = (a-1)x$. From this and Definition 3, we have

$$\begin{aligned}\cos_B(\overrightarrow{-x, y-x}) &= \frac{\|kx\|}{\|y-x\|}\operatorname{sgn} g(y-x,-x)\\ &= \frac{\min\{a, 1-a\}\,\|x\|}{\|y\|}\\ &= \cos_B(\overrightarrow{x,y}).\end{aligned}$$

□

We now assume that X is a $s.i.p.$ space.

Analogous to Definition 3 and Definition 4, in a $q.i.p.$ space, we will introduce a new definition of an oriented $g-$angle and the corresponding non oriented $g-$angle.

Definition 5. Let $x \ne 0, y \in X$ and $p = \frac{\tau}{2}$ (see Lemma 4). Then

$$\cos_g(\overrightarrow{x,y}) := \frac{\|px\|}{\|y\|}\operatorname{sgn}(\|x\|^2 g(x,y) + \|y\|^2 g(y,x)).$$

The number $\angle_g(\overrightarrow{x,y}) := \operatorname{arccos}_g(\overrightarrow{x,y})$ is the oriented $g-$angle between vector x and vector y.

We observe that, for all $\lambda \ne 0$,

$$y - px \perp_g px \Rightarrow \lambda y - \lambda px \perp_g \lambda px,$$

i.e., $P^g_x y = a \Rightarrow P^g_{\lambda x}\lambda y = ax$. Hence we have

(10) $$\cos_g(\overrightarrow{\lambda x, \lambda y}) = \cos_g(\overrightarrow{x,y})\ \operatorname{sgn}\lambda \quad (\lambda \ne 0).$$

Definition 6.

$$\cos_g(x,y) := \sqrt{\cos_g(\overrightarrow{x,y})\ \cos_g(\overrightarrow{y,x})}\operatorname{sgn}(\|x\|^2 g(x,y) + \|y\|^2 g(y,x)).$$

The number $\angle_g(x,y) := \operatorname{arccos}_g(x,y)$ is the non-oriented $g-$angle between x and y.

Clearly, in a $q.i.p.$ space we have $\cos_g(x,y) = \cos_g(y,x)$.

If X is an $i.p.$ space with $i.p.$ $\langle\cdot,\cdot\rangle$ we have

$$\begin{aligned}(y-px) \perp_g px \ &\Leftrightarrow\ \|y-px\|^2 g(y-px,px) + \|px\|^2 g(px, y-px) = 0\\ &\Leftrightarrow\ (\|y-px\|^2 + \|px\|^2)\,\langle x, y-px\rangle = 0\\ &\Leftrightarrow\ p = \frac{\langle x,y\rangle}{\|x\|^2}\\ &\Rightarrow\ \|px\| = \frac{|\langle x,y\rangle|}{\|x\|}\\ &\Rightarrow\ \cos_g(\overrightarrow{x,y}) = \frac{\|px\|}{\|y\|^2}\operatorname{sgn}((\|x\|^2 + \|y\|^2)\,\langle x,y\rangle) = \frac{\langle x,y\rangle}{\|x\|\ \|y\|}.\end{aligned}$$

Thus, Definition 5 and Definition 6 are correct.

Theorem 7. *Let X be a $q.i.p.$ space and $\|x\| = \|y\| = \|x-y\|$, i.e., let triangle $(0,x,y)$ be equilateral. Then*

$$\angle_g(\overrightarrow{x,y}) = \angle_g(x,y) = \angle_g(y,x) = \frac{\pi}{3}.$$

Proof. At first, from equations $\|x\| = \|y\| = \|y-x\|$ and inequalities (9) we get inequalities $0 < g(x,y)$ and $0 < g(y,x)$. By this $\operatorname{sgn}(\|x\|^2 g(x,y) + \|y\|^2 g(y,x)) = 1$.

Let c be the circle centred at $\frac{x}{2}$ with diameter $\|x\|$, (see Figure 4). Then $\frac{y}{2}$, $\frac{x+y}{2} \in c$. Ac-

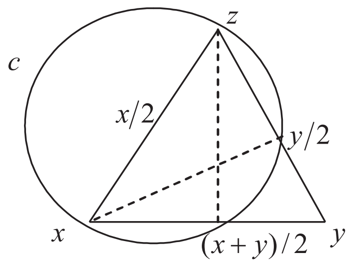

Figure 4:

cording to 1), Theorem 3, we have $(x - \frac{y}{2}) \perp_g \frac{y}{2}$ and $\frac{x+y}{2} \perp_g \frac{x-y}{2}$. That is, we have $P_x^g y = \frac{x}{2}$ and $P_y^g x = \frac{y}{2}$. By Definition 5 we get $\cos_g(\overrightarrow{x,y}) = \cos_g(y,x) = \frac{1}{2}$. Hence, by Definition 6, we have $\angle_g(x,y) = \frac{\pi}{3}$. □

Theorem 8. *Let $(0,x,y,x+y)$ be a $g-$quadrangle, i.e. let $\|x\| = \|y\| \wedge x \perp_g y$. Then $\angle_g(x, x+y) = \frac{\pi}{4}$, i.e., the non-oriented $g-$angle between a diagonal and a side is $\frac{\pi}{4}$.*

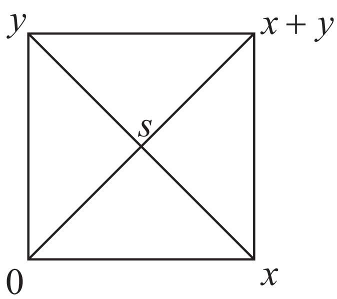

Figure 5:

Proof. We observe that in a $q.i.p.$ space

$$\operatorname{sgn}(\|x\|^2 g(x,y) + \|y\|^2 g(y,x)) = \operatorname{sgn}(\|x+y\| - \|x-y\|)$$

and that

$$\|2x+y\| - \|x\| \geq \|2x\| - \|y\| - \|y\| = 0.$$

Now consider Figure 5. Since $P_x^g(x+y) = x$, we have

$$\cos_g(\overrightarrow{x, x+y}) = \frac{\|x\|}{\|x+y\|} \operatorname{sgn}(\|2x+y\| - \|y\|) = \frac{\|x\|}{\|x+y\|}.$$

Let s be the crossing point of the diagonal $[0, x+y]$ and the diagonal $[x, y]$. Then, by Theorem 3, $P^g_{x+y}x = s$. It follows, by Definition 5, that

$$\begin{aligned}\cos_g(\overrightarrow{x+y, x}) &= \frac{\|s\|}{\|x\|}\operatorname{sgn}(\|s+x\| - \|s-x\|)\\ &= \frac{\|x+y\|}{2\,\|x\|}\operatorname{sgn}(\|2x\| - \|x\|)\\ &= \frac{\|x+y\|}{2\,\|x\|}.\end{aligned}$$

So, by Definition 6, we have

$$\begin{aligned}\cos_g(x, x+y) &= \sqrt{\cos_g(\overrightarrow{x, x+y})\ \cos_g(\overrightarrow{x+y, x})}\operatorname{sgn}(\|2x+y\| - \|y\|)\\ &= \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}.\end{aligned}$$

Hence $\angle_g(x, x+y) = \frac{\pi}{4}$. □

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