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SOME NEW DISCRETE INEQUALITIES AND THEIR APPLICATIONS

SH. SALEM AND K. R. RASLAN

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE,
AL-AZHAR UNIVERSITY, NASR CITY, CAIRO, EGYPT.

kamal_raslan@yahoo.com

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ABSTRACT. The aim of the present paper is to establish some new linear and nonlinear discrete inequalities in two independent variables. We give some examples in difference equations and we also give numerical test problems for our results.

Key words and phrases: Discrete inequalities, two independent variables, difference equations, nondecreasing.

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1. INTRODUCTION

The role played by linear and nonlinear discrete inequalities in one and more than one variable in the theory of difference equations and numerical analysis is well known. During the last few years there have been a number of papers written on the discrete inequalities of the Gronwall inequality and its nonlinear version to the Bhiari type, see [1, 2, 3, 4]. In this paper we present several new linear and nonlinear discrete inequalities in two independent variables. Finally, we give two examples to illustrate the importance of our results. Also, we give some numerical examples and compare our theoretical results with the numerical results.

2. LINEAR INEQUALITY IN TWO INDEPENDENT VARIABLES

Theorem 2.1. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$ be nonnegative functions and $a(m,n)$ nondecreasing for $m,n \in \mathbb{N}$. If*

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)u(s,t) \tag{2.1}$$

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for $m,n\in\mathbb{N}$, *then*

$$u(m,n)\le a(m,n)\prod_{t=0}^{n-1}\left[1+\sum_{s=0}^{m-1}b(s,t)\right]. \tag{2.2}$$

Proof. Define a function $z(m,n)$ by

$$z(m,n)=a(m,n)+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}b(s,t)u(s,t). \tag{2.3}$$

From (2.1) and (2.3), we have

$$u(m,n)\le z(m,n). \tag{2.4}$$

Since $a(m,n)$ is nonnegative for $m,n\in\mathbb{N}$, then from (2.3) and (2.4), we get

$$\frac{z(m,n)}{a(m,n)}\le 1+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}b(s,t)\frac{z(s,t)}{a(s,t)}. \tag{2.5}$$

Define a function $v(m,n)$ by

$$v(m,n)=1+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}b(s,t)\frac{z(s,t)}{a(s,t)}, \tag{2.6}$$

then, from (2.5) and (2.6), we get

$$z(m,n)\le a(m,n)v(m,n). \tag{2.7}$$

From (2.6), we obtain

$$\begin{aligned} v(m+1,n+1)&=1+b(m,n)\frac{z(m,n)}{a(m,n)}+\sum_{t=0}^{n-1}b(m,t)\frac{z(m,t)}{a(m,t)}\\ &\quad+\sum_{s=0}^{m-1}b(s,n)\frac{z(s,n)}{a(s,n)}+\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}b(s,t)\frac{z(s,t)}{a(s,t)}, \end{aligned} \tag{2.8}$$

then from (2.6) and (2.8), we get

$$v(m+1,n+1)-v(m,n)=b(m,n)\frac{z(m,n)}{a(m,n)}+\sum_{t=0}^{n-1}b(m,t)\frac{z(m,t)}{a(m,t)}+\sum_{s=0}^{m-1}b(s,n)\frac{z(s,n)}{a(s,n)}. \tag{2.9}$$

Also from (2.7), we have

$$v(m+1,n)-v(m,n)=\sum_{t=0}^{n-1}b(m,t)\frac{z(m,t)}{a(m,t)}, \tag{2.10}$$

and

$$v(m,n+1)-v(m,n)=\sum_{s=0}^{m-1}b(s,n)\frac{z(s,n)}{b(s,n)}. \tag{2.11}$$

From (2.9), (2.10) and (2.11), we get

$$[v(m+1,n+1)-v(m,n+1)]-[v(m+1,n)-v(m,n)]\le b(m,n)v(m,n). \tag{2.12}$$

Suppose n is fixed, then from (2.12), we get

$$v(m,n+1)\le\left[1+\sum_{s=0}^{m-1}b(s,n)\right]v(m,n),$$

from which we have

$$v(m,n) \le \prod_{t=0}^{n-1}\left(1+\sum_{s=0}^{m-1} b(s,n)\right). \tag{2.13}$$

The required inequality (2.2) follows from (2.4), (2.7) and (2.13). □

3. NONLINEAR INEQUALITIES IN TWO INDEPENDENT VARIABLES

Theorem 3.1. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$ be nonnegative functions and $a(m,n)$ nondecresing for $m,n \in \mathbb{N}$. If*

$$u^{m_1}(m,n) \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, u^{m_2}(s,t). \tag{3.1}$$

Then

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1}\left[1+\sum_{s=0}^{m-1} b(s,t)\right]^{\frac{1}{m_1}}; \quad m_1 = m_2, \tag{3.2}$$

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1}\left[1+\sum_{s=0}^{m-1} b(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right]^{\frac{m_2(n-t-1)}{m_1^2}}; \quad m_1 < m_2, \tag{3.3}$$

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1}\left[1+\sum_{s=0}^{m-1} b(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right]^{\frac{1}{m_1}}; \quad m_1 > m_2. \tag{3.4}$$

Proof. Define a function $z(m,n)$ by

$$z^{m_1}(m,n) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, u^{m_2}(s,t). \tag{3.5}$$

From (3.1), (3.5), we have

$$u(m,n) \le z(m,n). \tag{3.6}$$

Since $a(m,n)$ is nonnegative and nondecreasing for $m,n \in \mathbb{N}$; then we get

$$\frac{z^{m_1}(m,n)}{a(m,n)} \le 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, \frac{u^{m_2}(s,t)}{a(s,t)}. \tag{3.7}$$

Define function $v(m,n)$ by

$$v(m,n) = 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, \frac{z^{m_2}(s,t)}{a(s,t)}, \tag{3.8}$$

so, we obtain from (3.7) and (3.8) that

$$z^{m_1}(m,n) \le a(m,n)\, v(m,n). \tag{3.9}$$

As in Theorem 2.1, from (3.8), we get

$$\begin{aligned}[v(m+1,n+1) - v(m,n+1)] - [v(m+1,n) - v(m,n)]\\ \le b(m,n)\, a^{\frac{m_2-m_1}{m_1}}(m,n)\, v^{\frac{m_2}{m_1}}(m,n).\end{aligned} \tag{3.10}$$

Now, we consider the following cases:

Case 1. If $m_1 = m_2$, then from (3.10), we have

$$v(m+1,n+1) - v(m+1,n) - v(m,n+1) \leq (-1 + b(m,n))\, v(m,n), \tag{3.11}$$

keeping n fixed in (3.11), set $m = 0,1,2,\ldots,m-1$, then we get

$$v(m,n+1) \leq \left[1 + \sum_{s=0}^{m-1} b(s,n)\right] v(m,n). \tag{3.12}$$

From (3.12), we have

$$v(m,n) \leq \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t)\right]. \tag{3.13}$$

The required result (3.2) follows from (3.6),(3.9) and (3.13).

Case 2. If $m_2 > m_1$ then as in Case 1 from (3.10), we have

$$\begin{aligned} v(m+1,n+1) - v(m,n+1) - v(m+1,n) + v(m,n)] \\ \leq b(m,n)\, a^{\frac{m_2-m_1}{m_1}}(m,n)\, v^{\frac{m_2}{m_1}}(m,n), \end{aligned} \tag{3.14}$$

when n is fixed and $m = 0,1,2,\ldots,m-1$, we obtain from (3.14) that

$$v(m,n+1) \leq \left[1 + \sum_{s=0}^{m-1} b(s,n)\, a^{\frac{m_2-m_1}{m_1}}(s,n)\right] v^{\frac{m_2}{m_1}}(m,n)\,. \tag{3.15}$$

Lemma 3.2. *If*

$$v(m,n+1) \leq (1 + b(m,n))\, v^p(m,n);\ p > 1\,, \tag{3.16}$$

then

$$v(m,n) \leq \prod_{t=0}^{n-1} (1 + b(m,t))^{(n-t-1)p}. \tag{3.17}$$

Then from (3.15), (3.16), (3.17), we get

$$v(m,n) \leq \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right]^{\frac{m_2(n-t-1)}{m_1}}. \tag{3.18}$$

The required result (3.3) follows from (3.6), (3.9) and (3.18).

Case 3. If $m_2 < m_1$, then $v^{\frac{m_2}{m_1}}(m,n) \leq v(m,n)$, then, as in the last two cases, we get

$$v(m,n+1) \leq \left[1 + \sum_{s=0}^{m-1} b(s,n)\, a^{\frac{m_2-m_1}{m_1}}(s,n)\right] v(m,n)\,. \tag{3.19}$$

Then from (3.19), we obtain

$$v(m,n) \leq \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right]. \tag{3.20}$$

From (3.6), (3.9) and (3.20), we have

$$u(m,n) \leq a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right]^{\frac{1}{m_1}},$$

which is the required result (3.4).

□

Remark 3.3.

(1) If $m_1 = m_2 = 1$, then from (3.1) and (3.2), we get the same result as that of Theorem 2.1.

(2) If $m_1 = 1, m_2 > 1$, then from (3.1) and (3.2), we get if

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, u^{m_2}(s,t), \tag{3.21}$$

then

$$u(m,n) \le a(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t)\, a^{m_2-1}(s,t)\right]^{m_2(n-t-1)}. \tag{3.22}$$

(3) Let $m_2 = 1$, $m_1 > 1$, then from (3.1) and (3.4), we get if

$$u^{m_1}(m,n) \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, u(s,t)\,, \tag{3.23}$$

then

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t)\, a^{\frac{1-m_1}{m_1}}(s,t)\right]^{\frac{1}{m_1}}. \tag{3.24}$$

Theorem 3.4. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$ and $c(m,n)$ be nonnegative and $a(m,n)$ is non-decreasing for $m,n \in \mathbb{N}$, if $m_1, m_2 \in \mathbb{R}^+$, and*

$$u^{m_1}(m,n) \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\, u(s,t) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} c(s,t)\, u^{m_2}(s,t)\,, \tag{3.25}$$

then

$$u(m,n) \le a(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} (b(s,t) + c(s,t))\right], \quad m_1 = m_2 = 1, \tag{3.26}$$

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} (c(s,t) + b(s,t))\; a^{\frac{1-m_1}{m_1}}(s,t)\right]^{\frac{1}{m_1}}, \quad m_1 = m_2 > 1, \tag{3.27}$$

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} (c(s,t) + b(s,t))\, a^{\frac{1-m_1}{m_1}}(s,t)\right]^{\frac{n-t-1}{m_1^2}}, \quad 0 < m_1 = m_2 < 1, \tag{3.28}$$

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} \Big(b(s,t)\, a^{\frac{1-m_1}{m_1}}(s,t) + c(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\Big)\right]^{\frac{m_2(n-t-1)}{m_1^2}}, \quad m_2 > m_1, \tag{3.29}$$

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} \left(b(s,t)\, a^{\frac{1-m_1}{m_1}}(s,t) + c(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right)\right]^{\frac{1}{m_1}}, \quad 1 \le m_2 < m_1, \tag{3.30}$$

and

$$u(m,n) \le a^{\frac{1}{m_1}}(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} \left(b(s,t)\, a^{\frac{1-m_1}{m_1}}(s,t) + c(s,t)\, a^{\frac{m_2-m_1}{m_1}}(s,t)\right)\right]^{\frac{n-t-1}{m_1^2}}, \quad 0 < m_2 < m_1 < 1. \tag{3.31}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. Here we leave the details to the reader. □

Remark 3.5.

(1) If $c(m,n) = 0$, $m_1 = m_2$, then we get Theorem 2.1.
(2) If $b(m,n) = 0$, then we get Theorem 3.1.

4. SOME APPLICATIONS

There are many possible applications of the inequality established in this paper, but those presented here are sufficient to convey the importance of our results.

Example 4.1. Consider the difference equation

$$u(m,n) = a(m,n) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} k(s,t,u(s,t)). \tag{4.1}$$

Let

$$k(s,t,u(s,t)) \le t\, u(s,t), \tag{4.2}$$

from (4.1), (4.2), we get

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} t\, u(s,t). \tag{4.3}$$

From (2.1), (2.2) and (4.1) we get

$$u(m,n) \le a(m,n) \prod_{t=0}^{n-1} (1 + m\, t). \tag{4.4}$$

Remark 4.1.

(1) If

$$k(s,t,u(s,t)) \le 2\, s\, t\, u(s,t), \tag{4.5}$$

then, we get

$$u(m,n) \le a(m,n) \prod_{t=0}^{n-1} (1 + m(m-1)t). \tag{4.6}$$

(2) If

$$k(s,t,u(s,t)) \le u(s,t), \tag{4.7}$$

then, we get

$$u(m,n) \le a(m,n)\prod_{t=0}^{n-1}(1+m) = a(m,n)(1+m)^n. \tag{4.8}$$

Example 4.2. Consider the difference equation

$$u^{m_1}(m,n) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} k(s,t,u(s,t)). \tag{4.9}$$

let

$$k(s,t,u(s,t)) \le b(s,t)\,u(s,t), \tag{4.10}$$

if we consider $a(s,t) = b(s,t) = t$, from (3.23) and (3.24) we get

$$u(m,n) \le n^{\frac{1}{m_1}} \prod_{t=0}^{n-1}\left[1 + mt^{\frac{1}{m_1}}\right]^{\frac{1}{m_1}}. \tag{4.11}$$

Example 4.3. Consider the difference equation

$$u^{m_1}(m,n) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} k(s,t,u(s,t)). \tag{4.12}$$

Let

$$k(s,t,u(s,t)) \le b(s,t)\,u(s,t) + b(s,t)\,u^{m_2}(s,t), \tag{4.13}$$

if we take $m_1 = 3$, $m_2 = 2$, $a(s,t) = b(s,t) = c(s,t) = t^3$, then from (3.30) we have

$$u(m,n) \le n\prod_{t=0}^{n-1}[1 + mt(t+1)]^{\frac{1}{3}}, \tag{4.14}$$

As special cases of (4.14), let $m = 2$ and $n = 2$, then $u(2,2) \le 2\sqrt[3]{5}$, if we take $m = 2$ and $n = 3$, then $u(2,3) \le 3\sqrt[3]{45}$, also for $m = 3$ and $n = 2$ then $u(3,2) \le 2\sqrt[3]{7}$.

Example 4.4. Consider the difference inequality as in (2.1) with $a(s,t) = \alpha(st+5)$, $b(s,t) = \alpha(2t + s^2 + 1)$, $\alpha = 10^{-6}$, and we compute the values of $u(m,n)$ from (2.1) and also we find the values of $u(m,n)$ by using the result (2.2). In our computations we use (2.1) and (2.2) as equations and as we see in the Table 4.1 the computation values as in (2.1) are less than the values of the result (2.2).

Example 4.5. Consider the difference as in (3.1) with $a(s,t) = \alpha(t + s^2 + st)$, $b(s,t) = \beta(t + s^2 + 6)$, $\beta = 10^{-6}$, $\alpha = 10^{-5}$, and we compute the values of $u(m,n)$ from (3.1) and also we find the values of $u(m,n)$ by using the results (3.2) – (3.4) and tabled them in the following Table 4.2.

Example 4.6. Consider the difference as in (4.1) with $a(s,t) = \alpha(t^2 + s + st)$, $b(s,t) = \alpha(t^2 + s + 6)$, $c(s,t) = \alpha(s + t + 1)$, $\alpha = 10^{-6}$, and we compute the values of $u(m,n)$ from (3.25) and also we find the values of $u(m,n)$ by using the results (3.26) – (3.31) and tabled them in the following Table 4.3.

m,n	(2.1)	(2.2)	m,n	(2.1)	(2.2)
1,1	6.0e-6	6.0e-6	6,1	1.103019e-5	1.616208e-5
1,10	1.5e-5	1.513245e-5	6,5	3.503161e-5	5.506815e-5
2,1	7.000005e-5	7.082850e-5	6,10	6.504472e-5	1.124603e-4
2,5	1.500034e-5	1.537153e-5	7,1	1.204472e-5	2.119437e-5
2,10	2.500372e-5	2.623212e-6	7,5	4.004651e-5	7.685613e-5
3,1	8.003725e-6	8.445905e-6	7,10	7.506240e-5	1.616860e-4
3,5	2.000427e-5	2.166051e-5	8,5	4.506461e-5	1.099740e-4
3,10	3.500988e-5	3.945630e-5	8,10	8.508343e-5	2.380212e-4
4,1	9.009876e-6	1.024249e-6	9,1	1.408343e-5	4.035491e-5
4,5	2.501068e-5	2.958550e-5	9,5	5.008608e-5	1.621632e-4
4,10	4.501864e-5	5.640386e-5	9,10	9.510797e-5	3.608566e-4
5,1	1.001864e-5	1.269826e-5	10,1	1.510797e-5	5.891117e-5
5,5	3.001973e-5	4.018442e-5	10,5	5.511112e-5	2.474333e-4
5,10	5.503019e-5	7.947053e-5	10,10	1.051362e-4	5.659792e-4

Table 4.1:

Case	$m_1 = m_2 = 2$		$1 = m_1 < m_2 = 4$		$1 = m_1 > m_2 = 0.6$	
m,n	(3.1)	(3.2)	(3.1)	(3.3)	(3.1)	(3.4)
1,1	5.477225e-3	5.477390e-3	3.0e-5	3.0e-5	3.0e-5	3.0e-5
1,10	1.449138e-2	1.452727e-2	2.1e-4	2.1e-4	2.1e-4	2.1e-4
2,1	8.366600e-3	8.392021e-3	6.7e-5	6.7e-5	6.999999e-5	7.000654e-5
2,10	1.843944e-2	1.863311e-2	3.4e-4	3.4e-4	3.404703e-4	3.401309e-4
3,1	1.140233e-2	1.153497e-2	1.3e-4	1.3e-4	1.304703e-4	1.300566e-4
3,10	2.213686e-2	2.269679e-2	4.9e-4	4.9e-4	4.912776e-4	4.905238e-4
4,1	1.449278e-2	1.488674e-2	2.1e-4	2.1e-4	2.112776e-4	2.102450e-4
4,10	2.569232e-2	2.695817e-2	6.6e-4	6.6e-4	6.626532e-4	6.615494e-4
5,1	1.760952e-2	1.852931e-2	3.1e-4	3.1e-4	3.126533e-4	3.107818e-4
5,10	2.915815e-2	3.167530e-2	8.5e-4	8.5e-4	8.549279e-4	8.538832e-4
6,1	2.074121e-2	2.262540e-2	4.3e-4	4.3e-4	4.349280e-4	4.320908e-4
6,10	3.256347e-2	3.719871e-2	1.06e-3	1.06e-3	1.068537e-3	1.068736e-3
7,1	2.388262e-2	2.744415e-2	5.7e-4	5.7e-4	5.785374e-4	5.749682e-4
7,10	3.592609e-2	4.405006e-2	1.29e-3	1.29e-3	1.304028e-3	1.308159e-3
8,1	2.703117e-2	3.341747e-2	7.3e-4	7.3e-4	7.440277e-4	7.408159e-4
8,10	3.925774e-2	5.304486e-2	1.54e-3	1.54e-3	1.562061e-3	1.575471e-3
9,1	3.018559e-2	4.124418e-2	9,1e-4	9,1e-4	9.320608e-4	9.319749e-4
9,10	4.256656e-2	6.551374e-2	1.81e-3	1.81e-3	1.843420e-3	1.875835e-3
10,1	3.334534e-2	5.208647e-2	1.11e-3	1.11e-3	1.143420e-3	1.152192e-3
10,10	4.585852e-2	8.372936e-2	2.1e-3	2.1e-3	2.149017e-3	2.217082e-3

Table 4.2:

From Tables 4.1, 4.2 and 4.3, we can say that the numerical solution agrees with the analytical solution for some discrete linear and nonlinear inequalities. The programs for each case are written in the programming language Fortran.

Case	$m_1 = m_2 = 1$		$m_1 = m_2 = 2 > 1$		$0 < m_1 = m_2 = 0.5 < 1$	
m, n	(3.25)	(3.26)	(3.25)	(3.27)	(3.25)	(3.28)
1,1	3.000000e-5	3.000210e-5	5.477225e-3	5.477390e-3	9.0e-10	9.01946e-10
1,10	2.100000e-4	2.115060e-4	1.449138e-2	1.452747e-2	4.41e-8	4.649055e-8
2,10	3.400212e-4	3.508332e-4	1.843980e-2	1.863446e-2	1.156178e-7	1.562039e-7
3,10	4.900642e-4	5.285408e-4	2.213770e-2	2.268906e-2	2.401761e-7	5.146701e-7
4,10	6.601446e-4	7.631531e-4	2.569380e-2	2.688507e-2	4.358260e-7	1.866012e-6
5,10	8.502871e-4	1.087983e-3	2.916045e-2	3.137589e-2	7.230648e-7	7.930772e-6
6,10	1.060527e-3	1.564760e-3	3.256680e-2	3.631432e-2	1.124866e-6	4.124445e-5
7,10	1.210846e-3	2.312758e-3	3.593069e-2	4.187295e-2	1.666718e-6	2.713262e-4
8,10	1.541507e-3	3.575617e-3	3.926384e-2	4.826101e-2	2.376683e-6	2.321973e-3
9,10	1.812392e-3	5.886483e-3	4.257445e-2	5.574329e-2	3.285458e-6	2.650311e-2
10,10	2.103667e-3	1.051122e-2	4.586849e-2	6.466483e-2	4.426470e-6	4.128818e-1
Case	$2 = m_2 > m_1 = 1$		$1 \leq m_2 = 1.5 < m_1 = 2$		$0 < m_2 = 0.2 < m_1 = 0.8$	
m, n	(3.25)	(3.29)	(3.25)	(3.27)	(3.25)	(3.31)
1,1	3.000000e-5	3.003242e-5	5.477225e-3	5.477225e-3	2.220248e-6	2.220248e-6
1,10	2.100000e-4	2.156167e-4	1.449138e-2	1.449402e-2	2.527983e-5	2.530985e-5
2,10	3.400000e-4	3.952264e-4	1.844572e-2	1.845583e-2	4.996473e-5	4.658744e-5
3,10	4.900000e-4	7.221420e-4	2.215105e-2	2.218821e-2	8.272198e-5	7.483793e-5
4,10	6.600000e-4	1.427778e-3	2.571710e-2	2.581235e-2	1.247796e-4	1.120071e-4
5,10	8.500001e-4	3.288740e-3	2.919728e-2	2.939673e-2	1.776342e-4	1.617969e-4
6,10	1.060000e-3	9.571671e-3	3.262191e-2	3.299124e-2	2.431235e-4	2.313847e-4
7,10	1.290001e-3	3.864970e-2	3.601008e-2	3.664057e-2	3.234317e-4	3.350727e-4
8,10	1.450001e-3	2.411131e-1	3.937489e-2	4.039110e-2	4.211098e-4	5.026789e-4
9,10	1.810002e-3	2.625509	4.272596e-2	4.429572e-2	5.391026e-4	8.007679e-4
10,10	2.100004e-3	57.185780	4.607081e-2	4.831796e-2	6.807797e-4	1.392293e-3

Table 4.3:

5. CONCLUSIONS

This study presents the design and implementation of new discrete linear and nonlinear inequalities in one and two independent variables. We give new theoretical studies for those inequalities as in Section 3. We give test problems to demonstrate our results with different cases as we have shown in Section 4. We believe that the present studies can be useful for other applications and be extended to more complicated problems in higher dimensions.

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