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INCLUSION AND NEIGHBORHOOD PROPERTIES FOR CERTAIN CLASSES OF MULTIVALENTLY ANALYTIC FUNCTIONS ASSOCIATED WITH THE CONVOLUTION STRUCTURE

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ABSTRACT. Making use of the familiar convolution structure of analytic functions, in this paper we introduce and investigate two new subclasses of multivalently analytic functions of complex order. Among the various results obtained here for each of these function classes, we derive the coefficient bounds and coefficient inequalities, and inclusion and neighborhood properties, involving multivalently analytic functions belonging to the function classes introduced here.

Key words and phrases: Multivalently analytic functions, Hadamard product (or convolution), Coefficient bounds, Coefficient inequalities, Inclusion properties, Neighborhood properties.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}_p(n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \qquad (p < n;\ n, p \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

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which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

If $f \in \mathcal{A}_p(n)$ is given by (1.1) and $g \in \mathcal{A}_p(n)$ is given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k,$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined (as usual) by

$$(f * g)(z) := z^p + \sum_{k=n}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.2}$$

We denote by $\mathcal{T}_p(n)$ the subclass of $\mathcal{A}_p(n)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k \qquad \big(p < n;\ a_k \geqq 0 \ \ (k \geqq n);\ n, p \in \mathbb{N}\big), \tag{1.3}$$

which are p-valent in $\mathbb{U}$.

For a given function $g(z) \in \mathcal{A}_p(n)$ defined by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \qquad \big(p < n;\ b_k \geqq 0 \ \ (k \geqq n);\ n, p \in \mathbb{N}\big), \tag{1.4}$$

we introduce here a new class $\mathcal{S}_g(p, n, b, m)$ of functions belonging to the subclass of $\mathcal{T}_p(n)$, which consists of functions $f(z)$ of the form (1.3) satisfying the following inequality:

$$\left| \frac{1}{b} \left(\frac{z(f * g)^{(m+1)}(z)}{(f * g)^{(m)}(z)} - (p - m) \right) \right| < 1 \tag{1.5}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ p > m;\ b \in \mathbb{C} \setminus \{0\}).$$

We note that there are several interesting new or known subclasses of our function class $\mathcal{S}_g(p, n, b, m)$. For example, if we set

$$m = 0 \quad \text{and} \quad b = p(1 - \alpha) \qquad (p \in \mathbb{N};\ 0 \leqq \alpha < 1)$$

in (1.5), then $\mathcal{S}_g(p, n, b, m)$ reduces to the class studied very recently by Ali *et al.* [1]. On the other hand, if the coefficients b_k in (1.4) are chosen as follows:

$$b_k = \binom{\lambda + k - 1}{k - p} \qquad (\lambda > -p),$$

and n is replaced by $n + p$ in (1.2) and (1.3), then we obtain the class $\mathcal{H}^p_{n,m}(\lambda, b)$ of p-valently analytic functions (involving the familiar Ruscheweyh derivative operator), which was investigated by Raina and Srivastava [9]. Further, upon setting $p = 1$ and $n = 2$ in (1.2) and (1.3), if we choose the coefficients b_k in (1.4) as follows:

$$b_k = k^l \qquad (l \in \mathbb{N}_0),$$

then the class $\mathcal{S}_g(1, 2, 1 - \alpha, 0)$ would reduce to the function class $\mathcal{TS}^*_l(\alpha)$ (involving the familiar Sălăgean derivative operator [11]), which was studied in [1]. Moreover, when

$$g(z) = z^p + \sum_{k=n}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(k-p)!}\ z^k \tag{1.6}$$

$$\big(\alpha_j \in \mathbb{C} \ \ (j = 1, \ldots, q);\ \beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \ \ (j = 1, \ldots, s)\big),$$

with the parameters

$$\alpha_1, \ldots, \alpha_q \quad \text{and} \quad \beta_1, \ldots, \beta_s$$

being so chosen that the coefficients b_k in (1.4) satisfy the following condition:

$$b_k = \frac{(\alpha_1)_{k-p} \cdots (\alpha_q)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_s)_{k-p}(k-p)!} \geqq 0,$$

then the class $\mathcal{S}_g(p,n,b,m)$ transforms into a (presumably) new class $\mathcal{S}^*(p,n,b,m)$ defined by

$$\mathcal{S}^*(p,n,b,m) := \left\{ f : f \in \mathcal{T}_p(n) \text{ and } \left| \frac{1}{b} \left(\frac{z(H_s^q[\alpha_1]f)^{(m+1)}(z)}{(H_s^q[\alpha_1]f)^{(m)}(z)} - (p-m) \right) \right| < 1 \right\} \tag{1.7}$$

$$(z \in \mathbb{U};\ q \leqq s+1;\ m,q,s \in \mathbb{N}_0;\ p \in \mathbb{N};\ b \in \mathbb{C} \setminus \{0\}).$$

The operator

$$\left(H_s^q[\alpha_1]f\right)(z) := H_s^q(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)f(z),$$

involved in the definition (1.7), is the Dziok-Srivastava linear operator (see, for details, [4]; see also [5] and [6]), which contains such well-known operators as the Hohlov linear operator, Saitoh's generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized version, the Bernardi-Libera-Livingston operator, and the Srivastava-Owa fractional derivative operator. One may refer to the papers [4] to [6] for further details and references for these operators. The Dziok-Srivastava linear operator defined in [4] was further extended by Dziok and Raina [2] (see also [3] and [8]).

Following a recent investigation by Frasin and Darus [7], if $f(z) \in \mathcal{T}_p(n)$ and $\delta \geqq 0$, then we define the (q,δ)-neighborhood of the function $f(z)$ by

$$\mathcal{N}_{n,\delta}^q(f) := \left\{ h : h \in \mathcal{T}_p(n),\ h(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \text{ and } \sum_{k=n}^{\infty} k^{q+1}|a_k - c_k| \leqq \delta \right\}. \tag{1.8}$$

It follows from the definition (1.8) that, if

$$e(z) = z^p \qquad (p \in \mathbb{N}), \tag{1.9}$$

then

$$\mathcal{N}_{n,\delta}^q(e) = \left\{ h : h \in \mathcal{T}_p(n),\ h(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \text{ and } \sum_{k=n}^{\infty} k^{q+1}|c_k| \leqq \delta \right\}. \tag{1.10}$$

We observe that

$$\mathcal{N}_{2,\delta}^0(f) = \mathcal{N}_\delta(f)$$

and

$$\mathcal{N}_{2,\delta}^1(f) = \mathcal{M}_\delta(f),$$

where $\mathcal{N}_\delta(f)$ and $\mathcal{M}_\delta(f)$ denote, respectively, the δ-neighborhoods of the function

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ z \in \mathbb{U}), \tag{1.11}$$

as defined by Ruscheweyh [10] and Silverman [12].

Finally, for a given function

$$g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \in \mathcal{A}_p(n) \qquad \big(b_k > 0\ \ (k \geqq n)\big),$$

let $\mathcal{P}_g(p,n,b,m;\mu)$ denote the subclass of $\mathcal{T}_p(n)$ consisting of functions $f(z)$ of the form (1.3) which satisfy the following inequality:

$$\left|\frac{1}{b}\left[p(1-\mu)\left(\frac{(f*g)(z)}{z}\right)^{(m)}+\mu(f*g)^{(m+1)}(z)-(p-m)\right]\right|<p-m \tag{1.12}$$

$$(z\in\mathbb{U};\ m\in\mathbb{N}_0;\ p\in\mathbb{N}; p>m;\ b\in\mathbb{C}\setminus\{0\};\ \mu\geqq 0).$$

Our object in the present paper is to investigate the various properties and characteristics of functions belonging to the above-defined subclasses

$$\mathcal{S}_g(p,n,b,m) \qquad \text{and} \qquad \mathcal{P}_g(p,n,b,m;\mu)$$

of p-valently analytic functions in $\mathbb{U}$. Apart from deriving coefficient bounds and coefficient inequalities for each of these function classes, we establish several inclusion relationships involving the (n,δ)-neighborhoods of functions belonging to these subclasses.

2. Coefficient Bounds and Coefficient Inequalities

We begin by proving a necessary and sufficient condition for the function $f(z)\in\mathcal{T}_p(n)$ to be in each of the classes

$$\mathcal{S}_g(p,n,b,m) \quad \text{and} \quad \mathcal{P}_g(p,n,b,m;\mu).$$

Theorem 1. *Let $f(z)\in\mathcal{T}_p(n)$ be given by (1.3). Then $f(z)$ is in the class $\mathcal{S}_g(p,n,b,m)$ if and only if*

$$\sum_{k=n}^{\infty} a_k b_k (k-p+|b|)\binom{k}{m} \leqq |b|\binom{p}{m}. \tag{2.1}$$

Proof. Assume that $f(z)\in\mathcal{S}_g(p,n,b,m)$. Then, in view of (1.3) to (1.5), we obtain

$$\Re\left(\frac{z(f*g)^{(m+1)}(z)-(p-m)(f*g)^{(m)}(z)}{(f*g)^{(m)}(z)}\right)>-|b| \qquad (z\in\mathbb{U}),$$

which yields

$$\Re\left(\frac{\sum\limits_{k=n}^{\infty} a_k b_k (p-k)\binom{k}{m} z^{k-p}}{\binom{p}{m}-\sum\limits_{k=n}^{\infty} a_k b_k \binom{k}{m}\ z^{k-p}}\right)>-|b| \qquad (z\in\mathbb{U}). \tag{2.2}$$

Putting $z=r \quad (0\leqq r<1)$ in (2.2), the expression in the denominator on the left-hand side of (2.2) remains positive for $r=0$ and also for all $r\in(0,1)$. Hence, by letting $r\to 1-$, the inequality (2.2) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying the hypothesis (2.1) of Theorem 1, and setting $|z|=1$, we find that

$$\begin{aligned}\left|\frac{z(f*g)^{(m+1)}(z)}{(f*g)^{(m)}(z)}-(p-m)\right| &= \left|\frac{\sum\limits_{k=n}^{\infty} a_k b_k (k-p)\binom{k}{m} z^{k-m}}{\binom{p}{m} z^{p-m}-\sum\limits_{k=n}^{\infty} a_k b_k \binom{k}{m} z^{k-m}}\right| \\ &\leqq \frac{|b|\left[\binom{p}{m}-\sum\limits_{k=n}^{\infty} a_k b_k \binom{k}{m}\right]}{\binom{p}{m}-\sum\limits_{k=n}^{\infty} a_k b_k \binom{k}{m}} \\ &= |b|.\end{aligned}$$

Hence, by the *maximum modulus principle*, we infer that $f(z) \in \mathcal{S}_g(p,n,b,m)$, which completes the proof of Theorem 1. □

Remark 1. In the special case when

$$b_k = \binom{\lambda+k-1}{k-p} \qquad (\lambda > -p;\ k \geqq n;\ n,p \in \mathbb{N};\ n \mapsto n+p), \tag{2.3}$$

Theorem 1 corresponds to the result given recently by Raina and Srivastava [9, p. 3, Theorem 1]. Furthermore, if we set

$$m = 0 \quad \text{and} \quad b = p(1-\alpha) \qquad (p \in \mathbb{N};\ \ 0 \leqq \alpha < 1), \tag{2.4}$$

Theorem 1 yields a recently established result due to Ali *et al.* [1, p. 181, Theorem 1].

The following result involving the function class $\mathcal{P}_g(p,n,b,m;\mu)$ can be proved on similar lines as detailed above for Theorem 1.

Theorem 2. *Let $f(z) \in \mathcal{T}_p(n)$ be given by (1.3). Then $f(z)$ is in the class $\mathcal{P}_g(p,n,b,m;\mu)$ if and only if*

$$\sum_{k=n}^{\infty} a_k b_k [\mu(k-p)+p)] \binom{k-1}{m} \leqq (p-m) \left[\frac{|b|-1}{m!} + \binom{p}{m} \right]. \tag{2.5}$$

Remark 2. Making use of the same substitutions as mentioned above in (2.3), Theorem 2 yields the *corrected* version of another known result due to Raina and Srivastava [9, p. 4, Theorem 2].

3. INCLUSION PROPERTIES

We now establish some inclusion relationships for each of the function classes

$$\mathcal{S}_g(p,n,b,m) \quad \text{and} \quad \mathcal{P}_g(p,n,b,m;\mu)$$

involving the (n,δ)-neighborhood defined by (1.8).

Theorem 3. *If*

$$b_k \geqq b_n \quad (k \geqq n) \qquad \textit{and} \qquad \delta := \frac{n|b|\binom{p}{m}}{(n-p+|b|)\binom{n}{m} b_n} \quad (p > |b|), \tag{3.1}$$

then

$$\mathcal{S}_g(p,n,b,m) \subset \mathcal{N}^0_{n,\delta}(e). \tag{3.2}$$

Proof. Let $f(z) \in \mathcal{S}_g(p,n,b,m)$. Then, in view of the assertion (2.1) of Theorem 1, and the given condition that

$$b_k \geqq b_n \qquad (k \geqq n),$$

we get

$$(n-p+|b|)\binom{n}{m} b_n \sum_{k=n}^{\infty} a_k \leqq \sum_{k=n}^{\infty} a_k b_k (k-p+|b|) \binom{k}{m} < |b| \binom{p}{m},$$

which implies that

$$\sum_{k=n}^{\infty} a_k \leqq \frac{|b|\binom{p}{m}}{(n-p+|b|)\binom{n}{m} b_n}. \tag{3.3}$$

Applying the assertion (2.1) of Theorem 1 again, in conjunction with (3.3), we obtain

$$\begin{aligned}\binom{n}{m}b_n\sum_{k=n}^{\infty}ka_k &\leqq |b|\binom{p}{m}+(p-|b|)\binom{n}{m}b_n\sum_{k=n}^{\infty}a_k\\ &\leqq |b|\binom{p}{m}+(p-|b|)\binom{n}{m}b_n\frac{|b|\binom{p}{m}}{(n-p+|b|)\binom{n}{m}b_n}\\ &=\frac{n|b|\binom{p}{m}}{n-p+|b|}.\end{aligned}$$

Hence

$$\sum_{k=n}^{\infty}ka_k \leqq \frac{n|b|\binom{p}{m}}{(n-p+|b|)\binom{n}{m}b_n}=:\delta \qquad (p>|b|), \tag{3.4}$$

which, by virtue of (1.10), establishes the inclusion relation (3.2) of Theorem 3. □

In an analogous manner, by applying the assertion (2.5) of Theorem 2, instead of the assertion (2.1) of Theorem 1, to the functions in the class $\mathcal{P}_g(p,n,b,m;\mu)$, we can prove the following inclusion relationship.

Theorem 4. *If*

$$b_k \geqq b_n \quad (k \geqq n) \qquad \textit{and} \qquad \delta := \frac{n(p-m)\left[\frac{|b|-1}{m!}+\binom{p}{m}\right]}{[\mu(n-p)+p]\binom{n-1}{m}b_n} \quad (\mu>1), \tag{3.5}$$

then

$$\mathcal{P}_g(p,n,b,m;\mu)\subset \mathcal{N}^0_{n,\delta}(e). \tag{3.6}$$

Remark 3. Applying the parametric substitutions listed in (2.3), Theorem 3 yields a known result of Raina and Srivastava [9, p. 4, Theorem 3], while Theorem 4 would yield the *corrected* form of another known result [9, p. 5, Theorem 4].

4. NEIGHBORHOOD PROPERTIES

In this concluding section, we determine the neighborhood properties for each of the function classes

$$\mathcal{S}^{(\alpha)}_g(p,n,b,m) \quad \text{and} \quad \mathcal{P}^{(\alpha)}_g(p,n,b,m;\mu),$$

which are defined as follows.

A function $f(z)\in\mathcal{T}_p(n)$ is said to be in the class $\mathcal{S}^{(\alpha)}_g(p,n,b,m)$ if there exists a function $h(z)\in\mathcal{S}_g(p,n,b,m)$ such that

$$\left|\frac{f(z)}{h(z)}-1\right|<p-\alpha \qquad (z\in\mathbb{U};\ 0\leqq\alpha<p). \tag{4.1}$$

Analogously, a function $f(z)\in\mathcal{T}_p(n)$ is said to be in the class $\mathcal{P}^{(\alpha)}_g(p,n,b,m;\mu)$ if there exists a function $h(z)\in\mathcal{P}_g(p,n,b,m;\mu)$ such that the inequality (4.1) holds true.

Theorem 5. *If* $h(z)\in\mathcal{S}_g(p,n,b,m)$ *and*

$$\alpha = p-\frac{\delta}{n^{q+1}}\cdot\frac{(n-p+|b|)\binom{n}{m}b_n}{(n-p+|b|)\binom{n}{m}b_n-|b|\binom{n}{m}}, \tag{4.2}$$

then

$$\mathcal{N}^q_{n,\delta}(h)\subset\mathcal{S}^{(\alpha)}_g(p,n,b,m). \tag{4.3}$$

Proof. Suppose that $f(z) \in \mathcal{N}_{n,\delta}^{q}(h)$. We then find from (1.8) that

$$\sum_{k=n}^{\infty} k^{q+1}|a_k - c_k| \leqq \delta,$$

which readily implies that

$$\sum_{k=n}^{\infty} |a_k - c_k| \leqq \frac{\delta}{n^{q+1}} \qquad (n \in \mathbb{N}). \tag{4.4}$$

Next, since $h(z) \in \mathcal{S}_g(p,n,b,m)$, we find from (3.3) that

$$\sum_{k=n}^{\infty} c_k \leqq \frac{|b|\binom{p}{m}}{(n-p+|b|)\binom{n}{m}b_n}, \tag{4.5}$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{h(z)} - 1\right| &\leqq \frac{\sum_{k=n}^{\infty} |a_k - c_k|}{1 - \sum_{k=n}^{\infty} c_k} \\
&\leqq \frac{\delta}{n^{q+1}} \cdot \frac{1}{1 - \frac{|b|\binom{p}{m}}{(n-p+|b|)\binom{n}{m}b_n}} \\
&\leqq \frac{\delta}{n^{q+1}} \cdot \frac{(n-p+|b|)\binom{n}{m}b_n}{(n-p+|b|)\binom{n}{m}b_n - |b|\binom{n}{m}} \\
&= p - \alpha,
\end{aligned}$$

provided that α is given by (4.2). Thus, by the above definition, $f \in \mathcal{S}_g^{(\alpha)}(p,n,b,m)$, where α is given by (4.2). This evidently proves Theorem 5. □

The proof of Theorem 6 below is similar to that of Theorem 5 above. We, therefore, omit the details involved.

Theorem 6. *If* $h(z) \in \mathcal{P}_g(p,n,b,m;\mu)$ *and*

$$\alpha = p - \frac{\delta}{n^{q+1}} \cdot \frac{[\mu(n-p)+p]\binom{n-1}{m}b_n}{\left[[\mu(n-p)+p]\binom{n-1}{m}b_n - (p-m)\left(\frac{|b|-1}{m!} + \binom{p}{m}\right)\right]}, \tag{4.6}$$

then

$$\mathcal{N}_{n,\delta}^{q}(h) \subset \mathcal{P}_g^{(\alpha)}(p,n,b,m;\mu). \tag{4.7}$$

Remark 4. Applying the parametric substitutions listed in (2.3), Theorems 5 and 6 would yield the corresponding results of Raina and Srivastava [9, p. 6, Theorem 5 and (the *corrected* form of) Theorem 6].

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