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A DIRECT PROOF OF THE EQUIVALENCE BETWEEN THE ENTROPY SOLUTIONS OF CONSERVATION LAWS AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN ONE-SPACE VARIABLE

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ABSTRACT. We establish a direct proof of the well known equivalence between the Crandall-Lions viscosity solution of the Hamilton-Jacobi equation $w_t + H(w_x) = 0$ and the Kružkov-Vol'pert entropy solution of conservation law $u_t + H(u)_x = 0$. To reach at the purpose we work directly with defining entropy and viscosity inequalities, and using the front tracking method, and do not, as is usually done, exploit the convergence of the viscosity method.

Key words and phrases: Hamilton-Jacobi equation, Conservation law, Viscosity solution, Entropy solution, Front tracking method.

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1. INTRODUCTION

In this paper we present a direct proof of the equivalence between the unique viscosity solution [4, 2, 3] of the Hamilton-Jacobi equation of the form

$$w_t + H(w_x) = 0, \quad w(x,0) = w_0(x), \tag{1.1}$$

and the unique entropy solution [7, 13] of the conservation law of the form

$$u_t + H(u)_x = 0, \quad u(x,0) = u_0(x), \tag{1.2}$$

where $H : \mathbb{R} \to \mathbb{R}$ is a given function of class C^2 and $w_0 \in BUC(\mathbb{R})$, the space of all bounded uniformly continuous functions, and $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, the space of all integrable functions of bounded total variation. It is well known that if $u_0 = \frac{d}{dx} w_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, the solutions $u(\cdot,t) \in BV(\mathbb{R}), w(\cdot,t) \in BUC(\mathbb{R})$ of both problems are related by the transformation $u(\cdot,t) = w_x(\cdot,t)$. The usual proof in the one dimensional case of this relation exploits

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the known results about existence, uniqueness, and convergence of the viscosity method. As is usually done, the proof of this relation exploits the convergence of the viscosity method; it is known that the solutions u^ϵ, w^ϵ of

$$u_t^\epsilon + H(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \quad u^\epsilon(x,0) = u_0(x) \in L^1(\mathbb{R}) \cap BV(\mathbb{R}),$$

and

$$w_t^\epsilon + H(w_x^\epsilon) = \epsilon w_{xx}^\epsilon, \quad w^\epsilon(x,0) = w_0(x) \in BUC(\mathbb{R}),$$

converge to the entropy and viscosity solutions u, w of (1.1) and (1.2) respectively. If $w_{0x} \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and $u_0 = \frac{d}{dx} w_0$, the regularity of w^ϵ permits the relation $u^\epsilon = w_x^\epsilon$ which, after letting ϵ tend to 0 gives the desired result $u = w_x$. In this paper we are going to prove that the unique viscosity solution w of (1.1) is related to the unique entropy solution u of (1.2) by the identity $u = w_x$ - when $u_0 = \frac{d}{dx} w_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$- by a direct analysis without using the convergence of the viscosity method but instead using the defining viscosity and entropy inequalities directly. We recall that a function $w \in BUC(\mathbb{R} \times]0,T[)$ is a viscosity solution of the initial problem (1.1) if $w(x,0) = w_0(x)$ and w is simultaneously a (viscosity) sub-solution and a (viscosity) super-solution in$\mathbb{R} \times]0,T[$:

Sub-solution: For each $\varphi \in C^1(\mathbb{R} \times]0,T[)$,

$$\text{if } \; w - \varphi \text{ has a local maximum point at a point } (x_0,t_0) \in \mathbb{R} \times]0,T[,$$
$$\text{then } \varphi_t(x_0,t_0) + H(\varphi_x(x_0,t_0)) \le 0.$$

Super-solution: For each $\varphi \in C^1(\mathbb{R} \times]0,T[)$,

$$\text{if } w - \varphi \text{ has a local minimum point at a point } (x_0,t_0) \in \mathbb{R} \times]0,T[,$$
$$\text{then } \; \varphi_t(x_0,t_0) + H(\varphi_x(x_0,t_0)) \ge 0.$$

The existence, uniqueness and stability properties of the viscosity solutions were systematically studied by Kružkov, Crandall, Evans, Lions, Souganidis, and Ishii, [7, 10, 4, 2, 12, 3].

We recall that $u \in L^\infty(\mathbb{R} \times]0,T[)$ is an entropy solution of the initial problem (1.2) if: $||u(\cdot,t) - u_0(\cdot)||_{L^1_{loc}(\mathbb{R})} \to 0$ as $t \to 0$ and, for all convex entropy-entropy flux pairs $(U,F) : \mathbb{R} \to \mathbb{R}^2$ with $U'H' = F'$, we have:

$$\partial_t U(p) + \partial_x F(p) \le o \text{ in the distributional sense.}$$

In view that a continuous convex function U can be a uniform limit of a sequence of convex piece-affine functions of the form

$$U_\epsilon(x) = a_0 + a_1 + \Sigma_i a_i |x - k_i|, k_i \text{ constants } \in \mathbb{R},$$

then the convex pair (U,F) can be replaced by the Kružkov-pair [7]

$$(|\cdot - k|, \operatorname{sgn}(\cdot,k)(H(\cdot) - H(k)),$$

which is simple to manipulate. Therefore, using Kružkov-pair, the definition of the entropy solution can be presented as

$$\int_0^T \int_{\mathbb{R}} (|u-k|\varphi_t + \operatorname{sgn}(u-k)(H(u) - H(k))\varphi_x) dx dt \ge 0,$$

for all positive $\varphi \in C_c^1(\mathbb{R} \times]0,T[)$, and constants $k \in \mathbb{R}$.

For the existence, uniqueness, and stability results of the entropy solution we refer to Lax [8, 9], Vol'pert [13], and Kružkov [7].

The main purpose of the present paper is to give a direct proof of the the equivalence between viscosity solutions of the Hamilton-Jacobi equation (1.1) and entropy solutions of conservation law (1.2). There exist very few references which prove this relation without using the convergence of the viscosity method. The main result of the paper is contained in the following theorem:

Theorem 1.1. *Let w be the unique viscosity solution of the Hamilton-Jacobi equation (1.1) and let u be the unique entropy solution to the conservation law*

$$u_t + H(u)_x = 0,$$

with initial data

$$u(x,0) = \frac{d}{dx} w_0(x).$$

If $w_0 \in BUC(\mathbb{R})$, or $u(x,0) \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, then $w_x(x,t) = u(x,t)$ almost everywhere.

To show Theorem 1.1, we use the front tracking method, proposed firstly by Dafermos [5]. This is a numerical method for scalar conservation laws (1.2), which yield exact entropy solutions in the initial data u_0, is piecewise constant, and the flux function H piecewise linear. We then note that this method translates into a method that gives the exact viscosity solutions to the Hamilton-Jacobi equation (1.1) if w_0 and H are piecewise linear and Lipschitz continuous. This gives Theorem 1.1 in the case of piecewise linear/constant initial data, and piecewise linear Hamiltonians/flux functions. To extend the result to more general problems, we take the L^∞/L^1 closure of the set of the piecewise linear/constant initial data, and the Sup/Lip norm closure of the set of the piecewise linear Hamiltonians/flux functions, utilizing stability estimates from [12] and [6] for conservation laws and Hamilton-Jacobi equations respectively. Note that the front tracking method was translated to the system of conservation laws (see, e.g., [1], [11]).

The paper is organized as follows. In Section 2 we start by describing the front tracking for scalar conservation law (1.2), we treat firstly the linear case in Subsection 2.1, and in Subsection 2.2 we extend the method to more general problems. Section 3 focuses on the Hamilton-Jacobi equation (1.1), for which we translate the front tracking construction. Also we start by translating for the linear case in Subsection 3.1, Subsection 3.2 extends the construction to more general Hamiltonians. The end of Subsection 3.2 is devoted to the main result of the paper (Theorem 1.1).

2. FRONT TRACKING METHOD FOR THE SCALAR CONSERVATION LAW

2.1. **The linear case.** We start by describing front tracking for scalar conservation laws in the linear case, i.e., we assume that H is a piecewise linear continuous function and u_0 is a piecewise constant function with bounded support taking a finite number of values. To solve the initial value problem (2.1),

$$u_t + H(u)_x = 0, \quad u(x,0) = u_0(x), \tag{2.1}$$

we start by solving the Riemann problem, i.e., where u_0 is given by

$$u_0(x) = \begin{cases} u_l, & \text{for} \quad x < 0, \\ u_r, & \text{for} \quad x \geq 0. \end{cases}$$

By breakpoints of H we mean the points where H' is discontinuous. Let now $H^\smile$ be the lower convex envelope of H between u_l and u_r, i.e.,

$$H^\smile(u, u_l, u_r) = \sup\{h(u) | h"(u) \geq 0,\ h(u) \leq H(u) \text{ between } u_l \text{ and } u_r\}.$$

Let also $H^\frown$ be the upper concave envelope of H between u_l and u_r,

$$H^\frown(u, u_l, u_r) = \inf\{h(u) | h"(u) < 0,\ h(u) \geq H(u) \text{ between } u_l \text{ and } u_r\}.$$

Now set

$$H^\sharp(u,u_l,u_r)=\begin{cases} H^\smile(u,u_l,u_r), & \text{if} \quad u_l\le u_r;\\ H^\frown(u,u_l,u_r), & \text{if} \quad u_l> u_r.\end{cases}$$

Since H is assumed to be piecewise linear and continuous, $H^\sharp$ will also be linear and continuous. We suppose that $H^\sharp$ has $N-1$ breakpoints between u_l and u_r, call these $u_2,\dots,u_{N-1}$ and set $u_1=u_l$ and $u_N=u_r$, such that $u_i\le u_{i+1}$ if $u_l\le u_r$ and $u_i>u_{i+1}$ if $u_l>u_r$. We assume that $u_i\in[-M,M]$, for all $i=0,1,\dots,N$, where M is constant. Now set

$$\sigma_i=\begin{cases} -\infty, & \text{if} \quad i=0,\\ \frac{H_{i+1}-H_i}{u_{i+1}-u_i}, & \text{if} \quad i=1,\dots,N-1,\\ +\infty, & \text{if} \quad i=N,\end{cases}$$

where $H_i=H^\sharp(u_i;u_l,u_r)=H(u_i)$.
Let

$$\Omega_i=\{(x,t)|0\le t\le T, \text{ and } t\sigma_{i-1}<x\le t\sigma_i\}.$$

Then the following proposition holds:

Proposition 2.1. *Set*

$$u(x,t)=u_i \quad \textit{for } (x,t)\in\Omega_i,$$

then u is the entropy solution of the Riemann problem (2.1).

Proof. We show the proposition in the case where $u_l\le u_r$, the other case is similar. First note that the definition of the lower convex envelope implies that for $k\in[u_i,u_{i+1}]$,

$$\begin{aligned} H(k)&\ge H_i+(k-u_i)\sigma_i\\ &\ge H_{i+1}+(k-u_{i+1})\sigma_i\\ &\ge \frac12(H_{i+1}+H_i)+\left(k-\frac12(u_{i+1}+u_i)\right)\sigma_i.\end{aligned}$$

To show that u is the entropy solution desired, we have to prove that for each non-negative test function φ,

$$\begin{aligned} &-\int_{\Omega_T}\int(|u-k|\varphi_t+\operatorname{sgn}(u-k)(H(u)-H(k))\varphi_x)dxdt\\ &\qquad +\int_{\mathbb{R}}|u(x,T)-k|\varphi(x,T)-|u_0(x)-k|\varphi(x,0)dx\le 0, \end{aligned}\tag{2.2}$$

where $\Omega_T=\mathbb{R}\times[0,T]$, and $\operatorname{sgn}(u-k)=1$ if $u-k\ge 0$, and $=-1$ if $u-k<0$. The first term in (2.2) is given by

$$\begin{aligned} &-\int_{\Omega_T}\int(|u-k|\varphi_t+\operatorname{sgn}(u-k)(H(u)-H(k))\varphi_x)dxdt\\ &\quad=-\sum_{i=1}^N\int\int_{\Omega_i}|u_i-k|\varphi_t+\operatorname{sgn}(u_i-k)(H(u_i)-H(k))\varphi_x dxdt\\ &\quad=-\int_{\mathbb{R}}|u(x,T)-k|\varphi(x,T)-|u_0(x)-k|\varphi(x,0)\\ &\qquad\quad-\sum_{i=1}^{N-1}\int_0^T\{\sigma_i(|u_{i+1}-k|-|u_i-k|)-(\operatorname{sgn}(u_{i+1}-k)(H(u_{i+1})-H(k))\\ &\qquad\quad-\operatorname{sgn}(u_i-k)(H(u_i)-H(k)))\varphi(\sigma_i t,t)\}dt, \end{aligned}$$

by Green's formula applied to each Ω_i. Considering the integrand in the last term, we find that, if $k > u_{i+1}$ or $k < u_i$

$$\sigma_i(|u_{i+1} - k| - |u_i - k|) - (\operatorname{sgn}(u_{i+1} - k)(H(u_{i+1}) - H(k)) \\ - \operatorname{sgn}(u_i - k)(H(u_i) - H(k))) = 0.$$

Otherwise, we find that

$$\sigma_i(|u_{i+1} - k| - |u_i - k|) - (\operatorname{sgn}(u_{i+1} - k)(H(u_{i+1}) \\ - H(k)) - \operatorname{sgn}(u_i - k)(H(u_i) - H(k))) \\ = \frac{1}{2}(H_{i+1} + H_i) + (k - \frac{1}{2}(u_{i+1} + u_i))\sigma_i \geq 0,$$

since for $k \in [u_i, u_{i+1}]$,

$$H(k) \geq \frac{1}{2}(H_{i+1} + H_i) + \left(k - \frac{1}{2}(u_{i+1} + u_i)\right)\sigma_i.$$

This implies that u, defined in Proposition 2.1, is an entropy solution of the Riemann problem. □

For a more general initial problem, i.e., when u_0 has more than one discontinuous point, one defines a series of Riemann problems. Note that the initial value function is piecewise constant, and the construction of the solutions of this problem leads to defining the speeds $\sigma_i, i = 1, ..., N-1$, for each Riemann problem.
The solution $u(x,t)$ will be piecewise constant, with discontinuities on lines emanating from the discontinuities of u_0. These discontinuities are called fronts. In fact, the solution consists of constant states separated by these discontinuities:

$$u(x,t) = u_1, \text{ for } x < x_1(t), \\ u(x,t) = u_i, \text{ for } x_{i-1} < x < x_i, \quad i = 2, ..., N-1, \\ u(x,t) = u_N, \text{ for } x > x_{N-1}(t),$$

where each front (path of discontinuity) is given by:

$$x_i(t) = x_0 + \sigma_i(t - t_0).$$

The next proposition sums up the properties of the front tracking method.

Proposition 2.2. *Let H be a continuous and piecewise linear continuous function with a finite number of breakpoints in the interval $[-M, M]$, where M is some constant. Assume that u_0 is piecewise constant function with a finite number of discontinuities taking values in the interval $[-M, M]$. Then the initial value problem*

$$u_t + H(u)_x = 0, \quad u(x,0) = u_0(x)$$

has an entropy solution which can be constructed by front tracking. The construction solution $u(x,t)$ is a piecewise constant function of x for each t, and $u(x,t)$ takes values in finite set

$$\{u_0(x)\} \cup \{\textit{breakpoints of } H\}.$$

Furthermore, there are only a finite number of collisions between fronts in u.
If $\overline{H}$ is another piecewise linear continuous function with a finite number of breakpoints in the interval $[-M, M]$ and $\overline{u}_0$ is a piecewise constant function with a finite number of discontinuities taking values in the interval $[-M, M]$, set $\overline{u}$ to be the entropy solution to

$$\overline{u}_t + H(\overline{u})_x = 0, \quad \overline{u}(x,0) = \overline{u}_0(x).$$

If u_0 *and* $\overline{u}_0$ *are in* $L^1\mathbb{R} \cap BV(\mathbb{R})$*, then*

$$||u(\cdot,T) - \overline{u}(\cdot,T)||_{L^1(\mathbb{R})} \\ \leq ||u_0 - \overline{u}_0||_{L^1(\mathbb{R})} + T(\inf\{|u_0|_{BV(\mathbb{R})}, |\overline{u}_0|_{BV(\mathbb{R})}\})||H - \overline{H}||_{Lip([-M,M])}.$$

The proof of Proposition 2.2 can be found in [5, 6].

2.2. **The general case.** To deal with the general case, i.e, when the data of the problem is given by

$$H \in C^2 \text{ function and, } u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R}),$$

we construct a piecewise linear continuous flux H^δ, as:

$$H^\delta(u) = H(i\delta) + (u - i\delta)\frac{H((i+1)\delta) - H(i\delta)}{\delta}, \quad \text{for} \quad i\delta \leq u < (i+1)\delta. \tag{2.3}$$

Then if $\eta > \delta > 0$,

$$\begin{aligned} ||H^\eta - H^\delta||_{Lip([-M,M])} &\leq \sup_{u\in[-M,M]} |(H^\eta)'(u) - (H^\delta)'(u)| \\ &\leq \sup_{|u-v|\leq\eta} |H'(u) - H'(v)| \\ &\leq \sup_{|u-v|\leq\eta} \int_u^v |H''(r)|dr \\ &\leq ||H''||_{L^\infty([-M,M])}\eta. \end{aligned}$$

Thus $(H^\eta)_{\eta\in\mathbb{N}}$ is a Cauchy sequence (by the Lip-norm).
If furthermore, $u_0(x) \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$, set

$$u_0^h(x) = \frac{1}{h}\int_{ih}^{(i+1)h} u_0(\kappa)d\kappa, \quad \text{for} \quad ih \leq x < (i+1)h, \tag{2.4}$$

we have that,

$$\begin{aligned} ||u_0^h - u_0||_{L^1(\mathbb{R})} &= \sum_i \int_{ih}^{(i+1)h} |u_0(x) - \frac{1}{h}\int_{ih}^{(i+1)h} u_0(z)dz|dx \\ &\leq \sum_i \frac{1}{h}\int_{ih}^{(i+1)h}\int_{ih}^{(i+1)h} |u_0(x) - u_0(z)|dzdx \\ &\leq \sum_i \frac{1}{h}\int_{ih}^{(i+1)h}\int_{ih}^{(i+1)h}\int_z^x |u_0'(y)|dydzdx \\ &\leq \sum_i h\int_{ih}^{(i+1)h} |u_0'(y)|dy \\ &\leq h|u_0|_{BV(\mathbb{R})}. \end{aligned}$$

Therefore if $h \geq l > 0$,

$$\begin{aligned} ||u_0^h - u_0^l||_{L^1(\mathbb{R})} &\leq ||u_0^h - u_0||_{L^1(\mathbb{R})} + ||u_0^l - u_0||_{L^1(\mathbb{R})} \\ &\leq (h+l)|u_0|_{BV(\mathbb{R})} \leq 2h|u_0|_{BV(\mathbb{R})}. \end{aligned}$$

Proposition 2.3. *Let* $u^{\eta,h}$ *be the entropy solution to*

$$u_t^{\eta,h} + H^\eta(u^{\eta,h})_x = 0, \quad u^{\eta,h}(x,0) = u_0^h(x). \tag{2.5}$$

The sequence $(u^{\eta,h})_{\eta,h}$ *is a Cauchy sequence in* $L^1(\mathbb{R})$ *since*

$$||u^{\eta,h}(\cdot,T) - u^{\delta,l}(\cdot,T)||_{L^1(\mathbb{R})} \leq (2h + T||H''||_{L^\infty([-M,M])}\eta)|u_0|_{BV(\mathbb{R})}. \tag{2.6}$$

The proof of Proposition 2.3 can be easily deduced from Proposition 2.2. Now, using Proposition 2.3, we can define the L^1 limit

$$u = \lim_{(\eta,h)\to 0} u^{\eta,h}.$$

To prove that u is the entropy solution of the problem (1.2), we have to prove that u satisfies the entropy condition, i.e., for each test function φ non-negative in $C_c^1(\Omega_T)$, we have:

$$\begin{aligned} (2.7)\quad -\int_{\Omega_T}\int (|u-k|\varphi_t + \operatorname{sgn}(u-k)(H(u)-H(k))\varphi_x)dxdt \\ + \int_{\mathbb{R}} |u(x,T)-k|\varphi(x,T) - |u_0(x)-k|\varphi(x,0)dx \leq 0. \end{aligned}$$

For the linear case we have:

$$\begin{aligned} (2.8)\quad -\int_{\Omega_T}\int (|u^{\eta,h}-k|\varphi_t + \operatorname{sgn}(u^{\eta,h}-k)(H^\eta(u^{\eta,h})-H^\eta(k))\varphi_x)dxdt \\ + \int_{\mathbb{R}} |u^{\eta,h}(x,T)-k|\varphi(x,T) - |u_0^h(x)-k|\varphi(x,0)dx \leq 0. \end{aligned}$$

Since $|u^{\eta,h} - k| \to |u-k|$ and $H^\eta \to H$, then it easily follows that the limit function u is an entropy solution to

$$u_t + H(u)_x = 0,\ \ u(x,0) = u_0(x).$$

In the next section we will describe how the front tracking construction translates to the Hamilton-Jacobi equation (1.1).

3. FRONT TRACKING METHOD FOR THE HAMILTON-JACOBI EQUATIONS

3.1. **The linear case.** We deal now with the Hamilton-Jacobi equation when the data of the problem (1.1) is linear. Now set

$$w_t + H(w_x) = 0, \quad w(x,0) = w_0(x). \tag{3.1}$$

We assume that H is piecewise linear and continuous, and w_0 is also piecewise linear and continuous, i.e., $\frac{\partial}{\partial x}w_0$ is bounded and piecewise constant.

First we study the Riemann problem for (3.1) which is the initial value problem

$$w_0(x) = w_0(0) + \begin{cases} u_l x, & \text{for} \quad x < 0, \\ u_r x, & \text{for} \quad x \geq 0. \end{cases}$$

where u_l and u_r are constants, c.f. (2.3). Now let $u(x,t)$ denote the entropy solution of the corresponding Riemann problem for the conservation law (2.1). In the linear case, using the Hopf-Lax formula [10], the viscosity solution of (3.1) is given by

$$w(x,t) = w_0(0) + xu(x,t) - tH(u(x,t)). \tag{3.2}$$

Note that in the case where H is convex, this formula can be derived from the Hopf-Lax formula for the solution (3.1). Also note that $(H^\sharp)'(u)$ is monotone between u_l and u_r, hence we can define its inverse and set

$$u(x,t) = \begin{cases} u_l & \text{for} \quad x < t\min((H^\sharp)'(u_l),(H^\sharp)'(u_r)), \\ ((H^\sharp)')^{-1}(x/t), & \text{for} \quad t\min((H^\sharp)'(u_l),(H^\sharp)'(u_r)) \le x, \\ ((H^\sharp)')^{-1}(x/t), & \text{for} \quad x < t\max((H^\sharp)'(u_l),(H^\sharp)'(u_r)), \\ u_r & \text{for} \quad x \ge t\max((H^\sharp)'(u_l),(H^\sharp)'(u_r)). \end{cases}$$

Although u is discontinuous, a closer look at the formula (3.2) reveals that w is uniformly continuous. Indeed, for fixed t, $w(x,t)$ is piecewise linear in x, with breakpoints located at the fronts in u. Hence, when computing w, one only needs to keep a record of how w changes at the fronts. Along a front with speed σ_i, w is given by

$$w(\sigma_i t,t) = w_0(0) + t(\sigma_i u_i - H(u_i)) = w_0(0) + t(\sigma_i u_{i+1} - H(u_{i+1})). \tag{3.3}$$

Now we can use the front tracking construction for conservation laws to define a solution to the general initial value problem (3.1). We track the fronts as for the conservation law, but update w along each front by (3.3). Note that if for some (x,t), $w(x,t)$ is determined by the solution of the Riemann problem at (x_i,t_j), then

$$w(x,t) = w(x_j,t_j) + (x-x_j)u(x,t) - (t-t_j)H(u(x,t)), \tag{3.4}$$

where u is the solution of the initial value problem for (2.1) with the initial values given by

$$u(x,0) = \frac{d}{dx}w_0(x).$$

Analogously to Proposition 2.3 we have:

Proposition 3.1. *The piecewise linear function $w(x,t)$ is the viscosity solution of (3.1). Furthermore $w(x,t)$ is piecewise linear on a finite number of polygons in $\mathbb{R}\times\mathbb{R}_0^+$. If w_0 is bounded and uniformly continuous (BUC), then $w \in BUC(\mathbb{R}\times[0,T])$ for any $T<\infty$. If $\overline{H}$ is another Lipschitz continuous piecewise linear function with a finite number of breakpoints, and $\overline{w}$ is the viscosity solution of*

$$\overline{w}_t + \overline{H}(\overline{w}_x) = 0, \quad \overline{w}(x,0) = \overline{w}_0(x),$$

and w_0 and $\overline{w}_0$ are bounded and uniformly continuous (BUC), then

$$||w(\cdot,T) - \overline{w}(\cdot,T)||_{L^\infty(\mathbb{R})} \le ||w_0 - \overline{w}_0||_{L^\infty(\mathbb{R})} + T\sup_{|u|\le M}|\overline{H}(u) - H(u)|, \tag{3.5}$$

where $M = \min(||w_{0x}||, ||\overline{w}_{0x}||)$.

Proof. We first show that w is a viscosity solution. We have that w is determined by the solution of a finite number of Riemann problems at the points $(\overline{x}_i,\overline{t}_j)$. Given a point (x,t) in where $t>0$, we can find a j such that $w(x,t)$ is determined by the Riemann problem solved at $(\overline{x}_i,\overline{t}_j)$.

Set $u = w_x$. Let φ be a C^1-function, assume that (x_0,t_0) is the maximum point of $w-\varphi$. Since w is piecewise linear, we can define the following limits

$$\lim_{x\to x_0^-} w_x(x,t_0) - \varphi_x(x_0,t_0) \ge 0, \qquad \lim_{x\to x_0^+} w_x(x,t_0) - \varphi_x(x_0,t_0) \le 0.$$

Or

$$u_l \le \varphi_x(x_0,t_0) \le u_r, \tag{3.6}$$

where $u_{l,r} = \lim_{x\to x_0^\pm} u(x,t_0^-)$. Set

$$\sigma = \begin{cases} \frac{H(u_l)-H(u_r)}{u_l-u_r} & \text{if} \quad u_l \ne u_r, \\ H^{\sharp'}(u_l), & \text{if} \quad u_l = u_r. \end{cases}$$

Since $u_l \le \varphi_x(x_0,t_0) \le u_r$, the construction of $H^\sharp$ implies that

$$H(\varphi_x(x_0,t_0)) \ge H(u_l) + \sigma(\varphi_x(x_0,t_0) - u_l). \tag{3.7}$$

Now choose (x,t) sufficiently close to (x_0,t_0) such that

$$\sigma = \frac{x_0 - x}{t_0 - t}$$

and $w(x,t)$ is also determined by the solution of the Riemann problem at $(\overline{x}_j, \overline{t}_j)$, and $t < t_0$. **If** $t_0 > 0$, we have:

$$\frac{1}{t_0 - t}(w(x_0,t_0) - w(x,t)) \ge \frac{1}{t_0 - t}(\varphi(x_0,t_0) - \varphi(x,t)). \tag{3.8}$$

Using (3.4) we have that

$$w(x_0,t_0) = w(x,t) + (x_0 - x)u_l - (t_0 - t)H(u_l).$$

Hence, by letting $t \to t_{0-}$, we find that

$$\begin{aligned}\sigma u_l - H(u_l) &\ge \varphi_t(x_0,t_0) + \sigma\varphi_x(x_0,t_0)\\ &\ge \varphi_t(x_0,t_0) + H(\varphi_x(x_0,t_0)) + \sigma u_l - H(u_l),\end{aligned}$$

which implies that w is a sub-solution. A similar argument is applied to show that w is super-solution.

If $t_0 = 0$, assume that $(x_0, 0)$ is a maximum point of $w - \varphi$. Set $u_{l,r} = \lim_{x \to x_0 \mp} u(x, 0+)$. Then

$$w(x,t) = w(x_0,0) + (x - x_0)u_l - tH(u_l),$$

where $\sigma = (x - x_0)/t$ and (x,t) is sufficiently close to $(x_0, 0)$. Now, using (3.7) as before gives the conclusion. Note that this also demonstrates the solution of the Riemann problem (3.1).

Next we show the stability estimate (3.5). This is a consequence of Proposition 1.4 in [12], which in our context says that

$$\begin{aligned}&\sup_{(x,y)\in D_\epsilon} \{|w(x,t) - \overline{w}(y,t)| + 3R\beta_\epsilon(x-y)\}\\ &\qquad\le \sup_{(x,y)\in D_\epsilon} \{|w_0(x) - \overline{w}_0(y)| + 3R\beta_\epsilon(x-y)\} + t \sup_{|u|\le M} |H(u) - \overline{H}(u)|,\end{aligned}$$

where $\beta_\epsilon(x-y) = \beta(x/\epsilon)$ for some C_c^∞ function $\beta(x)$ with $\beta(0) = 1$ and $\beta(x) = 0$ for $|x| > 1$. Furthermore, $R = \max(||w||, ||\overline{w}||)$. Consequently,

$$\begin{aligned}&||w(\cdot,t) - \overline{w}(\cdot,t)||_{L^\infty(\mathbb{R})} + \sup_{(x,y)\in D_\epsilon} \{3R\beta_\epsilon(x-y) - |\overline{w}(x,t) - \overline{w}(y,t)\}\\ &\qquad\le ||w_0 - w_0^h||_{L^\infty(\mathbb{R})} + 3R + t \sup_{|u|\le M} |H(u) - \overline{H}(u)|.\end{aligned}$$

The inequality of the lemma now follows by noting that $\overline{w}$ is in $BUC(\mathbb{R} \times [0,T])$, and taking the limit as $\epsilon \to 0$ on the left side. □

Now we are able to explicitly construct a viscosity solution to all problems of the type (3.1) where H and u_0 are piecewise linear and Lipschitz continuous with a finite number of break-points. In the next subsection we extend the result to the more general case.

3.2. **The general case.** Now we pass to the general case. We assume that

$$H \in C^2 \text{ and } w_0 \in BUC(\mathbb{R}).$$

First, we construct a piecewise linear continuous Hamiltonian H^δ defined as follows:

$$H^\delta(u) = H(i\delta) + (u - i\delta)\frac{H((i+1)\delta) - H(i\delta)}{\delta}, \quad \text{for } i\delta \le u < (i+1)\delta. \tag{3.9}$$

and let

$$w_0^h = w_0(ih) + (x - ih)\frac{w_0((i+1)h) - w_0(ih)}{h}, \quad \text{for } ih \le x < (i+1)h. \tag{3.10}$$

Set $w^{\delta,h}$ to be the viscosity solution of

$$w_t^{\delta,h} + H^\delta(w_x^{\delta,h}) = 0, \; w^{\delta,h}(x,0) = w_0^h(x).$$

Then for $\eta > \delta > 0$ and $h > l > 0$,

$$\begin{aligned} ||w^{\delta,h}(\cdot,T) - w^{\eta,l}(\cdot,T)||_{L^\infty(\mathbb{R})} &\le ||w_0^h - w_0^l||_{L^\infty(\mathbb{R})} + T \sup_{|u|\le M} \left|H^\delta(u) - H^\eta(u)\right| \\ &\le h||w_0||_{Lip} + \eta||H||_{Lip}. \end{aligned}$$

Thus, the sequence $w^{\delta,h}$ is a Cauchy sequence in L^∞. Since H^δ converges uniformly to H on $[-M, M]$, we can use the stability result of the Hamiltonians in [3] to conclude that

$$w(x,t) = \lim_{(\delta,h)\to 0} w^{\delta,h}(x,t)$$

is a viscosity solution of

$$w_t + H(w_x) = 0, \quad w(x,0) = w_0(x). \tag{3.11}$$

Now we can state the main result.

Theorem 3.2. *Let w be the unique viscosity solution of the Hamilton-Jacobi equation (3.11), where w_0 is in $BUC(\mathbb{R})$, and let u be the unique entropy solution to the conservation law*

$$u_t + H(u)_x = 0, \quad u(x,0) = u_0(x), \tag{3.12}$$

with initial data

$$u_0(x) = \frac{d}{dx}w_0(x).$$

Then for $t > 0$, $w_x(x,t) = u(x,t)$ almost everywhere.

Proof. Fix z, by construction we have that

$$w^{\delta,h}(x,t) = w^{\delta,h}(z,t) + \int_z^x u^{\delta,h}(y,t)dy$$

as $(\delta, h) \to 0$, we have

$$\begin{aligned} w^{\delta,h}(x,t) &\to w(x,t), \\ w^{\delta,h}(z,t) &\to w(z,t), \\ u^{\delta,h}(y,t) &\to u(y,t), \end{aligned}$$

by the Lebesgue convergence theorem. Hence the theorem holds. □

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