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A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS ASSOCIATED WITH CERTAIN FRACTIONAL CALCULUS OPERATORS

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ABSTRACT. In this paper, we introduce a new class of functions which are analytic and univalent with negative coefficients defined by using a certain fractional calculus and fractional calculus integral operators. Characterization property,the results on modified Hadamard product and integrals transforms are discussed. Further, distortion theorem and radii of starlikeness and convexity are also determined here.

Key words and phrases: Univalent, Convex, Starlike, Uniformly convex , Fractional calculus integral operator.

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1. INTRODUCTION AND PRELIMINARIES

Fractional calculus operators have recently found interesting applications in the theory of analytic functions. The classical definition of fractional calculus and its other generalizations have fruitfully been applied in obtaining, the characterization properties, coefficient estimates and distortion inequalities for various subclasses of analytic functions.

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Denote by A the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $E = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. Also denote by T [11] the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{1.2}$$

A function $f \in A$ is said to be in the class of uniformly convex functions of order α, denoted by $UCV(\alpha)$ [9] if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|, \tag{1.3}$$

and is said to be in a corresponding subclass of $UCV(\alpha)$ denote by $S_p(\alpha)$ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1.4}$$

where $-1 \leq \alpha \leq 1$ and $z \in E$.

The class of uniformly convex and uniformly starlike functions has been extensively studied by Goodman [3, 4] and Ma and Minda [6].

If f of the form (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ are two functions in A, then the Hadamard product (or convolution) of f and g is denoted by $f * g$ and is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.5}$$

Let $\phi(a, c; z)$ be the incomplete beta function defined by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_n}{(c)_n} z^n, \quad c \neq 0, -1, -2, \dots, \tag{1.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma functions, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1 & \text{n = 0} \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & n \in N\} \end{array} \right\}$$

Further suppose

$$L(a, c)f(z) = \phi(a, c; z) * f(z), \quad \text{for} \quad f \in A$$

where $L(a, c)$ is called Carlson - Shaffer operator [2].

For real number μ $(-\infty < \mu < 1)$ and γ $(-\infty < \gamma < 1)$ and a positive real number η, we define the operator

$$U_{0,z}^{\mu,\gamma,\eta} : \mathcal{A} \longrightarrow \mathbb{A}$$

by

$$U_{0,z}^{\mu,\gamma,\eta} = z + \sum_{n=2}^{\infty} \frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}} a_n z^n, \tag{1.7}$$

which for $f(z) \neq 0$ may be written as

$$(1.8) \qquad U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \dfrac{\Gamma(2-\gamma)\Gamma(2-\mu+\gamma)}{\Gamma(2-\gamma+\eta)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta} f(z); & 0 \leq \mu < 1 \\ \dfrac{\Gamma(2-\gamma)\Gamma(2-\mu+\gamma)}{\Gamma(2-\gamma+\eta)} z^{\gamma} I_{0,z}^{-\mu,\gamma,\eta} f(z); & -\infty \leq \mu < 0 \end{cases}$$

where $J_{0,z}^{\mu,\gamma,\eta}$ and $I_{0,z}^{-\mu,\gamma,\eta}$ are fractional differential and fractional integral operators [12] respectively.

It is interesting to observe that

$$(1.9) \qquad \begin{aligned} U_z^{\mu} f(z) &= \Gamma(2-\mu) z^{\mu} D_z^{\mu} f(z), \quad -\infty < \mu < 1 \\ &= \Omega_z^{\mu} f(z) \end{aligned}$$

and D_z^{μ} is due to Owa [7]. U_z^{μ} is called a fractional integral operator of order μ, if $-\infty < \mu < 0$ and is called fractional differential operator of order μ if $0 \leq \mu < 1$.

Further note that

$$\begin{aligned} U_{0,z}^{\mu,\gamma,\eta} f(z) &= f(z) \quad \text{if} \quad \mu = \gamma = 0 \\ U_{0,z}^{\mu,\gamma,\eta} f(z) &= z f'(z) \quad \text{if} \quad \mu = \gamma = 1. \end{aligned}$$

For $-1 \leq \alpha < 1$, a function $f \in A$ is said to be in $S^*_{\mu,\gamma,\eta}(\alpha)$, if and only if

$$(1.10) \qquad \operatorname{Re}\left\{ \frac{z(U_{0,z}^{\mu,\gamma,\eta} f(z))'}{U_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha \right\} \geq \left| \frac{z(U_{0,z}^{\mu,\gamma,\eta} f(z))'}{U_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right|, \quad z \in \Delta.$$

where $-\infty < \mu < 1$, $-\infty < \gamma < 1$, and $\eta \in \mathbb{R}_+$.

Now let us write $TR(\mu,\gamma,\eta,\alpha) = S^*_{\mu,\gamma,\eta}(\alpha) \cap T$.

It follows from the statement, that for $\mu = \gamma = 0$, we have

$$S^*_{\mu,\gamma,\eta}(\alpha) = S_p(\alpha)$$

and for $\mu = \gamma \longrightarrow 1$, we have

$$S^*_{\mu,\gamma,\eta}(\alpha) = UCV(\alpha).$$

The classes $S_p(\alpha)$ and $UCV(\alpha)$ are introduced and studied by various authors including [8], [9] and [1].

2. CHARACTERIZATION PROPERTY

We now investigate the characterization property for the function f to belong to the class $S^*_{\mu,\gamma,\eta}(\alpha)$, by obtaining the coefficient bounds.

Definition 2.1. A function f is in $TR(\mu,\gamma,\eta,\alpha)$ if f satisfies the analytic characterization

$$(2.1) \qquad \operatorname{Re}\left\{ \frac{z(U_{0,z}^{\mu,\gamma,\eta} f(z))'}{U_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha \right\} > \left| \frac{z(U_{0,z}^{\mu,\gamma,\eta} f(z))'}{U_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right|,$$

where $0 \leq \alpha < 1$, $-\infty < \mu < 1$, $-\infty < \gamma < 1$, and $\eta \in \mathbb{R}$.

Theorem 2.1 (Coefficient Bounds)**.** *A function f defined by (1.2) is in the class $TR(\mu,\gamma,\eta,\alpha)$ if and only if*

$$(2.2) \qquad \sum_{n=2}^{\infty} \frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}} \cdot \frac{2n-1-\alpha}{1-\alpha} |a_n| \leq 1$$

where $0 \leq \alpha < 1,\ -\infty < \mu < 1,\ -\infty < \gamma < 1,$ *and* $\eta \in \mathbb{R}$.

Proof. It suffices to show that

$$\left|\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right| \leq \operatorname{Re}\left\{\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-\alpha\right\},$$

and we have

$$\left|\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right| \leq \operatorname{Re}\left\{\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right\}+(1-\alpha).$$

That is

$$\begin{aligned}\left|\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right| - \operatorname{Re}\left\{\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right\} &\leq 2\left|\frac{z(U_{0,z}^{\mu,\gamma,\eta}f(z))'}{U_{0,z}^{\mu,\gamma,\eta}f(z)}-1\right| \\ &\leq \frac{\sum_{n=2}^{\infty}(n-1)\psi(n)|a_n|}{1-\sum_{n=2}^{\infty}\psi(n)|a_n|}\end{aligned}$$

where

$$\psi(n)=\frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}}.$$

The above expression is bounded by $(1-\alpha)$ and hence the assertion of the result.

Now we need to show that $f \in TR(\mu,\gamma,\eta,\alpha)$ satisfies the coefficient inequality. If $f \in TR(\mu,\gamma,\eta,\alpha)$ and z is real then (2.1) yields

$$\frac{1-\sum_{n=2}^{\infty}n\psi(n)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}\psi(n)a_nz^{n-1}}-\alpha \geq \frac{1-\sum_{n=2}^{\infty}(n-1)\psi(n)a_nz^{n-1}}{1-\sum_{n=2}^{\infty}\psi(n)a_nz^{n-1}}.$$

Letting $z \to 1$ along the real axis leads to the desired inequality

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\psi(n)a_n \leq 1-\alpha.$$

□

Corollary 2.2. *Let a function f defined by (1.2) belong to the class $TR(\mu,\gamma,\eta,\alpha)$. Then*

$$a_n \leq \frac{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}}{(2-\gamma+\eta)_{n-1}(2)_{n-1}}\cdot\frac{1-\alpha}{2n-1-\alpha}, \quad n \geq 2.$$

Next we consider the growth and distortion theorem for the class $TR(\mu,\gamma,\eta,\alpha)$. We shall omit the proof as the techniques are similar to various other papers.

Theorem 2.3. *Let the function f defined by (1.2) be in the class $TR(\mu,\gamma,\eta,\alpha)$. Then*

$$\begin{aligned}|z|-|z|^2\frac{(2-\gamma)(2-\mu+\eta)(1-\alpha)}{2(2-\gamma+\eta)(3-\alpha)} &\leq |U_{0,z}^{\mu,\gamma,\eta}f(z)| \\ &\leq |z|+|z|^2\frac{(2-\gamma)(2-\mu+\eta)(1-\alpha)}{2(2-\gamma+\eta)(3-\alpha)}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}1-|z|\frac{(2-\gamma)(2-\mu+\eta)(1-\alpha)}{(2-\gamma+\eta)(3-\alpha)} &\leq |(U_{0,z}^{\mu,\gamma,\eta}f(z))'| \\ &\leq 1+|z|\frac{(2-\gamma)(2-\mu+\eta)(1-\alpha)}{(2-\gamma+\eta)(3-\alpha)}.\end{aligned} \tag{2.4}$$

The bounds (2.3) and (2.4) are attained for functions given by

$$f(z) = z - \frac{(2-\gamma)(2-\mu+\eta)(1-\alpha)z^2}{2(2-\gamma+\eta)(3-\alpha)}. \tag{2.5}$$

Theorem 2.4. *Let a function f be defined by (1.2) and*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{2.6}$$

be in the class $TR(\mu,\gamma,\eta,\alpha)$. Then the function h defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} q_n z^n \tag{2.7}$$

where $q_n = (1-\lambda)a_n + \lambda b_n,\ \ 0 \le \lambda \le 1$ is also in the class $TR(\mu,\gamma,\eta,\alpha)$.

Proof. The result follows easily by using (2.2) and (2.7). □

We prove the following theorem by defining functions $f_j(z)$ $(j = 1, 2, \ldots, m)$ of the form

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad \text{for} \quad a_{n,j} \ge 0,\ z \in U. \tag{2.8}$$

Theorem 2.5 (Closure theorem)**.** *Let the functions $f_j(z)$ $(j = 1, 2 \ldots, m)$ defined by (2.8) be in the classes $TR(\mu,\gamma,\eta,\alpha_j)$ $(j = 1, 2, \ldots, m)$ respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} \right) a_{n,j} z^n$$

is in the class $TR(\mu,\gamma,\eta,\alpha)$ where

$$\alpha = \min_{1 \le j \le m} \{\alpha_j\} \quad \textit{with} \quad 0 \le \alpha_j < 1. \tag{2.9}$$

Proof. Since $f_j \in TR(\mu,\gamma,\eta,\alpha_j)$ $(j = 2, \ldots, m)$ by applying Theorem 2.1, we observe that

$$\begin{aligned} \sum_{n=2}^{\infty} \psi(n)(2n-1-\alpha) \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) &= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{n=2}^{\infty} \psi(n)(2n-1-\alpha) a_{n,j} \right) \\ &\le \frac{1}{m} \sum_{j=1}^{m} (1-\alpha_j) \le 1-\alpha, \end{aligned}$$

which in view of Theorem 2.1, again implies that $h \in TR(\mu,\gamma,\eta,\alpha)$ and the proof is complete. □

3. RESULTS INVOLVING MODIFIED HADAMARD PRODUCTS

We let

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

be the modified Hadamard product of functions f and g defined by (1.2) and (2.6) respectively. The following results are proved using the techniques of Schild and Silverman [10].

Theorem 3.1. *For functions $f_j(z)$ $(j = 1, 2)$ defined by (2.8), let $f_1(z) \in TR(\mu, \gamma, \eta, \alpha)$ and $f_2(z) \in TR(\mu, \gamma, \eta, \beta)$. Then $f_1 * f_2 \in TR(\mu, \gamma, \eta, \xi)$ where*

$$\xi = \xi(\mu, \gamma, \eta, \beta) = 1 - \frac{2(1-\alpha)(1-\beta)}{(3-\alpha)(3-\beta)\psi(2) - (1-\alpha)(1-\beta)}, \tag{3.1}$$

where $\psi(2) = \frac{(2-\gamma+\eta)(2)}{(2-\gamma)(2-\mu+\eta)}$. The result is the best possible for

$$f_1(z) = z - \frac{1-\alpha}{(3-\alpha)\psi(2)} z^2,$$
$$f_2(z) = z - \frac{1-\beta}{(3-\beta)\psi(2)} z^2,$$

where $\psi(2) = \frac{(2-\gamma+\eta)(2)}{(2-\gamma)(2-\mu+\eta)}$.

Proof. In the view of Theorem 2.1, it suffices to prove that

$$\sum_{n=2}^{\infty} \frac{2n-1-\xi}{1-\xi} \psi(n) a_{n,1} a_{n,2} \leq 1,$$

where ξ is defined by (3.1) under the hypothesis, it follows from (2.1) and the Cauchy-Schwarz inequality that

$$\sum_{n=2}^{\infty} \frac{[2n-1-\alpha]^{1/2}[2n-1-\beta]^{1/2}}{\sqrt{(1-\alpha)(1-\beta)}} \psi(n) \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{3.2}$$

Thus we need to find largest ξ such that

$$\sum_{n=2}^{\infty} \frac{2n-1-\xi}{1-\xi} \psi(n) a_{n,1} a_{n,2} \leq \sum_{n=2}^{\infty} \frac{[2n-1-\alpha]^{1/2}[2n-1-\beta]^{1/2}}{\sqrt{(1-\alpha)(1-\beta)}} \psi(n) \sqrt{a_{n,1} a_{n,2}} \leq 1$$

or, equivalently that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{[2n-1-\alpha]^{1/2}[2n-1-\beta]^{1/2}}{\sqrt{(1-\alpha)(1-\beta)}} \cdot \frac{1-\xi}{2n-1-\xi} \quad \text{for} \quad n \geq 2.$$

By virtue of (3.2) it is sufficient to find the largest ψ such that

$$\frac{\sqrt{(1-\alpha)(1-\beta)}}{[2n-1-\alpha]^{1/2}[2n-1-\beta]^{1/2}\psi(n)} \leq \frac{[2n-1-\alpha]^{1/2}[2n-1-\beta]^{1/2}}{\sqrt{(1-\alpha)(1-\beta)}} \cdot \frac{1-\xi}{2n-1-\xi} \quad \text{for} \quad n \geq 2$$

which yields

$$\xi \leq 1 - \frac{2(n-1)(1-\alpha)(1-\beta)}{(2n-1-\alpha)(2n-1-\beta)\psi(n) - (1-\alpha)(1-\beta)},$$

where

$$\psi(n) = \frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}} \quad \text{for} \quad n \geq 2. \tag{3.3}$$

Since $\psi(n)$ is a decreasing function of n $(n \geq 2)$, we have

$$\xi = \xi(\mu, \gamma, \eta, \alpha, \beta) = 1 - \frac{2(1-\alpha)(1-\beta)}{(3-\alpha)(3-\beta)\psi(2) - (1-\alpha)(1-\beta)},$$

where $\psi(2) = \frac{(2-\gamma+\eta)(2)}{(2-\gamma)(2-\mu+\eta)}$. Thus completes the proof. □

Theorem 3.2. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by (2.8) be in the class* $TR(\mu, \gamma, \eta, \alpha)$. *Then* $(f_1 * f_2)(z) \in TR(\mu, \gamma, \eta, \delta)$, *where*

$$\delta = 1 - \frac{2(1-\alpha)^2}{(3-\alpha)^2\psi(2) - (1-\alpha)^2},$$

with $\psi(2) = \frac{(2-\gamma+\eta)(2)}{(2-\gamma)(2-\mu+\eta)}$.

Proof. By taking $\beta = \alpha$ in the above theorem, the results follows. □

Theorem 3.3. *Let the function* f *defined by (1.2) be in the class* $TR(\mu, \gamma, \eta, \alpha)$. *Also let*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad \textit{for} \quad |b_n| \leq 1.$$

Then $(f * g)(z) \in TR(\mu, \gamma, \eta, \alpha)$.

Proof. Since

$$\begin{aligned}\sum_{n=2}^{\infty} \psi(n)(2n-1-\alpha)|a_n b_n| &= \sum_{n=2}^{\infty} \psi(n)(2n-1-\alpha)a_n|b_n| \\ &\leq \sum_{n=2}^{\infty} \psi(n)(2n-1-\alpha)a_n \\ &\leq 1-\alpha \quad \text{(by Theorem 2.1)},\end{aligned}$$

where $\psi(n)$ is defined by (3.3). Hence it follows that $(f * g)(z) \in TR(\mu, \gamma, \eta, \alpha)$. □

Corollary 3.4. *Let the function* f *defined by (1.2) be in the class* $TR(\mu, \gamma, \eta, \alpha)$. *Also let* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ *for* $0 \leq b_n \leq 1$. *Then* $(f * g)(z) \in TR(\mu, \gamma, \eta, \alpha)$.

For functions in the class $TR(\mu, \gamma, \eta, \alpha)$ we can prove the following inclusion property also.

Theorem 3.5. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by (2.5) be in the class* $TR(\mu, \gamma, \eta, \alpha)$. *Then the function* h *defined by*

$$h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n$$

is in the class $TR(\mu, \gamma, \eta, \Delta)$ *where*

$$\Delta = 1 - \frac{4(1-\alpha)^2}{(3-\alpha)^2\psi(2) - 2(1-\alpha)^2} \quad \textit{with}$$

$$\psi(2) = \frac{2(2-\gamma+\eta)}{(2-\gamma)(2-\mu+\eta)}.$$

Proof. In view of Theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} \psi(n) \frac{2n-1-\Delta}{1-\Delta} (a_{n,1}^2 + a_{n,2}^2) \leq 1 \tag{3.4}$$

where $f_j(z) \in TR(\mu, \gamma, \eta, \alpha)$ $(j = 1, 2)$. We find from (2.8) and Theorem 2.1, that

$$\sum_{n=2}^{\infty} \left[\psi(n) \frac{2n-1-\alpha}{1-\alpha}\right]^2 a_{n,j}^2 \leq \sum_{n=2}^{\infty} \left[\psi(n) \frac{2n-1-\alpha}{1-\alpha} a_{n,j}\right]^2 \leq 1, \tag{3.5}$$

which would yield

$$\sum_{n=2}^{\infty} \frac{1}{2}\left[\psi(n)\frac{2n-1-\alpha}{1-\alpha}\right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{3.6}$$

On comparing (3.5) and (3.6) it can be seen that inequality (3.4) will be satisfied if

$$\psi(n)\frac{2n-1-\Delta}{1-\Delta}(a_{n,1}^2 + a_{n,2}^2) \leq \frac{1}{2}\left[\psi(n)\frac{2n-1-\alpha}{1-\alpha}\right]^2 (a_{n,1}^2 + a_{n,2}^2).$$

That is, if

$$\Delta \leq 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2\psi(n) - 2(1-\alpha)^2}, \tag{3.7}$$

where $\psi(n)$ is given by (3.3). Hence we conclude from (3.7)

$$\Delta = \Delta(\mu,\gamma,\eta,\alpha) = 1 - \frac{4(1-\alpha)^2}{(3-\alpha)^2}\psi(2) - 2(1-\alpha)^2,$$

where $\psi(2) = \frac{2(2-\gamma+\eta)}{(2-\gamma)(2-\mu+\eta)}$ which completes the proof. □

4. INTEGRAL TRANSFORM OF THE CLASS $TR(\mu,\gamma,\eta,\alpha)$

For $f \in TR(\mu,\gamma,\eta,\alpha)$ we define the integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt,$$

where λ is a real valued, non-negative weight function normalized so that $\int_0^1 \lambda(t)dt = 1$. Since special cases of $\lambda(t)$ are particularly interesting such as $\lambda(t) = (1+c)t^c$, $c > -1$, for which V_λ is known as the Bernardi operator, and

$$\lambda(t) = \frac{(c+1)^\delta}{\lambda(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1}, \quad c > -1,\ \delta \geq 0$$

which gives the Komatu operator. For more details see [5].

First we show that the class $TR(\mu,\gamma,\eta,\alpha)$ is closed under $V_\lambda(f)$.

Theorem 4.1. *Let $f \in TR(\mu,\gamma,\eta,\alpha)$. Then $V_\lambda(f) \in TR(\mu,\gamma,\eta,\alpha)$.*

Proof. By definition, we have

$$\begin{aligned} V_\lambda(f) &= \frac{(c+1)^\delta}{\lambda(\delta)}\int_0^1 (-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1}\right)dt \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\lambda(\delta)}\lim_{r\to 0^+}\left[\int_r^1 t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1}\right)dt\right], \end{aligned}$$

and a simple calculation gives

$$V_\lambda(f)(z) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty}\frac{2n-1-\alpha}{1-\alpha}\cdot\frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}}\left(\frac{c+1}{c+n}\right)^\delta a_n < 1. \tag{4.1}$$

On the other hand by Theorem 2.1, $f \in TR(\mu, \gamma, \eta, \alpha)$ if and only if

$$\sum_{n=2}^{\infty} \frac{2n-1-\alpha}{1-\alpha} \cdot \frac{(2-\gamma+\eta)_{n-1}(2)_{n-1}}{(2-\gamma)_{n-1}(2-\mu+\eta)_{n-1}} < 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (4.1) holds and the proof is complete. □

Next we provide a starlike condition for functions in $TR(\mu, \gamma, \eta, \alpha)$ and $V_\lambda(f)$.

Theorem 4.2. *Let $f \in TR(\mu, \gamma, \eta, \alpha)$. Then $V_\lambda(f)$ is starlike of order $0 \le \gamma < 1$ in $|z| < R_1$ where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^{\delta} \cdot \frac{1-\gamma(2n-1-\alpha)}{(n-\gamma)(1-\alpha)} \phi(n) \right]^{\frac{1}{n-1}}.$$

Proof. It is sufficient to prove

$$\left| \frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)} - 1 \right| < 1-\gamma. \tag{4.2}$$

For the left hand side of (4.2) we have

$$\begin{aligned} \left| \frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty}(1-n)(\frac{c+1}{c+n})^{\delta} a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta} a_n z^{n-1}} \right| \\ &\le \frac{\sum_{n=2}^{\infty}(n-1)(\frac{c+1}{c+n})^{\delta} a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty}(\frac{c+1}{c+n})^{\delta} a_n |z|^{n-1}}. \end{aligned}$$

This last expression is less than $(1-\gamma)$ since

$$|z|^{n-1} < \left(\frac{c+1}{c+n} \right)^{\delta} \frac{(1-\gamma)[2n-1-\alpha]}{(n-\gamma)(1-\alpha)} \phi(n).$$

Therefore the proof is complete. □

Using the fact that f is convex if and only if zf' is starlike, we obtain the following:

Theorem 4.3. *Let $f \in TR(\mu, \gamma, \eta, \alpha)$. Then $V_\lambda(f)$ is convex of order $0 \le \gamma < 1$ in $|z| < R_2$ where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^{\delta} \frac{(1-\gamma)[2n-1-\alpha]}{n(n-\gamma)(1-\alpha)} \phi(n) \right]^{\frac{1}{n-1}}.$$

We omit the proof as it is easily derived.

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