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POINCARÉ INEQUALITIES FOR THE HEISENBERG GROUP TARGET

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ABSTRACT. In this paper some Poincaré type inequalities are obtained for the maps of the Heisenberg group target.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbf{H}^m$ ([1]) denote a Heisenberg group which is a Lie group that has algebra $\mathbf{g} = \mathbb{R}^{2m+1}$, with a non-abelian group law:

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(y_2 x_1 - x_2 y_1)), \tag{1.1}$$

for every $u_1 = (x_1, y_1, t_1), u_2 = (x_2, y_2, t_2) \in \mathbf{H}^m$. The Lie algebra is generated by the left invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \qquad i = 1, 2, \ldots, m, \tag{1.2}$$

and $T = \frac{\partial}{\partial t}$. For every $u_1 = (x_1, y_1, t_1)$, $u_2 = (x_2, y_2, t_2) \in \mathbf{H}^m$, the metric $d(u_1, u_2)$ in the Heisenberg group $\mathbf{H}^m$ is defined as ([2])

$$d(u_1, u_2) = \left| u_2 u_1^{-1} \right| = \left[((x_2 - x_1)^2 + (y_2 - y_1)^2)^2 + (t_2 - t_1 + 2(x_2 y_1 - x_1 y_2))^2 \right]^{\frac{1}{4}}. \tag{1.3}$$

We see that $\mathbf{H}^m$ possesses the nonlinear structure of group laws. It is one of the differences between $\mathbf{H}^m$ and general Riemann manifolds.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded and connected Lipschitz domain . Let $2 \leq p < \infty$.

L. Capogna and Fang-Hua Lin [3] have provided the characterizations for the Sobolev space $W^{1,p}(\Omega, \mathbf{H}^m)$, proved the existence theorem for the minimizer, and established that all critical

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points for the energy are Lipschitz continuous in the 2-dimensional case. However, the higher dimensional regularity problem is still open.

In this paper, we shall give some Poincaré type inequalities for the maps of the Heisenberg group target. The statements of these results are similar to the ones in the classical case. However, since the metric possesses the nonlinear structure of the group law, we require the use of a few techniques in the proofs for our Poincaré type inequalities.

Definition 1.1. Let $2 \le p < \infty$. A function $u = (z,t) : \Omega \to \mathbf{H}^m$ is in $L^p(\Omega, \mathbf{H}^m)$ if for some $h_0 \in \Omega$, one has

$$\int_\Omega (d(u(h), u(h_0))^p dh < \infty. \tag{1.4}$$

A function $u = (z,t) : \Omega \to \mathbf{H}^m$ is in the Sobolev space $W^{1,p}(\Omega, \mathbf{H}^m)$ if $u \in L^p(\Omega, \mathbf{H}^m)$ and

$$E_{p,\Omega}(u) = \sup_{f \in C_c(\Omega), 0 \le f \le 1} \limsup_{\epsilon \to 0} \int_\Omega f(h) e_{u,\epsilon}(h) dh < \infty, \tag{1.5}$$

where

$$e_{u,\epsilon}(h) = \int_{|h-q|=\epsilon} \left(\frac{d(u(h), u(q))}{\epsilon} \right)^p \frac{d\sigma_\epsilon(q)}{\epsilon^{n-1}}.$$

$E_{p,\Omega}(u)$ is called the p-energy of u on Ω.

Lemma 1.1. *If $u = (x,y,t) \in W^{1,p}(\Omega, \mathbf{H}^m)$, then*

$$\nabla t = 2(y \nabla x - x \nabla y) \quad \textit{in} \quad L^{\frac{p}{2}}(\Omega). \tag{1.6}$$

The maps satisfying (1.6) are called Legendrian maps.

Lemma 1.2. *If $u = (z,t) = (x,y,t) \in W^{1,p}(\Omega, \mathbf{H}^m)$, then*

$$E_{p,\Omega}(u) = \omega_{n-1} \int_\Omega |\nabla z|^p(q) dq.$$

Lemma 1.1 and Lemma 1.2 are due to L. Capogna and Fang-Hua Lin [3].

Lemma 1.3 ([4])**.** *(C_p-inequality) Let $p > 0$. Then for any $a_i \in \mathbb{R}$,*

$$\left(\sum_{i=1}^n |a_i| \right)^p \le C_p \sum_{i=1}^n |a_i|^p,$$

where $C_p = 1$ if $0 < p < 1$ and $C_p = n^{p-1}$ if $p \ge 1$.

Lemma 1.4 ([5])**.** *(Poincaré Inequality in the classical case) Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^m$. Let $p > 1$. Then there exists a constant C depending only on Ω, m and p, such that for every function $u \in W^{1,p}(\Omega, \mathbb{R})$, we have*

$$\int_\Omega |u(x) - \lambda_u|^p dx \le C \int_\Omega |\nabla u|^p dx,$$

where $\lambda_u = \frac{1}{|\Omega|} \int_\Omega u(x) dx$.

2. THE POINCARÉ TYPE INEQUALITIES FOR THE HEISENBERG GROUP TARGET

Theorem 2.1 (Poincaré type inequality)**.** *Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^n$. Then there exists a constant C depending only on Ω, n, m and p, such that for every function $u = (x, y, t) = (z, t) \in W^{1,p}(\Omega, \mathbf{H}^m)$,*

$$\int_\Omega (d(u(q), \lambda_u))^p dq \le C_\Omega E_{p,\Omega}(u) = C_\Omega \int_\Omega |\nabla z|^p(q) dq. \tag{2.1}$$

Here $\lambda_u = (\lambda_x, \lambda_y, \lambda_t)$ and $\lambda_f = \frac{1}{|\Omega|}\int_\Omega f(q)dq$.

Proof. Obviously, $\lambda_u \in W^{1,p}(\Omega, \mathbf{H}^m)$. From (1.3), using the C_p-inequality, we have

$$\begin{aligned}
&(d(u(q), \lambda_u))^p \\
&\quad = \left[|z(q) - \lambda_z|^4 + (t(q) - \lambda_t + 2(\lambda_x y(q) - \lambda_y x(q)))^2\right]^{\frac{p}{4}} \\
&\quad \le C_p \left[|x(q) - \lambda_x|^p + |y(q) - \lambda_y|^p + |t(q) - \lambda_t + 2(\lambda_x y(q) - \lambda_y x(q))|^{\frac{p}{2}}\right], \qquad (2.2)
\end{aligned}$$

where C_p depends on p. By the Poincaré inequality in the classical case, noting that

$$2(\lambda_x y(q) - \lambda_y x(q)) = 2\lambda_x(y(q) - \lambda_y) - 2\lambda_y(x(q) - \lambda_x),$$

we obtain

$$\begin{aligned}
&\int_\Omega (d(u(q), \lambda_u))^p dq \\
&\le C_p \left[\int_\Omega \left(|x(q) - \lambda_x|^p + |y(q) - \lambda_y|^p + |t(q) - \lambda_t + 2(\lambda_x y(q) - \lambda_y x(q))|^{\frac{p}{2}}\right) dq\right] \\
&\le C_1 \int_\Omega |\nabla x|^p(q) dq + C_2 \int_\Omega |\nabla y|^p(q) dq + C_3 \int_\Omega |\nabla t + 2(\lambda_x \nabla y - \lambda_y \nabla x)|^{\frac{p}{2}} dq.
\end{aligned}$$

By virtue of the Legendrian condition $\nabla t = 2(y\nabla x - x\nabla y)$, using the Hölder inequality, noting that $|\nabla x| \le |\nabla z|$ and $|\nabla y| \le |\nabla z|$, we have

$$\begin{aligned}
&\int_\Omega (d(u(q), \lambda_u))^p dq \\
&\quad \le C_1 \int_\Omega |\nabla x|^p dq + C_2 \int_\Omega |\nabla y|^p(q) \\
&\qquad\qquad + C_3 2^{\frac{p}{2}} \int_\Omega |\nabla y(x - \lambda_x) - \nabla x(y - \lambda_y)|^{\frac{p}{2}} dq \\
&\quad \le C_1 \int_\Omega |\nabla x|^p dq + C_2 \int_\Omega |\nabla y|^p dq \\
&\qquad\qquad + C_4 \left(\int_\Omega |\nabla x|^{\frac{p}{2}} |y - \lambda_y|^{\frac{p}{2}} dq + \int_\Omega |\nabla y|^{\frac{p}{2}} |x - \lambda_x|^{\frac{p}{2}} dq\right) \\
&\quad \le C_1 \int_\Omega |\nabla x|^p dq + C_2 \int_\Omega |\nabla y|^p dq + C_5 \left(\int_\Omega |\nabla x|^p dq \int_\Omega |\nabla y|^p dq\right)^{\frac{1}{2}} \\
&\quad \le C_1 \int_\Omega |\nabla x|^p dq + C_2 \int_\Omega |\nabla y|^p dq + C_6 \left(\int_\Omega |\nabla x|^p dq + \int_\Omega |\nabla y|^p dq\right) \\
&\quad \le C \int_\Omega |\nabla z|^p(q) dq,
\end{aligned}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ and C are dependent on Ω, n, m and p. □

Corollary 2.2. *If $u \in W^{1,p}(B(h_0,r),\mathbf{H}^m)$, then*

$$\int_{B(h_0,r)} (d(u(q),\lambda_u))^p dq \le Cr^p E_{p,B(h_0,r)}(u) = Cr^p \int_{B(h_0,r)} |\nabla z|^p(q)dq. \tag{2.3}$$

Proof. Observe that

$$\begin{aligned}(d(u(q),\lambda_u))^p &= \left[|z(q)-\lambda_z|^4 + (t(q)-\lambda_t+2(\lambda_x y(q)-\lambda_y x(q)))^2\right]^{\frac{p}{4}} \\ &\le C_p\left[|x(q)-\lambda_x|^p + |y(q)-\lambda_y|^p + |t(q)-\lambda_t+2(\lambda_x y(q)-\lambda_y x(q))|^{\frac{p}{2}}\right].\end{aligned}$$

Here C_p depends on p. By the Poincaré inequality in the classical case, noting that

$$2(\lambda_x y(q)-\lambda_y x(q)) = 2\lambda_x(y(q)-\lambda_y) - 2\lambda_y(x(q)-\lambda_x),$$

we deduce

$$\begin{aligned}&\int_{B_r(h_0)} (d(u(q),\lambda_u))^p dq \\ &\quad\le C_p\left[\int_{B_r(h_0)} (|x-\lambda_x|^p + |y-\lambda_y|^p + |t-\lambda_t+2(\lambda_x y-\lambda_y x)|^{\frac{p}{2}})dq\right] \\ &\quad\le C_1 r^p \int_{B_r(h_0)} |\nabla x|^p(q)dq + C_2 r^p \int_{B_r(h_0)} |\nabla y|^p(q)dq \\ &\qquad + C_3 r^{\frac{p}{2}} \int_{B_r(h_0)} |\nabla t + 2(\lambda_x \nabla y - \lambda_y \nabla x)|^{\frac{p}{2}} dq.\end{aligned}$$

By virtue of the Legendrian condition $\nabla t = 2(y\nabla x - x\nabla y)$, using Hölder's inequality, noting that $|\nabla x| \le |\nabla z|$ and $|\nabla y| \le |\nabla z|$, we can obtain

$$\begin{aligned}&\int_{B_r(h_0)} (d(u(q),\lambda_u))^p dq \\ &\quad\le C_1 r^p \int_{B_r(h_0)} |\nabla x|^p(q)dq + C_2 r^p \int_{B_r(h_0)} |\nabla y|^p(q)dq \\ &\qquad + C_3 r^{\frac{p}{2}} \int_{B_r(h_0)} |\nabla y(q)(x(q)-\lambda_x) - \nabla x(q)(y(q)-\lambda_y)|^{\frac{p}{2}} dq \\ &\quad\le C_1 r^p \int_{B_r(h_0)} |\nabla x|^p(q)dq + C_2 r^p \int_{B_r(h_0)} |\nabla y|^p(q)dq \\ &\qquad + C_4 r^p \left(\int_{B_r(h_0)} |\nabla x|^p dq \int_{B_r(h_0)} |\nabla y|^p dq\right)^{\frac{1}{2}} \\ &\quad\le Cr^p \int_\Omega |\nabla z|^p(q)dq,\end{aligned}$$

where C_1, C_2, C_3, C_4 and C depend on Ω, n, m and p. □

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