

Journal of Inequalities in Pure and Applied Mathematics

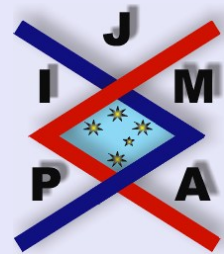
INEQUALITIES ARISING OUT OF THE VALUE DISTRIBUTION OF A DIFFERENTIAL MONOMIAL

INDRAJIT LAHIRI AND SHYAMALI DEWAN

Department of Mathematics,
University of Kalyani,
West Bengal 741235, INDIA.

EMail: indrajit@cal2.vsnl.net.in

©2000 Victoria University
ISSN (electronic): 1443-5756
066-02



volume 4, issue 2, article 27,
2003.

*Received 4 June, 2002;
accepted 23 January, 2003.*

Communicated by: A. Sofo

Abstract

Contents

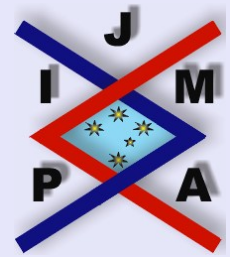


Home Page

Go Back

Close

Quit



Abstract

In the paper we derive two inequalities that describe the value distribution of a differential monomial generated by a transcendental meromorphic function and which improve some earlier results.

2000 Mathematics Subject Classification: 30D35, 30A10.

Key words: Meromorphic function, Monomial, Value distribution, Inequality.

The second author is thankful to C.S.I.R. for awarding her a junior research fellowship.

The authors are thankful to the referee for valuable comments towards the improvement of the paper.

Contents

1	Introduction and Definitions	3
2	Lemmas	5
3	Theorems	8
References		

Inequalities Arising out of the Value Distribution of a Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 2 of 13

1. Introduction and Definitions

Let f be a transcendental meromorphic function defined in the open complex plane \mathbb{C} . We do not explain the standard definitions and notations of the value distribution theory as these are available in [3].

Definition 1.1. A meromorphic function $\alpha \equiv \alpha(z)$ defined in \mathbb{C} is called a small function of f if $T(r, \alpha) = S(r, f)$.

Hiong [5] proved the following inequality.

Theorem A. If a, b, c are three finite complex numbers such that $b \neq 0$, $c \neq 0$ and $b \neq c$ then

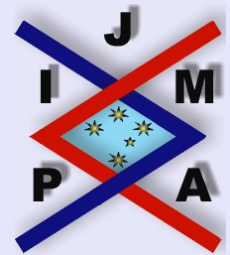
$$T(r, f) \leq N(r, a; f) + N(r, b; f^{(k)}) + N(r, c; f^{(k)}) - N(r, 0; f^{(k+1)}) + S(r, f).$$

Improving Theorem A, K.W.Yu [7] proved the following result.

Theorem B. Let $\alpha (\neq 0, \infty)$ be a small function of f , then for any finite non-zero distinct complex numbers b and c and any positive integer k for which $\alpha f^{(k)}$ is non-constant, we obtain

$$T(r, f) \leq N(r, 0; f) + N(r, b; \alpha f^{(k)}) + N(r, c; \alpha f^{(k)}) - N(r, \infty; f) - N(r, 0; (\alpha f^{(k)})') + S(r, f).$$

Recently K.W.Yu [8] has further improved Theorem B and has proved the following result.



Inequalities Arising out of the
Value Distribution of a
Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 3 of 13

Theorem C. Let $\alpha (\neq 0, \infty)$ be a small function of f . Suppose that b and c are any two finite non-zero distinct complex numbers and $k (\geq 1)$, $n (\geq 0)$ are integers. If $n = 0$ or $n \geq 2 + k$ then

$$(1.1) \quad (1+n)T(r, f) \leq (1+n)N(r, 0; f) + N(r, b; \alpha(f)^n f^{(k)}) + N(r, c; \alpha(f)^n f^{(k)}) - N(r, \infty; f) - N\left(r, 0; (\alpha(f)^n f^{(k)})'\right) + S(r, f).$$

If, in particular, f is entire, then (1.1) is true for all non-negative integers $n (\neq 1)$.

Yu [8] also remarked that inequality (1.1) might be valid even for $n = 1$ if f is entire.

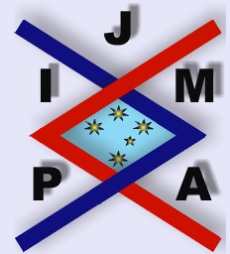
In this paper we first show that inequality (1.1) is valid for all integers $n (\geq 0)$ and $k (\geq 1)$ even if f is meromorphic.

Next we prove that the following inequality of Q.D. Zhang [9] can be extended to a differential monomial of the form $\alpha(f)^n (f^{(k)})^p$, where $\alpha (\neq 0, \infty)$ is a small function of f and $n (\geq 0)$, $p (\geq 1)$, $k (\geq 1)$ are integers.

Theorem D. [9] Let $\alpha (\neq 0, \infty)$ be a small function of f , then

$$2T(r, f) \leq \overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + \overline{N}(r, 1; \alpha f f') + S(r, f).$$

Definition 1.2. For a positive integer k we denote by $N_k(r, 0; f)$ the counting function of zeros of f , where a zero with multiplicity q is counted q times if $q \leq k$ and is counted k times if $q > k$.



Inequalities Arising out of the Value Distribution of a Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 4 of 13

2. Lemmas

In this section we discuss some lemmas which will be needed in the sequel.

Lemma 2.1. [4] Let $A > 1$, then there exists a set $M(A)$ of upper logarithmic density at most $\delta(A) = \min\{(2e^{A-1} - 1)^{-1}, 1 + e(A - 1) \exp(e(1 - A))\}$ such that for $k = 1, 2, 3, \dots$

$$\limsup_{r \rightarrow \infty, r \notin M(A)} \frac{T(r, f)}{T(r, f^{(k)})} \leq 3eA.$$

Lemma 2.2. Let f be a transcendental meromorphic function and $\alpha (\neq 0, \infty)$ be a small function of f , then $\psi = \alpha(f)^n (f^{(k)})^p$ is non-constant, where $n (\geq 0)$, $p (\geq 1)$ and $k (\geq 1)$ are integers.

Proof. We consider the following two cases.

Case 2.1. Let $n = 0$.

If possible suppose that ψ is a constant, then we get

$$T(r, (f^{(k)})^p) \leq T(r, \alpha) + O(1) = S(r, f)$$

i.e.,

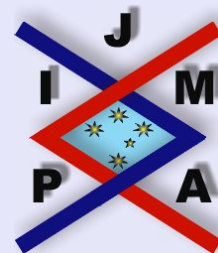
$$T(r, f^{(k)}) = S(r, f),$$

which is impossible by Lemma 2.1. Hence ψ is non-constant in this case.

Case 2.2. Let $n \geq 1$.

Since

$$\left(\frac{1}{f}\right)^{p+n} = \alpha \left(\frac{f^{(k)}}{f}\right)^p \frac{1}{\psi},$$



Inequalities Arising out of the
Value Distribution of a
Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 5 of 13

it follows, by the first fundamental theorem and the Milloux theorem ([3, p.55]), that

$$\begin{aligned}
 (2.1) \quad (p+n)T(r, f) &\leq T(r, \alpha) + pT\left(r, \frac{f^{(k)}}{f}\right) + T(r, \psi) + O(1) \\
 &= pN\left(r, \frac{f^{(k)}}{f}\right) + T(r, \psi) + S(r, f) \\
 &\leq pk\{\bar{N}(r, 0; f) + \bar{N}(r, \infty; f)\} + T(r, \psi) + S(r, f).
 \end{aligned}$$

We note that if all the zeros (poles) of $(f)^n(f^{(k)})^p$ are poles (zeros) of α in the same multiplicities then

$$\bar{N}(r, 0; f) \leq N(r, 0; (f)^n(f^{(k)})^p) = N(r, \infty; \alpha) = S(r, f)$$

and

$$\bar{N}(r, \infty; f) \leq N(r, \infty; (f)^n(f^{(k)})^p) = N(r, 0; \alpha) = S(r, f),$$

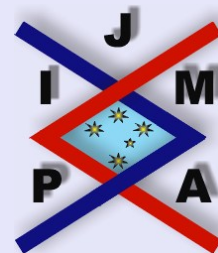
because $n \geq 1$. Since $n \geq 1$, it follows that

$$\bar{N}(r, 0; f) \leq N(r, 0; \psi) + S(r, f) \quad \text{and} \quad \bar{N}(r, \infty; f) \leq N(r, \infty; \psi) + S(r, f).$$

Hence, from (2.1), we get

$$\begin{aligned}
 (p+n)T(r, f) &\leq pk\{N(r, 0; \psi) + N(r, \infty; \psi)\} + T(r, \psi) + S(r, f) \\
 &\leq (2pk+1)T(r, \psi) + S(r, f),
 \end{aligned}$$

which shows that ψ is non-constant. This proves the lemma. \square



Inequalities Arising out of the
Value Distribution of a
Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

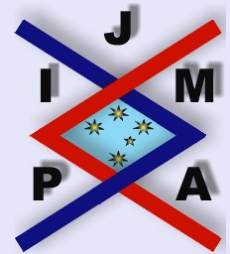
Close

Quit

Page 6 of 13

Lemma 2.3. [1] Let f be a transcendental meromorphic function and $\alpha (\neq 0, \infty)$ be a small function of f . If $\psi = \alpha(f)^n (f^{(k)})^p$, where $n(\geq 0)$, $p(\geq 1)$ and $k(\geq 1)$ are integers, then

$$T(r, \psi) \leq \{n + (1 + k)p\}T(r, f) + S(r, f).$$



**Inequalities Arising out of the
Value Distribution of a
Differential Monomial**

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 7 of 13

3. Theorems

In this section we prove the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function and $\alpha (\neq 0, \infty)$ be a small function of f . Suppose that b and c are any two finite non-zero distinct complex numbers. If $\psi = \alpha(f)^n (f^{(k)})^p$, where $n(\geq 0)$, $p(\geq 1)$ and $k(\geq 1)$ are integers, then*

$$(p+n)T(r, f) \leq (p+n)N(r, 0; f) + N(r, b; \psi) + N(r, c; \psi) - N(r, \infty; f) - N(r, 0; \psi') + S(r, f).$$

Proof. By Lemma 2.2 we see that ψ is non-constant. We now get

$$m\left(r, \frac{1}{\alpha(f)^{p+n}}\right) \leq m(r, 0; \psi) + m\left(r, \left(\frac{f^{(k)}}{f}\right)^p\right) + O(1),$$

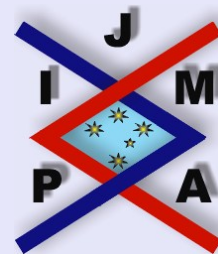
$$m\left(r, \frac{1}{\alpha(f)^{p+n}}\right) = T(r, \alpha(f)^{p+n}) - N(r, 0; \alpha(f)^{p+n}) + O(1)$$

and

$$m(r, 0; \psi) = T(r, \psi) - N(r, 0; \psi) + O(1).$$

Hence we obtain

$$(3.1) \quad T(r, \alpha(f)^{p+n}) \leq N(r, 0; \alpha(f)^{p+n}) + T(r, \psi) - N(r, 0; \psi) + m\left(r, \left(\frac{f^{(k)}}{f}\right)^p\right) + O(1) = N(r, 0; \alpha(f)^{p+n}) + T(r, \psi) - N(r, 0; \psi) + S(r, f).$$



Inequalities Arising out of the Value Distribution of a Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 8 of 13

By the second fundamental theorem we get

$$(3.2) \quad T(r, \psi) \leq N(r, 0; \psi) + N(r, b; \psi) + N(r, c; \psi) - N_1(r, \psi) + S(r, \psi),$$

where $N_1(r, \psi) = 2N(r, \infty; \psi) - N(r, \infty; \psi') + N(r, 0; \psi')$.

Let z_0 be a pole of f with multiplicity $q (\geq 1)$. ψ and ψ' have a pole with multiplicities $nq + (q + k)p + t$ and $nq + (q + k)p + 1 + t$ respectively, where $t = 0$ if z_0 is neither a pole nor a zero of α , $t = s$ if z_0 is a pole of α with multiplicity s and $t = -s$ if z_0 is a zero of α with multiplicity s , where s is a positive integer.

Thus,

$$\begin{aligned} 2\{nq + (q + k)p + t\} - \{nq + (q + k)p + 1 + t\} \\ &= nq + (q + k)p + t - 1 \\ &= q + t + nq + (q + k)p - q - 1 \\ &\geq q + t \end{aligned}$$

because

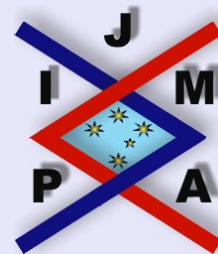
$$nq + (q + k)p - q - 1 \geq k - 1 \geq 0.$$

Since $T(r, \alpha) = S(r, f)$, it follows that

$$(3.3) \quad N_1(r, \psi) \geq N(r, \infty; f) + N(r, 0; \psi') + S(r, f).$$

Now, we get from (3.1), (3.2) and (3.3) in view of Lemma 2.3

$$\begin{aligned} T(r, \alpha(f)^{p+n}) \leq N(r, 0; \alpha(f)^{p+n}) + N(r, b; \psi) + N(r, c; \psi) \\ - N(r, \infty; f) - N(r, 0; \psi') + S(r, f) \end{aligned}$$



Inequalities Arising out of the
Value Distribution of a
Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 9 of 13

i.e.,

$$(p+n)T(r, f) \leq (p+n)N(r, 0; f) + N(r, b; \psi) + N(r, c; \psi) - N(r, \infty; f) - N(r, 0; \psi') + S(r, f).$$

This proves the theorem. \square

Theorem 3.2. *Let f be a transcendental meromorphic function and $\alpha (\neq 0, \infty)$ be a small function of f . If $\psi = \alpha(f)^n (f^{(k)})^p$, where $n (\geq 0)$, $p (\geq 1)$, $k (\geq 1)$ are integers, then for any small function $a (\neq 0, \infty)$ of ψ ,*

$$(p+n)T(r, f) \leq \bar{N}(r, \infty; f) + \bar{N}(r, 0; f) + pN_k(r, 0; f) + \bar{N}(r, a; \psi) + S(r, f).$$

Proof. Since by Lemma 2.2 ψ is non-constant, by Nevanlinna's three small functions theorem ([3, p. 47]) we get

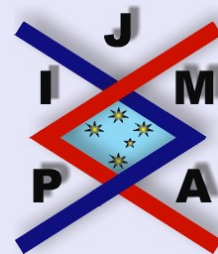
$$T(r, \psi) \leq \bar{N}(r, 0; \psi) + \bar{N}(r, \infty; \psi) + \bar{N}(r, a; \psi) + S(r, \psi).$$

So from (3.1) we obtain

$$T(r, \alpha(f)^{p+n}) \leq N(r, 0; \alpha(f)^{p+n}) + \bar{N}(r, 0; \psi) + \bar{N}(r, \infty; \psi) + \bar{N}(r, a; \psi) - N(r, 0; \psi) + S(r, \psi).$$

Since by Lemma 2.3 we can replace $S(r, \psi)$ by $S(r, f)$ and $\bar{N}(r, \infty; \psi) = \bar{N}(r, \infty; f) + S(r, f)$, we get

$$(3.4) \quad (p+n)T(r, f) \leq N(r, 0; (f)^{p+n}) + \bar{N}(r, 0; \psi) + \bar{N}(r, \infty; f) + \bar{N}(r, a; \psi) - N(r, 0; \psi) + S(r, f).$$



Inequalities Arising out of the Value Distribution of a Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 10 of 13

Let z_0 be a zero of f with multiplicity $q (\geq 1)$. It follows that z_0 is a zero of ψ with multiplicity $nq + t$ if $q \leq k$ and $nq + (q - k)p + t$ if $q \geq 1 + k$, where $t = 0$ if z_0 is neither a pole nor a zero of α , $t = s$ if z_0 is a zero of α with multiplicity s and $t = -s$ if z_0 is a pole of α with multiplicity s , where s is a positive integer.

Hence $(p + n)q + 1 - nq - t = pq + 1 - t$ if $q \leq k$ and $(p + n)q + 1 - nq - (q - k)p - t = pk + 1 - t$ if $q \geq 1 + k$. Since $T(r, \alpha) = S(r, f)$, we get

$$(3.5) \quad N(r, 0; \alpha(f)^{p+n}) + \overline{N}(r, 0; \psi) - N(r, 0; \psi) \leq \overline{N}(r, 0; f) + pN_k(r, 0; f) + S(r, f).$$

Now the theorem follows from (3.4) and (3.5). This proves the theorem. \square

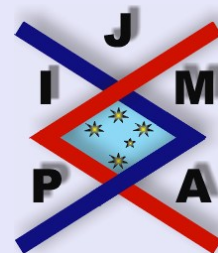
Hayman [2] proved that if f is a transcendental meromorphic function and $n (\geq 3)$ is an integer then $(f)^n f'$ assumes all finite values, except possibly zero, infinitely often.

In the following corollary of *Theorem 3.2* we improve this result.

Corollary 3.3. *Let f be a transcendental meromorphic function and $\psi = \alpha(f)^n (f^{(k)})^p$, where $n (\geq 3)$, $k (\geq 1)$, $p (\geq 1)$ are integers and $\alpha (\neq 0, \infty)$ is a small function of f , then*

$$\Theta(a; \psi) \leq \frac{(1 + k)p + 2}{(1 + k)p + n}$$

for any small function $a (\neq 0, \infty)$ of f .



Inequalities Arising out of the Value Distribution of a Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 11 of 13

Proof. Since for $n \geq 1$,

$$(3.6) \quad T(r, f) \leq BT(r, \psi)$$

holds except possibly for a set of r of finite linear measure, where B is a constant (see [6]), it follows that if $a(\not\equiv 0, \infty)$ is a small function of f , then it is also a small function of ψ .

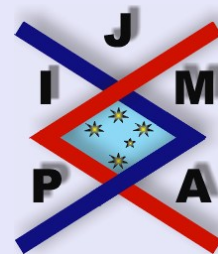
Hence by *Theorem 3.2* we get

$$(n - 2)T(r, f) \leq \bar{N}(r, a; \psi) + S(r, f),$$

and so by *Lemma 2.3* and (3.6) we obtain

$$\frac{n - 2}{(1 + k)p + n} T(r, \psi) \leq \bar{N}(r, a; \psi) + S(r, \psi),$$

from which the corollary follows. This proves the corollary. □



**Inequalities Arising out of the
Value Distribution of a
Differential Monomial**

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

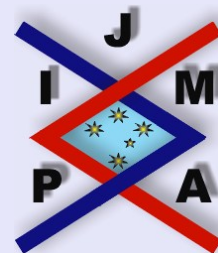
Close

Quit

Page 12 of 13

References

- [1] W. DOERINGER, Exceptional values of differential polynomials, *Pacific J. Math.*, **98**(1) (1982), 55–62.
- [2] W.K. HAYMAN, Picard values of meromorphic functions and their derivatives, *Ann. Math.*, **70** (1959), 9–42.
- [3] W.K. HAYMAN, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).
- [4] W.K. HAYMAN AND J. MILES, On the growth of a meromorphic function and its derivatives, *Complex Variables*, **12** (1989), 245–260.
- [5] K.L. HIONG, Sur la limitation de $T(r, f)$ sans intervention des pôles, *Bull. Sci. Math.*, **80** (1956), 175–190.
- [6] L.R. SONS, Deficiencies of monomials, *Math. Z.*, **111** (1969), 53–68.
- [7] K.W. YU, A note on the product of meromorphic functions and its derivatives, *Kodai Math. J.*, **24**(3) (2001), 339–343.
- [8] K.W. YU, On the value distribution of $\phi(z)[f(z)]^{n-1}f^{(k)}(z)$, *J. Inequal. Pure Appl. Math.*, **3**(1) (2002), Article 8. [ONLINE http://jipam.vu.edu.au/v3n1/037_01.html]
- [9] Q.D. ZHANG, On the value distribution of $\phi(z)f(z)f'(z)$ (in Chinese), *Acta Math. Sinica*, **37** (1994), 91–97.



Inequalities Arising out of the
Value Distribution of a
Differential Monomial

Indrajit Lahiri and Shyamali Dewan

Title Page

Contents



Go Back

Close

Quit

Page 13 of 13