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CESÁRO MEANS OF N-MULTIPLE TRIGONOMETRIC FOURIER SERIES

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Abstract

Zhizhiashvili proved sufficient condition for the Cesáro summability by negative order of N-multiple trigonometric Fourier series in the space $L^p, 1 \leq p \leq \infty$. In this paper we show that this condition cannot be improved .

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Let R^N be N-dimensional Euclidean space. The elements of R^N are denoted by $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$, ... For any $x, y \in R^N$ the vector $(x_1 + y_1, \ldots, x_N + y_N)$ of the space R^N is denoted by $x + y$. Let $\|x\| = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$.

Denote by $C\left([0,2]^N\right)$ the space of continuous on $[0, 2\pi]^N$, 2π-periodic relative to each variable functions with the following norm

$$\|f\|_C = \sup_{x \in [0,2\pi]^N} |f(x)|$$

and $L^p\left([0, 2\pi]^N\right)$, $(1 \leq p \leq \infty)$ are the collection of all measurable, 2π-periodic relative to each variable functions f defined on $[0, 2\pi]^N$, with the norms

$$\|f\|_p = \left(\int_{[0,2\pi]^N} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

For the case $p=\infty$, by $L^p\left([0,2\pi]^N\right)$ we mean $C\left([0,2\pi]^N\right)$.

Let $M := \{1,2,\dots,N\}$, $B := \{s_1,\dots,s_r\}$, $s_k < s_{k+1}$, $k=1,\dots,r-1$, $B\subset M$, $B' := M\backslash B$. Let

$$\Delta^{\{s_i\}}\left(f,x,h_{s_i}\right) := f\left(x_1,\dots,x_{s_i-1},x_{s_i}+h_{s_i},x_{s_i+1},\dots,x_N\right) - f\left(x_1,\dots,x_{s_i-1},x_{s_i},x_{s_i+1},\dots,x_N\right).$$

The expression we get by successive application of operators $\Delta^{\{s_1\}}\left(f,x,h_{s_1}\right)$, $\dots$, $\Delta^{\{s_r\}}\left(f,x,h_{s_r}\right)$ will be denoted by $\Delta^B\left(f,x,h_{s_1},\dots,h_{s_r}\right)$, i. e.

$$\Delta^B\left(f,x,h_{s_1},\dots,h_{s_r}\right) := \Delta^{\{s_r\}}\left(\Delta^{B\backslash\{s_r\}},x,h_{s_r}\right).$$

Let $f\in L^p\left([0,2\pi]^N\right)$. The expression

$$\omega_B\left(\delta_{s_1},\dots,\delta_{s_r};f\right) := \sup_{\left|h_{s_i}\right|\le\delta_{s_i},i=1,\dots,r}\left\|\Delta^B\left(f,\cdot,h_{s_1},\dots,h_{s_r}\right)\right\|_p$$

is called a mixed or a particular modulus of continuity in the L^p norm, when $\operatorname{card}(B)\in[2,N]$ or $\operatorname{card}(B)=1$.

The total modulus of continuity of the function $f\in L^p\left([0,2\pi]^N\right)$ in the L^p norm is defined by

$$\omega(\delta,f)_p = \sup_{\|h\|\le\delta}\|f(\cdot+h)-f(\cdot)\|_p \qquad (1\le p\le\infty).$$

Suppose that f is a Lebesgue integrable function on $[0,2\pi]^N$, 2π periodic relative to each variable.Then its N-dimensional Fourier series with respect to the trigonometric system is defined by

$$\sum_{i_1=0}^{\infty}\cdots\sum_{i_N=0}^{\infty} 2^{-\lambda(i)}\sum_{B\subset M} a_{i_1,\dots,i_N}^{(B)}\prod_{j\in B'}\cos i_jx_j\prod_{k\in B}\sin i_kx_k,$$

where

$$a_{i_1,\dots,i_N}^{(B)} = \frac{1}{\pi^N}\int\limits_{[0,2\pi]^N} f(x)\prod_{j\in B'}\cos i_jx_j\prod_{k\in B}\sin i_kx_k dx$$

is the Fourier coefficient of f and $\lambda(i)$ is the number of those coordinates of the vector $i := (i_1,\dots,i_N)$ which are equal to zero.

Let $S_{p_1,\dots,p_N}(f,x)$ denote the $(p_1,\dots,p_N)$-th rectangular partial sums of the N-dimensional Fourier series with respect to the trigonometric system, i. e.

$$S_{p_1,\dots,p_N}(f,x) := \sum_{i_1=0}^{p_1}\cdots\sum_{i_N=0}^{p_N} A_{i_1,\dots,i_N}(f,x),$$

where

$$A_{i_1,\dots,i_N}(f,x) := 2^{-\lambda(i)}\sum_{B\subset M} a_{i_1,\dots,i_N}^{(B)}\prod_{j\in B'}\cos i_jx_j\prod_{k\in B}\sin i_kx_k.$$

The Cesáro $(C;\alpha_1,\dots,\alpha_N)$-means of N-multiple trigonometric Fourier series defined by

$$\sigma_{m_1,\dots,m_N}^{\alpha_1,\dots,\alpha_N}(f,x) = \left(\prod_{i=1}^{N} A_{m_i}^{\alpha_i}\right)^{-1}\sum_{p_1=0}^{m_1}\cdots\sum_{p_N=0}^{m_N}\prod_{j=1}^{N} A_{m_j-p_j}^{\alpha_j-1} S_{p_1,\dots,p_N}(f,x),$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, \qquad \alpha \neq -1, -2, \dots, \quad n = 0, 1, \dots .$$

It is well-known that [4]

(1) $$c_1(\alpha)\, n^\alpha \le A_n^\alpha \le c_2(\alpha)\, n^\alpha .$$

For the uniform summability of Cesáro means of negative order of one-dimensional trigonometric Fourier series the following result of Zygmund [3] is well-known: if

$$\omega(\delta, f)_C = o(\delta^\alpha)$$

and $\alpha \in (0,1)$, then the trigonometric Fourier series of the function f is uniformly $(C, -\alpha)$ summable to f.

In [2] Zhizhiashvili proved sufficient conditions for the convergence of Cesáro means of negative order of N-multiple trigonometric Fourier series in the space $L^p\left([0,2\pi]^N\right)$, $(1 \le p \le \infty)$. The following is proved.

Theorem A (Zhizhiashvili). *Let $f \in L^p\left([0,2\pi]^N\right)$ for some $p \in [1, +\infty]$ and $\alpha_1 + \cdots + \alpha_N < 1$, where $\alpha_i \in (0,1)$, $i = 1, 2, \dots, N$. If*

$$\omega(\delta, f)_p = o\left(\delta^{\alpha_1 + \cdots + \alpha_N}\right),$$

then

$$\left\| \sigma_{m_1,\dots,m_N}^{-\alpha_1,\dots,-\alpha_N}(f) - f \right\|_p \to 0 \qquad \text{as} \quad m_i \to \infty, i = 1, \dots, N.$$

In case $p = \infty$ the sharpness of Theorem A has been proved by Zhizhiashvili [2]. The following theorem shows that Theorem A cannot be improved in cases $1 \le p < \infty$. Moreover, we prove the following

Theorem 1 (for $N = 1$ see [1]). *Let $\alpha_1 + \cdots + \alpha_N < 1$ and $\alpha_i \in (0,1)$, $i = 1, 2, \ldots, N$, then there exists the function $f_0 \in C\left([0, 2\pi]^N\right)$ for which*

$$\omega\left(\delta, f_0\right)_C = O\left(\delta^{\alpha_1 + \cdots + \alpha_N}\right) \tag{2}$$

and

$$\overline{\lim_{m \to \infty}} \left\| \sigma_{m,\ldots,m}^{-\alpha_1,\ldots,-\alpha_N}\left(f_0\right) - f_0 \right\|_1 > 0.$$

Proof. We can define the sequence $\{n_k : k \le 1\}$ satisfying the properties

$$\sum_{j=k+1}^{\infty} \frac{1}{n_j^{\alpha_1 + \cdots + \alpha_N}} = O\left(\frac{1}{n_k^{\alpha_1 + \cdots + \alpha_N}}\right), \tag{3}$$

$$\sum_{j=1}^{k-1} n_j^{1-(\alpha_1 + \cdots + \alpha_N)} = O\left(n_k^{1-(\alpha_1 + \cdots + \alpha_N)}\right), \tag{4}$$

$$\frac{n_{k-1}}{n_k} < \frac{1}{k}. \tag{5}$$

Consider the function f_0 defined by

$$f_0\left(x_1, \ldots, x_N\right) := \sum_{j=1}^{\infty} f_j\left(x_1, \ldots, x_N\right),$$

where

$$f_j(x_1,\dots,x_N) := \frac{1}{n_j^{\alpha_1+\dots+\alpha_N}} \prod_{i=1}^{N} \sin n_j x_i.$$

From (3) it is easy to show that $f_0 \in C\left([0,2\pi]^N\right)$. First we shall prove that

(6) $$\omega_i(\delta, f)_C = O\left(\delta^{\alpha_1+\dots+\alpha_N}\right), \qquad i = 1,\dots,N.$$

Let $\frac{1}{n_k} \le \delta < \frac{1}{n_{k-1}}$. Then from (3) and (4) we can write that

$$\begin{aligned}
&\left|f_0(x_1,\dots,x_{i-1},x_i+\delta,x_{i+1},\dots,x_N) - f_0(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_N)\right| \\
&\le \sum_{j=1}^{\infty} \frac{1}{n_j^{\alpha_1+\dots+\alpha_N}} \left|\sin n_j(x_i+\delta) - \sin n_j x_i\right| \\
&\le \sum_{j=1}^{k-1} \frac{1}{n_j^{\alpha_1+\dots+\alpha_N}} \left|\sin n_j(x_i+\delta) - \sin n_j x_i\right| + 2\sum_{j=k}^{\infty} \frac{1}{n_j^{\alpha_1+\dots+\alpha_N}} \\
&\le \sum_{j=1}^{k-1} \frac{n_j \delta}{n_j^{\alpha_1+\dots+\alpha_N}} + O\left(\frac{1}{n_k^{\alpha_1+\dots+\alpha_N}}\right) \\
&= O\left(\delta n_{k-1}^{1-(\alpha_1+\dots+\alpha_N)}\right) + O\left(\frac{1}{n_k^{\alpha_1+\dots+\alpha_N}}\right) \\
&= O\left(\delta^{\alpha_1+\dots\alpha_N}\right),
\end{aligned}$$

which proves (6).

Since

$$\omega(\delta, f)_C \leq \sum_{i=1}^{N} \omega_i(\delta, f)_C ,$$

we obtain the proof of estimation (2).

Next we shall prove that $\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_0)$ diverge in the metric of $L^1\left([0,2\pi]^N\right)$. It is clear that

$$\begin{aligned}(7)\quad \left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_0) - f_0\right\|_1 &\geq \left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_k)\right\|_1 \\ &\quad - \sum_{j=1}^{k-1}\left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j) - f_j\right\|_C \\ &\quad - \sum_{j=k+1}^{\infty}\left\|\sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j)\right\|_C - \sum_{j=k}^{\infty}\|f_j\|_C \\ &= I - II - III - IV.\end{aligned}$$

It is evident that

$$(8)\qquad \sigma_{n_k,\ldots,n_k}^{-\alpha_1,\ldots,-\alpha_N}(f_j) = 0, \qquad j = k+1, k+2, \ldots.$$

Using (3) for IV we have

$$(9)\qquad IV \leq \sum_{j=k}^{\infty}\frac{1}{n_j^{\alpha_1+\cdots+\alpha_N}} = O\left(\frac{1}{n_k^{\alpha_1+\cdots+\alpha_N}}\right).$$

Since [2]

$$\left\|\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_j)-f_j\right\|_C=O\left(\sum_{B\subset M}\omega_B\left(\frac{1}{n_k},f_j\right)_C n_k^{\sum_{s\in B}\alpha_s}\right)$$

and

$$\omega_i\left(\frac{1}{n_k},f_j\right)=O\left(\frac{1}{n_j^{\alpha_1+\dots+\alpha_N}}\frac{n_j}{n_k}\right),$$

from (4) and (5) we get

$$\begin{aligned}(10)\qquad II&=O\left(\frac{1}{n_k^{1-(\alpha_1+\dots+\alpha_N)}}\sum_{j=1}^{k-1}n_j^{1-(\alpha_1+\dots+\alpha_N)}\right)\\&=O\left(\frac{1}{n_k^{1-(\alpha_1+\dots+\alpha_N)}}\sum_{j=1}^{k-2}n_j^{1-(\alpha_1+\dots+\alpha_N)}+\frac{n_{k-1}^{1-(\alpha_1+\dots+\alpha_N)}}{n_k^{1-(\alpha_1+\dots+\alpha_N)}}\right)\\&=O\left(\frac{n_{k-1}^{1-(\alpha_1+\dots+\alpha_N)}}{n_k^{1-(\alpha_1+\dots+\alpha_N)}}\right)\\&=O\left(\left(\frac{1}{k}\right)^{1-(\alpha_1+\dots+\alpha_N)}\right)=o(1)\quad\text{as}\quad k\to\infty.\end{aligned}$$

Since

$$a_{i_1,\dots,i_N}^{(B)}(f_k)=0,\quad\text{for}\quad B\subset M,B\neq M$$

and

$$a_{i_1,\dots,i_N}^{(M)}(f_k)=\begin{cases}n_k^{-\alpha_1-\dots-\alpha_N}, & \text{for}\quad i_1=\dots=i_N=n_k;\\ 0, & \text{otherwise},\end{cases}$$

from (1) we have

$$
\begin{aligned}
(11)\ &\left\|\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_k)\right\|_1 \\
&= \int_0^{2\pi}\cdots\int_0^{2\pi}\left|\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_k;x_1,\dots,x_N)\right| dx_1\cdots dx_N \\
&\geq \left|\int_0^{2\pi}\cdots\int_0^{2\pi}\sigma_{n_k,\dots,n_k}^{-\alpha_1,\dots,-\alpha_N}(f_k;x_1,\dots,x_N)\prod_{i=1}^{N}\sin n_k x_i dx_1\cdots dx_N\right| \\
&= \left|\frac{1}{A_{n_k}^{-\alpha_1}}\cdots\frac{1}{A_{n_k}^{-\alpha_N}}\sum_{i_1=0}^{n_k}\cdots\sum_{i_N=0}^{n_k}\prod_{j=1}^{N}A_{n_k-i_j}^{-\alpha_1-1}\right. \\
&\qquad \left.\times\int_0^{2\pi}\cdots\int_0^{2\pi}S_{i_1,\dots,i_N}(f_k;x_1,\dots,x_N)\prod_{i=1}^{N}\sin n_k x_i dx_1\cdots dx_N\right| \\
&= \pi^N\frac{1}{A_{n_k}^{-\alpha_1}}\cdots\frac{1}{A_{n_k}^{-\alpha_N}}a_{n_k,\dots,n_k}^{(M)}(f_k) \\
&= \pi^N\frac{1}{A_{n_k}^{-\alpha_1}}\cdots\frac{1}{A_{n_k}^{-\alpha_N}}n_k^{-\alpha_1-\cdots-\alpha_N}\geq c(\alpha_1,\dots,\alpha_N)>0.
\end{aligned}
$$

Combining (7) – (11) we complete the proof of Theorem 1. □

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