Journal of Inequalities in Pure and Applied Mathematics

ON GRÜSS TYPE INEQUALITIES OF DRAGOMIR AND FEDOTOV

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ISSN (electronic): 1443-5756
065-03

volume 4, issue 5, article 91,
2003.

Received 20 May, 2003;
accepted 18 September, 2003.

Communicated by: S.S. Dragomir

Abstract

Weighted versions of Grüss type inequalities of Dragomir and Fedotov are given. Some related results are also obtained.

2000 Mathematics Subject Classification: 26D15.
Key words: Grüss type inequalities.

Contents

1. Introduction

In 1935, G. Grüss proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leqslant \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ satisfying the condition

$$\varphi \leqslant f(x) \leqslant \Phi \text{ and } \gamma \leqslant g(x) \leqslant \Gamma \text{ for all } x \in [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right).$$

The following result of Grüss type was proved by S.S. Dragomir and I. Fedotov [1]:

Theorem 1.1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is $L-$Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)| \leqslant L|x-y| \quad \textit{for all} \quad x \in [a,b], \tag{1.3}$$

f is Riemann integrable on $[a,b]$ and there exist the real numbers m,M so that

$$m \leqslant f(x) \leqslant M \quad \textit{for all} \quad x \in [a,b]. \tag{1.4}$$

Then we have the inequality

$$\left|\int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt\right| \leqslant \frac{1}{2}L(M-m)(b-a), \tag{1.5}$$

and the constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

The following result of Grüss' type was proved by S.S. Dragomir and I. Fedotov [2]:

Theorem 1.2. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is $L-$lipschitzian on $[a,b]$, and f is a function of bounded variation on $[a,b]$. Denote by $\bigvee_a^b f$ the total variation of f on $[a,b]$. Then the following inequality holds:*

$$\left|\int_a^b u(x)\,df(x) - \frac{f(b)-f(a)}{b-a}\cdot\int_a^b u(x)\,dx\right| \leqslant \frac{1}{2}L(b-a)\bigvee_a^b f. \tag{1.6}$$

The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller one.

Remark 1.1. *For other related results see [3].*

Let us also state that the weighted version of (1.1) is well known, that is we have with condition (1.2) the following generalization of (1.1):

$$|D(f,g;w)| \leqslant \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.7}$$

where

$$D(f,g;w) = A(f,g;w) - A(f;w)\,A(g;w),$$

and

$$A(f;w) = \frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x) dx}.$$

So, in this paper we shall show that corresponding weighted versions of (1.5) and (1.6) are also valid. Some related results will be also given.

2. Results

Theorem 2.1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that f is Riemann integrable on $[a,b]$ and u is $L-$Lipschitzian on $[a,b]$, i.e. (1.3) holds true. If $w:[a,b]\to\mathbb{R}$ is a positive weight function, then*

$$|T(f,u;w)|\leqslant L\int_a^b w(x)\,|f(x)-A(f;w)|\,dx, \tag{2.1}$$

where

$$T(f,u;w)=\int_a^b w(x)\,f(x)\,du(x) - \frac{1}{\int_a^b w(x)\,dx}\int_a^b w(x)\,du(x)\int_a^b w(x)\,f(x)\,dx. \tag{2.2}$$

Moreover, if there exist the real numbers m,M such that (1.4) is valid, then

$$|T(f,u;w)|\leqslant \frac{L}{2}(M-m)\int_a^b w(x)\,dx. \tag{2.3}$$

Proof. As in [1], we have

$$|T(f,u;w)|=\left|\int_a^b w(x)\,[f(x)-A(f;w)]\,du(x)\right| \leqslant L\int_a^b w(x)\,|f(x)-A(f;w)|\,dx.$$

That is, (2.1) is valid. Furthermore, from an application of Cauchy's inequality we have:

$$|T(f,u;w)| \leqslant L\left(\int_a^b w(x)\,dx \int_a^b w(x)\,(f(x)-A(f;w))^2\,dx\right)^{\frac{1}{2}}, \tag{2.4}$$

from where we obtain

$$|T(f,u;w)| \leqslant L\cdot(D(f,f;w))^{\frac{1}{2}}\cdot\int_a^b w(x)\,dx. \tag{2.5}$$

From (1.7) for $g \equiv f$ we get:

$$(D(f,f;w))^{\frac{1}{2}} \leqslant \frac{1}{2}(\Phi-\varphi). \tag{2.6}$$

Now, (2.4) and (2.5) give (2.3). □

Now, we shall prove the following result.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be $M-$Lipschitzian on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ be $L-$Lipschitzian on $[a,b]$. If $w : [a,b] \to \mathbb{R}$ is a positive weight function, then*

$$|T(f,u;w)| \leqslant L\cdot M\cdot\frac{\int_a^b\int_a^b w(x)\,w(x)\,|x-y|\,dxdy}{\int_a^b w(y)\,dy}. \tag{2.7}$$

Proof. It follows from (2.1)

$$
\begin{aligned}
|T(f,u;w)| &\leqslant L\cdot\int_a^b w(x)\left|\frac{\int_a^b w(y)\,(f(x)-f(y))\,dy}{\int_a^b w(y)\,dy}\right|dx\\
&\leqslant L\cdot\int_a^b w(x)\,\frac{\int_a^b w(y)\,|f(x)-f(y)|\,dy}{\int_a^b w(y)\,dy}dx\\
&\leqslant L\cdot M\cdot\frac{\int_a^b\int_a^b w(x)\,w(x)\,|x-y|\,dxdy}{\int_a^b w(y)\,dy}.
\end{aligned}
$$

□

If in the previous result we set $w(x)\equiv 1$, then we can obtain the following corollary:

Corollary 2.3. *Let f and u be as in Theorem 2.2, then,*

$$
\left|\int_a^b f(x)\,du(x)-\frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt\right|\leqslant\frac{L\cdot M\cdot(b-a)^2}{3}.
$$

Proof. The proof follows by the fact that

$$
\begin{aligned}
\int_a^b\int_a^b|x-y|\,dxdy &= \int_a^b\left(\int_a^b|x-y|\,dx\right)dy\\
&= \int_a^b\left(\int_a^y(y-x)\,dx+\int_y^b(x-y)\,dx\right)dy\\
&= \frac{1}{2}\int_a^b\left((y-a)^2+(b-y)^2\right)dy=\frac{1}{3}(b-a)^3.
\end{aligned}
$$

□

Theorem 2.4. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is $L-$Lipschitzian on $[a, b]$, and f is a function of bounded variation on $[a, b]$. If $w : [a, b] \to \mathbb{R}$ is a positive weight function, then the following inequality holds:*

$$|T(u, f; w)| \leqslant ML \bigvee_a^b g \leqslant WML \bigvee_a^b f,$$

where $T(u, f; w)$ is defined by (2.2), $g : [a, b] \to \mathbb{R}$ is the function $g(x) = \int_a^x w(t)\, df(t)$,

$$W = \sup_{x \in [a,b]} w(x), \qquad M = \max\left\{ \frac{\int_a^b w(t)(b-t)\, dt}{\int_a^b w(t)\, dt}, \frac{\int_a^b w(t)(t-a)\, dt}{\int_a^b w(t)\, dt} \right\},$$

and $\bigvee_a^b g$ and $\bigvee_a^b f$ denote the total variation of g and f on $[a, b]$, respectively.

Proof. We have

$$\begin{aligned}
&T(u, f; w) \\
&= \int_a^b w(x) u(x)\, df(x) - \frac{1}{\int_a^b w(x)\, dx} \int_a^b w(x)\, df(x) \int_a^b w(x) u(x)\, dx \\
&= \int_a^b w(x) \left(u(x) - \frac{\int_a^b w(t) u(t)\, dt}{\int_a^b w(t)\, dt} \right) df(x) \\
&= \int_a^b \left(\frac{\int_a^b w(t)(u(x) - u(t))\, dt}{\int_a^b w(t)\, dt} \right) w(x)\, df(x).
\end{aligned}$$

Using the fact that u is $L-$Lipschitzian on $[a,b]$, we can state that:

$$\begin{aligned}
|T(u,f;w)| &= \left| \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt} \right) w(x)\,df(x) \right| \\
&= \left| \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt} \right) d\left(\int_a^x w(t)\,df(t) \right) \right| \\
&\leqslant L \sup_{x\in[a,b]} \left(\frac{\int_a^b w(t)|x-t|\,dt}{\int_a^b w(t)\,dt} \right) \bigvee_a^b \left(\int_a^x w(t)\,df(t) \right) \\
&= ML \bigvee_a^b g.
\end{aligned}$$

The constant M has the value

$$M = \sup_{x\in[a,b]} \left(\frac{\int_a^b w(t)|x-t|\,dt}{\int_a^b w(t)\,dt} \right).$$

If we denote a new function $y(x)$ as:

$$\begin{aligned}
y(x) &= \int_a^b w(t)|x-t|\,dt \\
&= \int_a^x w(t)(x-t)\,dt + \int_x^b w(t)(t-x)\,dt,
\end{aligned}$$

then the first derivative of this function is:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\int_a^x w(t)\,tdt - \int_a^x tw(t)\,dt + \int_x^b w(t)\,tdt - x\int_x^b w(t)\,dt\right)$$

$$= \int_a^x w(t)\,dt + w(x)\,x - w(x)\,x - w(x)\,x - \int_x^b w(t)\,dt + w(x)\,x$$

$$= \int_a^x w(t)\,dt - \int_x^b w(t)\,dt;$$

and the second derivative is:

$$\frac{d^2y}{dx^2} = w(x) + w(x) = 2w(x) > 0.$$

Obviously f is a convex function, so we have:

$$M = \sup_{x\in[a,b]}\left(\frac{\int_a^b w(t)\,|x-t|\,dt}{\int_a^b w(t)\,dt}\right)$$

$$= \sup_{x\in[a,b]}\left(\frac{y(x)}{\int_a^b w(t)\,dt}\right)$$

$$= \max\left\{\frac{\int_a^b w(t)\,(b-t)\,dt}{\int_a^b w(t)\,dt}, \frac{\int_a^b w(t)\,(t-a)\,dt}{\int_a^b w(t)\,dt}\right\}.$$

That is:

$$
\begin{aligned}
|T(u,f;w)| &= \left| \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt} \right) w(x)\,df(x) \right| \\
&= \left| \int_a^b \left(\frac{\int_a^b w(t)(u(x)-u(t))\,dt}{\int_a^b w(t)\,dt} \right) w(x)\,df(x) \right| \\
&\leqslant \int_a^b \frac{\int_a^b w(t)\,|u(x)-u(t)|\,dt}{\int_a^b w(t)\,dt} w(x)\,|df(x)| \\
&\leqslant \sup_{x\in[a,b]} w(x)\ L \sup_{x\in[a,b]} \left(\frac{\int_a^b w(t)\,|x-t|\,dt}{\int_a^b w(t)\,dt} \right) \bigvee_a^b f \\
&= WML \bigvee_a^b f.
\end{aligned}
$$

□

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