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## CESÁRO MEANS OF $N$ -MULTIPLE TRIGONOMETRIC FOURIER SERIES

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Abstract

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## Abstract

Zhizhiashvili proved sufficient condition for the Cesáro summability by negative order of  $N$ -multiple trigonometric Fourier series in the space  $L^p, 1 \leq p \leq \infty$ . In this paper we show that this condition cannot be improved .

*2000 Mathematics Subject Classification:* 42B08.

*Key words:* Trigonometric system, Cesáro means, Summability.

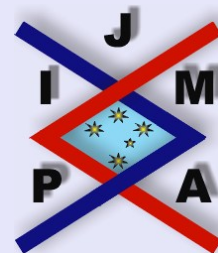
Let  $R^N$  be  $N$ -dimensional Euclidean space. The elements of  $R^N$  are denoted by  $x = (x_1, \dots, x_N)$ ,  $y = (y_1, \dots, y_N)$ , .... For any  $x, y \in R^N$  the vector  $(x_1 + y_1, \dots, x_N + y_N)$  of the space  $R^N$  is denoted by  $x + y$ . Let  $\|x\| = \left( \sum_{i=1}^N x_i^2 \right)^{1/2}$ .

Denote by  $C \left( [0, 2\pi]^N \right)$  the space of continuous on  $[0, 2\pi]^N$ ,  $2\pi$ -periodic relative to each variable functions with the following norm

$$\|f\|_C = \sup_{x \in [0, 2\pi]^N} |f(x)|$$

and  $L^p \left( [0, 2\pi]^N \right)$ ,  $(1 \leq p \leq \infty)$  are the collection of all measurable,  $2\pi$ -periodic relative to each variable functions  $f$  defined on  $[0, 2\pi]^N$ , with the norms

$$\|f\|_p = \left( \int_{[0, 2\pi]^N} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$



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For the case  $p = \infty$ , by  $L^p \left( [0, 2\pi]^N \right)$  we mean  $C \left( [0, 2\pi]^N \right)$ .

Let  $M := \{1, 2, \dots, N\}$ ,  $B := \{s_1, \dots, s_r\}$ ,  $s_k < s_{k+1}$ ,  $k = 1, \dots, r - 1$ ,  $B \subset M$ ,  $B' := M \setminus B$ . Let

$$\Delta^{\{s_i\}}(f, x, h_{s_i}) := f(x_1, \dots, x_{s_i-1}, x_{s_i} + h_{s_i}, x_{s_i+1}, \dots, x_N) - f(x_1, \dots, x_{s_i-1}, x_{s_i}, x_{s_i+1}, \dots, x_N).$$

The expression we get by successive application of operators  $\Delta^{\{s_1\}}(f, x, h_{s_1})$ ,  $\dots$ ,  $\Delta^{\{s_r\}}(f, x, h_{s_r})$  will be denoted by  $\Delta^B(f, x, h_{s_1}, \dots, h_{s_r})$ , i. e.

$$\Delta^B(f, x, h_{s_1}, \dots, h_{s_r}) := \Delta^{\{s_r\}}(\Delta^{B \setminus \{s_r\}}(f, x, h_{s_1}, \dots, h_{s_{r-1}})).$$

Let  $f \in L^p \left( [0, 2\pi]^N \right)$ . The expression

$$\omega_B(\delta_{s_1}, \dots, \delta_{s_r}; f) := \sup_{|h_{s_i}| \leq \delta_{s_i}, i=1, \dots, r} \left\| \Delta^B(f, \cdot, h_{s_1}, \dots, h_{s_r}) \right\|_p$$

is called a mixed or a particular modulus of continuity in the  $L^p$  norm, when  $\text{card}(B) \in [2, N]$  or  $\text{card}(B) = 1$ .

The total modulus of continuity of the function  $f \in L^p \left( [0, 2\pi]^N \right)$  in the  $L^p$  norm is defined by

$$\omega(\delta, f)_p = \sup_{\|h\| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p \quad (1 \leq p \leq \infty).$$

Suppose that  $f$  is a Lebesgue integrable function on  $[0, 2\pi]^N$ ,  $2\pi$  periodic relative to each variable. Then its  $N$ -dimensional Fourier series with respect to



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the trigonometric system is defined by

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_N=0}^{\infty} 2^{-\lambda(i)} \sum_{B \subset M} a_{i_1, \dots, i_N}^{(B)} \prod_{j \in B'} \cos i_j x_j \prod_{k \in B} \sin i_k x_k,$$

where

$$a_{i_1, \dots, i_N}^{(B)} = \frac{1}{\pi^N} \int_{[0, 2\pi]^N} f(x) \prod_{j \in B'} \cos i_j x_j \prod_{k \in B} \sin i_k x_k dx$$

is the Fourier coefficient of  $f$  and  $\lambda(i)$  is the number of those coordinates of the vector  $i := (i_1, \dots, i_N)$  which are equal to zero.

Let  $S_{p_1, \dots, p_N}(f, x)$  denote the  $(p_1, \dots, p_N)$ -th rectangular partial sums of the  $N$ -dimensional Fourier series with respect to the trigonometric system, i. e.

$$S_{p_1, \dots, p_N}(f, x) := \sum_{i_1=0}^{p_1} \cdots \sum_{i_N=0}^{p_N} A_{i_1, \dots, i_N}(f, x),$$

where

$$A_{i_1, \dots, i_N}(f, x) := 2^{-\lambda(i)} \sum_{B \subset M} a_{i_1, \dots, i_N}^{(B)} \prod_{j \in B'} \cos i_j x_j \prod_{k \in B} \sin i_k x_k.$$

The Cesáro  $(C; \alpha_1, \dots, \alpha_N)$ -means of  $N$ -multiple trigonometric Fourier series defined by

$$\sigma_{m_1, \dots, m_N}^{\alpha_1, \dots, \alpha_N}(f, x) = \left( \prod_{i=1}^N A_{m_i}^{\alpha_i} \right)^{-1} \sum_{p_1=0}^{m_1} \cdots \sum_{p_N=0}^{m_N} \prod_{j=1}^N A_{m_j-p_j}^{\alpha_j-1} S_{p_1, \dots, p_N}(f, x),$$



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where

$$A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, \dots, \quad n = 0, 1, \dots$$

It is well-known that [4]

$$(1) \quad c_1(\alpha) n^\alpha \leq A_n^\alpha \leq c_2(\alpha) n^\alpha.$$

For the uniform summability of Cesáro means of negative order of one-dimensional trigonometric Fourier series the following result of Zygmund [3] is well-known: if

$$\omega(\delta, f)_C = o(\delta^\alpha)$$

and  $\alpha \in (0, 1)$ , then the trigonometric Fourier series of the function  $f$  is uniformly  $(C, -\alpha)$  summable to  $f$ .

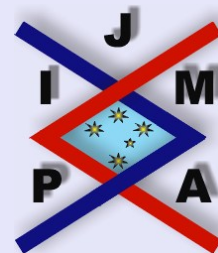
In [2] Zhizhiashvili proved sufficient conditions for the convergence of Cesáro means of negative order of  $N$ -multiple trigonometric Fourier series in the space  $L^p([0, 2\pi]^N)$ ,  $(1 \leq p \leq \infty)$ . The following is proved.

**Theorem A (Zhizhiashvili).** *Let  $f \in L^p([0, 2\pi]^N)$  for some  $p \in [1, +\infty]$  and  $\alpha_1 + \cdots + \alpha_N < 1$ , where  $\alpha_i \in (0, 1)$ ,  $i = 1, 2, \dots, N$ . If*

$$\omega(\delta, f)_p = o(\delta^{\alpha_1 + \cdots + \alpha_N}),$$

then

$$\left\| \sigma_{m_1, \dots, m_N}^{-\alpha_1, \dots, -\alpha_N}(f) - f \right\|_p \rightarrow 0 \quad \text{as} \quad m_i \rightarrow \infty, \quad i = 1, \dots, N.$$



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In case  $p = \infty$  the sharpness of Theorem A has been proved by Zhizhiashvili [2]. The following theorem shows that Theorem A cannot be improved in cases  $1 \leq p < \infty$ . Moreover, we prove the following

**Theorem 1 (for  $N = 1$  see [1]).** Let  $\alpha_1 + \dots + \alpha_N < 1$  and  $\alpha_i \in (0, 1)$ ,  $i = 1, 2, \dots, N$ , then there exists the function  $f_0 \in C([0, 2\pi]^N)$  for which

$$(2) \quad \omega(\delta, f_0)_C = O(\delta^{\alpha_1 + \dots + \alpha_N})$$

and

$$\overline{\lim}_{m \rightarrow \infty} \left\| \sigma_{m, \dots, m}^{-\alpha_1, \dots, -\alpha_N}(f_0) - f_0 \right\|_1 > 0.$$

*Proof.* We can define the sequence  $\{n_k : k \leq 1\}$  satisfying the properties

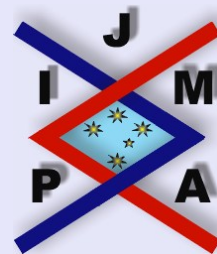
$$(3) \quad \sum_{j=k+1}^{\infty} \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N}} = O\left(\frac{1}{n_k^{\alpha_1 + \dots + \alpha_N}}\right),$$

$$(4) \quad \sum_{j=1}^{k-1} n_j^{1 - (\alpha_1 + \dots + \alpha_N)} = O\left(n_k^{1 - (\alpha_1 + \dots + \alpha_N)}\right),$$

$$(5) \quad \frac{n_{k-1}}{n_k} < \frac{1}{k}.$$

Consider the function  $f_0$  defined by

$$f_0(x_1, \dots, x_N) := \sum_{j=1}^{\infty} f_j(x_1, \dots, x_N),$$



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where

$$f_j(x_1, \dots, x_N) := \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N}} \prod_{i=1}^N \sin n_j x_i.$$

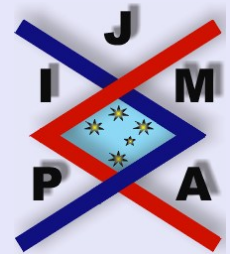
From (3) it is easy to show that  $f_0 \in C([0, 2\pi]^N)$ . First we shall prove that

$$(6) \quad \omega_i(\delta, f)_C = O(\delta^{\alpha_1 + \dots + \alpha_N}), \quad i = 1, \dots, N.$$

Let  $\frac{1}{n_k} \leq \delta < \frac{1}{n_{k-1}}$ . Then from (3) and (4) we can write that

$$\begin{aligned} & |f_0(x_1, \dots, x_{i-1}, x_i + \delta, x_{i+1}, \dots, x_N) - f_0(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N)| \\ & \leq \sum_{j=1}^{\infty} \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N}} |\sin n_j(x_i + \delta) - \sin n_j x_i| \\ & \leq \sum_{j=1}^{k-1} \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N}} |\sin n_j(x_i + \delta) - \sin n_j x_i| + 2 \sum_{j=k}^{\infty} \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N}} \\ & \leq \sum_{j=1}^{k-1} \frac{n_j \delta}{n_j^{\alpha_1 + \dots + \alpha_N}} + O\left(\frac{1}{n_k^{\alpha_1 + \dots + \alpha_N}}\right) \\ & = O\left(\delta n_{k-1}^{1 - (\alpha_1 + \dots + \alpha_N)}\right) + O\left(\frac{1}{n_k^{\alpha_1 + \dots + \alpha_N}}\right) \\ & = O(\delta^{\alpha_1 + \dots + \alpha_N}), \end{aligned}$$

which proves (6).



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Since

$$\omega(\delta, f)_C \leq \sum_{i=1}^N \omega_i(\delta, f)_C,$$

we obtain the proof of estimation (2).

Next we shall prove that  $\sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N}(f_0)$  diverge in the metric of  $L^1([0, 2\pi]^N)$ .

It is clear that

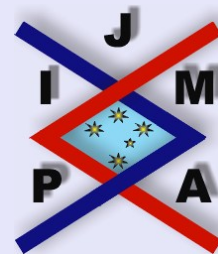
$$\begin{aligned} (7) \quad \|\sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N}(f_0) - f_0\|_1 &\geq \|\sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N}(f_k)\|_1 \\ &\quad - \sum_{j=1}^{k-1} \|\sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N}(f_j) - f_j\|_C \\ &\quad - \sum_{j=k+1}^{\infty} \|\sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N}(f_j)\|_C - \sum_{j=k}^{\infty} \|f_j\|_C \\ &= I - II - III - IV. \end{aligned}$$

It is evident that

$$(8) \quad \sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N}(f_j) = 0, \quad j = k + 1, k + 2, \dots$$

Using (3) for IV we have

$$(9) \quad IV \leq \sum_{j=k}^{\infty} \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N}} = O\left(\frac{1}{n_k^{\alpha_1 + \dots + \alpha_N}}\right).$$



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Since [2]

$$\left\| \sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N} (f_j) - f_j \right\|_C = O \left( \sum_{BCM} \omega_B \left( \frac{1}{n_k}, f_j \right)_C n_k^{\sum_{s \in B} \alpha_s} \right)$$

and

$$\omega_i \left( \frac{1}{n_k}, f_j \right) = O \left( \frac{1}{n_j^{\alpha_1 + \dots + \alpha_N} n_k} \right),$$

from (4) and (5) we get

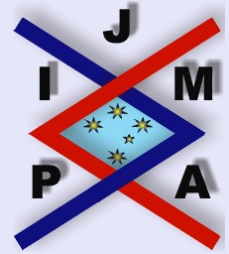
$$\begin{aligned} (10) \quad II &= O \left( \frac{1}{n_k^{1-(\alpha_1 + \dots + \alpha_N)}} \sum_{j=1}^{k-1} n_j^{1-(\alpha_1 + \dots + \alpha_N)} \right) \\ &= O \left( \frac{1}{n_k^{1-(\alpha_1 + \dots + \alpha_N)}} \sum_{j=1}^{k-2} n_j^{1-(\alpha_1 + \dots + \alpha_N)} + \frac{n_{k-1}^{1-(\alpha_1 + \dots + \alpha_N)}}{n_k^{1-(\alpha_1 + \dots + \alpha_N)}} \right) \\ &= O \left( \frac{n_{k-1}^{1-(\alpha_1 + \dots + \alpha_N)}}{n_k^{1-(\alpha_1 + \dots + \alpha_N)}} \right) \\ &= O \left( \left( \frac{1}{k} \right)^{1-(\alpha_1 + \dots + \alpha_N)} \right) = o(1) \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Since

$$a_{i_1, \dots, i_N}^{(B)} (f_k) = 0, \quad \text{for } B \subset M, B \neq M$$

and

$$a_{i_1, \dots, i_N}^{(M)} (f_k) = \begin{cases} n_k^{-\alpha_1 - \dots - \alpha_N}, & \text{for } i_1 = \dots = i_N = n_k; \\ 0, & \text{otherwise,} \end{cases}$$



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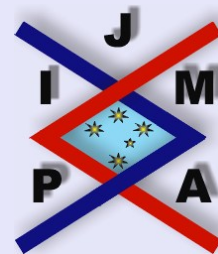
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from (1) we have

$$\begin{aligned}
 (11) & \left\| \sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N} (f_k) \right\|_1 \\
 &= \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N} (f_k; x_1, \dots, x_N) \right| dx_1 \cdots dx_N \\
 &\geq \left| \int_0^{2\pi} \cdots \int_0^{2\pi} \sigma_{n_k, \dots, n_k}^{-\alpha_1, \dots, -\alpha_N} (f_k; x_1, \dots, x_N) \prod_{i=1}^N \sin n_k x_i dx_1 \cdots dx_N \right| \\
 &= \left| \frac{1}{A_{n_k}^{-\alpha_1}} \cdots \frac{1}{A_{n_k}^{-\alpha_N}} \sum_{i_1=0}^{n_k} \cdots \sum_{i_N=0}^{n_k} \prod_{j=1}^N A_{n_k-i_j}^{-\alpha_1-1} \right. \\
 &\quad \times \left. \int_0^{2\pi} \cdots \int_0^{2\pi} S_{i_1, \dots, i_N} (f_k; x_1, \dots, x_N) \prod_{i=1}^N \sin n_k x_i dx_1 \cdots dx_N \right| \\
 &= \pi^N \frac{1}{A_{n_k}^{-\alpha_1}} \cdots \frac{1}{A_{n_k}^{-\alpha_N}} a_{n_k, \dots, n_k}^{(M)} (f_k) \\
 &= \pi^N \frac{1}{A_{n_k}^{-\alpha_1}} \cdots \frac{1}{A_{n_k}^{-\alpha_N}} n_k^{-\alpha_1 - \cdots - \alpha_N} \geq c(\alpha_1, \dots, \alpha_N) > 0.
 \end{aligned}$$

Combining (7) – (11) we complete the proof of Theorem 1. □



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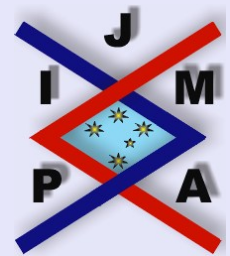
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