Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 4, Article 108, 2004

A GENERALIZATION OF AN INEQUALITY INVOLVING THE GENERALIZED ELEMENTARY SYMMETRIC MEAN

ZHI-HUA ZHANG AND ZHEN-GANG XIAO

ZIXING EDUCATIONAL RESEARCH SECTION
CHENZHOU, HUNAN 423400, CHINA.
zxzh1234@163.com

HUNAN INSTITUTE OF SCIENCE AND TECHNOLOGY
YUEYANG, HUNAN 414006, CHINA.
xiaozg@163.com

Received 23 May, 2003; accepted 09 October, 2004
Communicated by F. Qi

ABSTRACT. A generalization of an inequality involving the generalized elementary symmetric mean and its elementary proof are given.

Key words and phrases: Generalized elementary symmetric mean, Inequality.

2000 *Mathematics Subject Classification.* Primary 26D15.

1. INTRODUCTION

Let $a = (a_1, a_2, \cdots, a_n)$ and r be a nonnegative integer, where a_i for $1 \le i \le n$ are nonnegative real numbers. Then

$$E_n^{[r]} = E_n^{[r]}(a) = \sum_{\substack{i_1+i_2+\cdots+i_n=r,\\ i_1,i_2,\cdots,i_n\ge 0 \text{ are integers}}} \prod_{k=1}^{n} a_k^{i_k} \tag{1.1}$$

with $E_n^{[0]} = E_n^{[0]}(a) = 1$ for $n \ge 1$ and $E_n^{[r]} = 0$ for $r < 0$ or $n \le 0$ is called the rth generalized elementary symmetric function of a.

The rth generalized elementary symmetric mean of a is defined by

$$\sum_n^{[r]} = \sum_n^{[r]}(a) = \frac{E_n^{[r]}(a)}{\binom{n+r-1}{r}}. \tag{1.2}$$

ISSN (electronic): 1443-5756

The authors would like to thank Professor Feng Qi and the anonymous referee for some valuable suggestions which have improved the final version of this paper.

070-03

In 1934, I. Schur [5, p. 182] obtained the following

$$\sum_{n}^{[r]}(a) = (n-1)! \int \cdots \int \left(\sum_{i=1}^{n} a_i x_i \right)^r \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}, \tag{1.3}$$

where $x_n = 1-(x_1+x_2+\cdots+x_{n-1})$ and the integral is taken over $x_k \geq 0$ for $k = 1, 2, \ldots, n-1$. By using (1.3) and Cauchy integral inequality, he also proved that

$$\sum_{n}^{[r-1]}(a) \sum_{n}^{[r+1]}(a) \geq \left[\sum_{n}^{[r]}(a) \right]^2 . \tag{1.4}$$

In 1968, K.V. Menon [2] proved that for $n = 2$ or $n \geq 3$ and $r = 1, 2, 3$, inequality (1.4) is valid.

In [1], the generalized symmetric means of two variables was investigated.

In [3] and [4] a problem was posed: Does inequality (1.4) hold for arbitrary $n, r \in \mathbb{N}$?

In 1997, Zh.-H. Zhang generalized (1.3) in [7] and also proved (1.4) by a similar proof as in [5].

In [6], some inequalities of weighted symmetric mean were established.

In this paper, we shall obtain an identity relating $\sum_{n}^{[r]}(a)$ to $E_n^{[r]}(a)$ and give an elementary proof of an inequality which generalizes (1.4). Our main result is as follows.

Theorem 1.1. *If $r, s \in \mathbb{N}$ and $r > s$, then*

$$\sum_{n}^{[s]}(a) \sum_{n}^{[r+1]}(a) \geq \sum_{n}^{[r]}(a) \sum_{n}^{[s+1]}(a). \tag{1.5}$$

The equality in (1.5) *holds if and only if $a_1 = a_2 = \cdots = a_n$.*

Letting $s = r - 1$ in inequality (1.5) leads to inequality (1.4).

2. PROOF OF THEOREM 1.1

To prove inequality (1.5), the following properties for $E_n^{[r]}$ are necessary.

Property 1. If $n, r \in \mathbb{N}$, then

$$E_n^{[r]} = E_{n-1}^{[r]} + a_n E_n^{[r-1]} \tag{2.1}$$

and

$$E_n^{[r]} = \sum_{j=0}^{r} a_n^j E_{n-1}^{[r-j]}. \tag{2.2}$$

Proof. If $n = 1$ or $r = 0$, (2.1) holds trivially.

When $n > 1$ and $r \geq 1$, we have

$$\sum_{i_1,i_2,\cdots,i_n \geq 0}^{i_1+i_2+\cdots+i_n=r} \prod_{k=1}^{n} a_k^{i_k} = \sum_{\substack{i_1,i_2,\cdots,i_{n-1}\geq 0 \\ i_n=0}}^{i_1+i_2+\cdots+i_n=r} \prod_{k=1}^{n} a_k^{i_k} + a_n \sum_{i_1,i_2,\cdots,i_n \geq 0}^{i_1+i_2+\cdots+i_n=r-1} \prod_{k=1}^{n} a_k^{i_k}. \tag{2.3}$$

Combining the definition of $E_n^{[r]}$ and (2.3), identity (2.1) follows.

Identity (2.2) can be deduced from the recurrence of (2.1). □

Property 2. If r is an integer, then

$$(r+1)E_n^{[r+1]} = \sum_{k=0}^{r} \left(\sum_{i=1}^{n} a_i^{k+1} \right) E_n^{[r-k]}. \tag{2.4}$$

Proof. It will be verified by induction. It is clear that identity (2.4) holds trivially for $n = 1$. Suppose identity (2.4) is true for $n-1$ and nonnegative integers r.

By (2.2), for $0 \le k \le r$, we have

$$E_n^{[r-k]} = \sum_{j=0}^{r-k} a_n^j E_{n-1}^{[r-k-j]},$$

and

$$\begin{aligned}
\sum_{j=0}^{r} (j+1) a_n^{j+1} E_{n-1}^{[r-j]} &= a_n E_{n-1}^{[r]} + a_n^2 E_{n-1}^{[r-1]} + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]} \\
&\quad + a_n^2 E_{n-1}^{[r-1]} + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]} \\
&\quad \cdots\cdots \\
&\quad + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]} \\
&\quad + a_n^{r+1} E_{n-1}^{[0]} \\
&= \sum_{k=0}^{r} \sum_{j=0}^{r-k} a_n^{j+k+1} E_{n-1}^{[r-k-j]}.
\end{aligned}$$

According to the inductive hypothesis, for nonnegative integers r and $0 \le j \le r$, we have

$$(r-j+1)E_{n-1}^{[r+1-j]} = \sum_{k=0}^{r-j} \left(\sum_{i=1}^{n-1} a_i^{k+1} \right) E_{n-1}^{[r-k-j]}. \tag{2.5}$$

From Property 1 and the above formula, we have

$$\begin{aligned}
(r+1)E_n^{[r+1]} &= (r+1) \sum_{j=0}^{r+1} a_n^j E_{n-1}^{[r+1-j]} \\
&= \sum_{j=0}^{r} (r-j+1) a_n^j E_{n-1}^{[r-j+1]} + \sum_{j=1}^{r+1} j a_n^j E_{n-1}^{[r-j+1]} \\
&= \sum_{j=0}^{r} a_n^j \sum_{k=0}^{r-j} \left(\sum_{i=1}^{n-1} a_i^{k+1} \right) E_{n-1}^{[r-j-k]} + \sum_{j=0}^{r} (j+1) a_n^{j+1} E_{n-1}^{[r-j]} \\
&= \sum_{k=0}^{r} \sum_{j=0}^{r-k} a_n^j \left(\sum_{i=1}^{n-1} a_i^{k+1} \right) E_{n-1}^{[r-j-k]} + \sum_{k=0}^{r} \sum_{j=0}^{r-k} a_n^{j+k+1} E_{n-1}^{[r-k-j]} \\
&= \sum_{k=0}^{r} \left(\sum_{i=1}^{n-1} a_i^{k+1} + a_n^{k+1} \right) \left(\sum_{j=0}^{r-k} a_n^j E_{n-1}^{[r-k-j]} \right) \\
&= \sum_{k=0}^{r} \left(\sum_{i=1}^{n} a_i^{k+1} \right) E_n^{[r-k]}.
\end{aligned} \tag{2.6}$$

This shows that (2.4) holds for n. The proof is complete. □

Property 3. If $r, s \in \mathbb{N}$ and $r > s$, then

$$(2.7)\quad (r+1)(s+1)\binom{n+r}{r+1}\binom{n+s}{s+1}\left[\sum_n^{[s]}\sum_n^{[r+1]} - \sum_n^{[r]}\sum_n^{[s+1]}\right]$$
$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n}\left(E_n^{[s-k]}E_n^{[r-j]} - E_n^{[s-j]}E_n^{[r-k]}\right)\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right].$$

Proof. When $j > k$, we have

$$(2.8)\quad \sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1}$$
$$= \frac{1}{2}\sum_{v=1}^{n}\sum_{u=1}^{n}\left(a_v^k a_u^{j+1} + a_v^{j+1}a_u^k - a_v^j a_u^{k+1} - a_v^{k+1}a_u^j\right)$$
$$= \frac{1}{2}\sum_{v=1}^{n}\sum_{u=1}^{n}\left[a_v^k a_u^k\left(a_v^{j-k+1} + a_u^{j-k+1} - a_v^{j-k}a_u - a_v a_u^{j-k}\right)\right]$$
$$= \frac{1}{2}\sum_{v=1}^{n}\sum_{u=1}^{n}\left[a_v^k a_u^k\left(a_v^{j-k} - a_u^{j-k}\right)(a_v - a_u)\right]$$
$$= \sum_{1\le v<u\le n}\left[a_v^k a_u^k\left(a_v^{j-k} - a_u^{j-k}\right)(a_v - a_u)\right]$$

and

$$(2.9)\quad \left(a_v^{j-k} - a_u^{j-k}\right) = (a_v - a_u)\sum_{t=0}^{j-k-1} a_v^{j-k-1-t}a_u^t.$$

Therefore

$$(2.10)\quad \sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1} = \sum_{1\le v<u\le n}\left[\left(\sum_{t=0}^{k-j-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right].$$

When $k > j$, we have

$$(2.11)\quad \sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1} = -\sum_{1\le v<u\le n}\left[\left(\sum_{t=0}^{k-j-1} a_v^{j+t}a_u^{k-1-t}\right)(a_v-a_u)^2\right].$$

From Property 2, it is deduced that

$$(2.12)\quad (r+1)E_n^{[r+1]} = \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^{j+1}\right)E_n^{[r-j]}$$

and

$$(2.13)\quad (n+r)E_n^{[r]} = nE_n^{[r]} + rE_n^{[r]} = \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^j\right)E_n^{[r-j]}.$$

Hence, using the above formulas and noting $E_n^{[r-k]} = 0$ for $k > r$ yields

$$(2.14)\quad (r+1)(s+1)\binom{n+r}{r+1}\binom{n+s}{s+1}\left[\sum_n^{[s]}\sum_n^{[r+1]} - \sum_n^{[r]}\sum_n^{[s+1]}\right]$$

$$= (n+s)(r+1)E_n^{[s]}E_n^{[r+1]} - (n+r)(s+1)E_n^{[r]}E_n^{[s+1]}$$

$$= \sum_{k=0}^{s}\left(\sum_{i=1}^{n} a_i^k\right)E_n^{[s-k]}\sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^{j+1}\right)E_n^{[r-j]}$$

$$- \sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^j\right)E_n^{[r-j]}\sum_{k=0}^{s}\left(\sum_{i=1}^{n} a_i^{k+1}\right)E_n^{[s-k]}$$

$$= \sum_{k=0}^{s}\sum_{j=0}^{r}\left(\sum_{i=1}^{n} a_i^k\sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j\sum_{i=1}^{n} a_i^{k+1}\right)E_n^{[s-k]}\cdot E_n^{[r-j]}$$

$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n} E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$

$$- \sum_{k=0}^{s}\sum_{j=0}^{k}\left[\sum_{1\le v<u\le n} E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{k-j-1} a_v^{j+t}a_u^{k-1-t}\right)(a_v-a_u)^2\right]$$

$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n} E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$

$$- \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n} E_n^{[s-j]}E_n^{[r-k]}\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$

$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n}\left(E_n^{[s-k]}E_n^{[r-j]} - E_n^{[s-j]}E_n^{[r-k]}\right)\right.$$

$$\left.\times\left(\sum_{t=0}^{j-k-1} a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right],$$

which implies the expression (2.7). □

Property 4. If $r, s \in \mathbb{N}$ and $r > s$, then

$$(2.15)\qquad E_n^{[r-1]}E_n^{[s]} \ge E_n^{[r]}E_n^{[s-1]}.$$

The equality in (2.15) holds if and only if at least $n-1$ numbers equal zero among $\{a_1, a_2, \ldots, a_n\}$.

Proof. From Property 1, we have

$$(2.16)\quad E_n^{[r-1]}E_n^{[s]} - E_n^{[r]}E_n^{[s-1]}$$

$$= E_n^{[r-1]}\left(E_{n-1}^{[s]} + a_nE_n^{[s-1]}\right) - \left(E_{n-1}^{[r]} + a_nE_n^{[r-1]}\right)E_n^{[s-1]}$$

$$
\begin{aligned}
&= E_n^{[r-1]}E_{n-1}^{[s]} - E_{n-1}^{[r]}E_n^{[s-1]}\\
&= \left(\sum_{j=0}^{r-1} a_n^j E_{n-1}^{[r-1-j]}\right) E_{n-1}^{[s]} - E_{n-1}^{[r]} \left(\sum_{j=0}^{s-1} a_n^j E_{n-1}^{[s-1-j]}\right)\\
&= \sum_{j=0}^{s-1} a_n^j \left(E_{n-1}^{[r-1-j]}E_{n-1}^{[s]} - E_{n-1}^{[r]}E_{n-1}^{[s-1-j]}\right) + E_{n-1}^{[s]} \left(\sum_{j=s}^{r-1} a_n^j E_{n-1}^{[r-1-j]}\right).
\end{aligned}
$$

Since (2.15) holds for $n = 1$, it follows by induction that (2.15) holds for n. □

Property 5. If $r, s, j, k \in \mathbb{N}$ and $r > s > j > k$, then

$$E_n^{[s-k]}E_n^{[r-j]} \geq E_n^{[s-j]}E_n^{[r-k]}. \tag{2.17}$$

The equality in (2.17) is valid if and only if at least $n-1$ numbers equal zero among $\{a_1, a_2, \dots, a_n\}$

Proof. From Property 4, if $r-(k+1) > s-(k+1), r-(k+2) > s-(k+2), \dots, r-j > s-j$, then

$$\prod_{m=k+1}^{j} \left(E_n^{[r-m]}E_n^{[s-m+1]}\right) \geq \prod_{m=k+1}^{j} \left(E_n^{[r-m+1]}E_n^{[s-m]}\right). \tag{2.18}$$

This implies (2.17).

It is easy to see that the equality in (2.17) is valid. The proof is completed. □

Proof of Theorem 1.1. Combination of Property 3 and Property 5 easily leads to Theorem 1.1. □

Remark 2.1. Finally, we pose an open problem: Give an explicit expression of $E_n^{[r-1]}E_n^{[s]} - E_n^{[r]}E_n^{[s-1]}$ in terms of $a_1, a_2, \dots, a_n$.

REFERENCES

[1] D.W. DETEMPLE AND J.M. ROBERTSON, On generalized symmetric means of two variables, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, **634–672** (1979), 236–238

[2] K.V. MENON, Inequalities for symmetric functions, *Duke Math. J.*, **35** (1968), 37–45.

[3] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer, 1970.

[4] J.-Ch. KUANG, *Chángyòng Búděngshì* (*Applied Inequalities*), 2nd Ed. Hunan Education Press, Changsha, China, 1993. (Chinese)

[5] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, 2nd Ed. Cambridge Univ. Press, 1952.

[6] Zh.-G. XIAO, Zh.-H. ZHANG AND X.-N. LU, A class of inequalities for weighted symmetric mean, *J. Hunan Educational Institute*, **17**(5) (1999), 130–134. (Chinese)

[7] Zh.-H. ZHANG, Three classes of new means in $n+1$ variables and their applications, *J. Hunan Educational Institute*, **15**(5) (1997), 130–136. (Chinese)