A NOTE ON THE MAGNITUDE OF WALSH FOURIER COEFFICIENTS

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Abstract: In this note, the order of magnitude of Walsh Fourier coefficients for functions of the classes $BV^{(p)}(p \geq 1)$, ϕBV, $\Lambda BV^{(p)}$ $(p \geq 1)$ and $\phi\Lambda BV$ is studied. For the classes $BV^{(p)}$ and ϕBV, Taibleson-like technique for Walsh Fourier coefficients is developed.
However, for the classes $\Lambda BV^{(p)}$ and $\phi\Lambda BV$ this technique seems to be not working and hence classical technique is applied. In the case of ΛBV, it is also shown that the result is best possible in a certain sense.

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1. Introduction

It appears that while the study of the order of magnitude of the trigonometric Fourier coefficients for the functions of various classes of generalized variations such as $BV^{(p)}$ $(p \geq 1)$ [9], ϕBV [2], ΛBV [8], $\Lambda BV^{(p)}$ $(p \geq 1)$ [5], $\phi\Lambda BV$ [4], etc. has been carried out, such a study for the Walsh Fourier coefficients has not yet been done. The only result available is due to N.J. Fine [1], who proves, using the second mean value theorem that, if $f \in BV[0,1]$ then its Walsh Fourier coefficients $\hat{f}(n) = O(\frac{1}{n})$. In this note we carry out this study. Interestingly, here, no use of the second mean value theorem is made. We also prove that for the class ΛBV, our result is best possible in a certain sense.

Definition 1.1. *Let* $I = [a,b]$, $p \geq 1$ *be a real number,* $\{\lambda_k\}$, $k \in \mathbb{N}$, *be a sequence of non-decreasing positive real numbers such that* $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ *diverges and* $\phi : [0,\infty) \to \mathbb{R}$, *be a strictly increasing function. We say that:*

1. $f \in BV^{(p)}(I)$ *(that is,* f *is of* $p-$*bounded variation over* I*) if*

$$V(f,p,I) = \sup_{\{I_k\}} \left\{ \left(\sum_k |f(b_k) - f(a_k)|^p \right)^{\frac{1}{p}} \right\} < \infty,$$

2. $f \in \phi BV(I)$ *(that is,* f *is of* $\phi-$ *bounded variation over* I*) if*

$$V(f,\phi,I) = \sup_{\{I_k\}} \left\{ \sum_k \phi(|f(b_k) - f(a_k)|) \right\} < \infty,$$

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3. *$f \in \Lambda BV^{(p)}(I)$ (that is, f is of $p - \Lambda-$ bounded variation over I) if*

$$V_\Lambda(f,p,I) = \sup_{\{I_k\}} \left\{ \left(\sum_k \frac{|f(b_k) - f(a_k)|^p}{\lambda_k} \right)^{\frac{1}{p}} \right\} < \infty,$$

4. *$f \in \phi\Lambda BV(I)$ (that is, f is of $\phi - \Lambda-$ bounded variation over I) if*

$$V_\Lambda(f,\phi,I) = \sup_{\{I_k\}} \left\{ \sum_k \frac{\phi(|f(b_k) - f(a_k)|)}{\lambda_k} \right\} < \infty,$$

in which $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of I.

In (2) and (4), it is customary to consider ϕ a convex function such that

$$\phi(0) = 0, \qquad \frac{\phi(x)}{x} \to 0 \quad (x \to 0_+), \qquad \frac{\phi(x)}{x} \to \infty \quad (x \to \infty);$$

such a function is necessarily continuous and strictly increasing on $[0, \infty)$.

Let $\{\varphi_n\}$ $(n = 0, 1, 2, 3, \ldots)$ denote the complete orthonormal Walsh system [7], where the subscript denotes the number of zeros (that is, sign-changes) in the interior of the interval $[0, 1]$. For a 1-periodic f in $L[0, 1]$ its Walsh Fourier series is given by

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)\varphi_n(x),$$

where the n^{th} Walsh Fourier coefficient $\hat{f}(n)$ is given by

$$\hat{f}(n) = \int_0^1 f(x)\varphi_n(x)dx \qquad (n = 0, 1, 2, 3, \ldots).$$

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The Walsh system can be realized [1] as the full set of characters of the dyadic group $G = Z_2^{\infty}$, in which $Z_2 = \{0, 1\}$ is the group under addition modulo 2. We denote the operation of G by $\dot{+}$. $(G, \dot{+})$ is identified with $([0, 1], +)$ under the usual convention for the binary expansion of elements of $[0, 1]$ [1].

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2. Results

We prove the following theorems. In Theorem 2.5 it is shown that Theorem 2.3 with $p = 1$ is best possible in a certain sense.

Theorem 2.1. *If* $f \in BV^{(p)}[0,1]$ *then* $\hat{f}(n) = O\left(1 \Big/ \left(n^{\frac{1}{p}}\right)\right)$.

Note. Theorem 2.1 with $p = 1$ gives the result of Fine [1, Theorem VI].

Theorem 2.2. *If* $f \in \phi BV[0,1]$ *then* $\hat{f}(n) = O(\phi^{-1}(1/n))$.

Theorem 2.3. *If* $1-$*periodic* $f \in \Lambda BV^{(p)}[0,1]$ $(p \geq 1)$ *then*

$$\hat{f}(n) = O\left(1 \Bigg/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)^{\frac{1}{p}}\right).$$

Theorem 2.4. *If* $1-$*periodic* $f \in \phi\Lambda BV[0,1]$ *then*

$$\hat{f}(n) = O\left(\phi^{-1}\left(1 \Bigg/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)\right)\right).$$

Theorem 2.5. *If* $\Gamma BV[0,1] \supseteq \Lambda BV[0,1]$ *properly then*

$$\exists f \in \Gamma BV[0,1] \ni \hat{f}(n) \neq O\left(1 \Bigg/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)\right).$$

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Proof of Theorem 2.1. Let $n \in \mathbb{N}$. Let $k \in \mathbb{N} \cup \{0\}$ be such that $2^k \le n < 2^{k+1}$ and put $a_i = (i/2^k)$ for $i = 0, 1, 2, 3, \dots, 2^k$. Since φ_n takes the value 1 on one half of each of the intervals $(a_{i-1},\ a_i)$ and the value -1 on the other half, we have

$$\int_{a_{i-1}}^{a_i} \varphi_n(x)dx = 0, \quad \text{for all } i = 1, 2, 3, \dots, 2^k.$$

Define a step function g by $g(x) = f(a_{i-1})$ on $[a_{i-1},\ a_i)$, $i = 1, 2, 3, \dots, 2^k$. Then

$$\int_0^1 g(x)\varphi_n(x)dx = \sum_{i=1}^{2^k} f(a_{i-1}) \int_{a_{i-1}}^{a_i} \varphi_n(x)dx = 0.$$

Therefore,

$$\begin{aligned} |\hat{f}(n)| &= \left| \int_0^1 [f(x) - g(x)]\varphi_n(x)dx \right| \\ &\le \int_0^1 |f(x) - g(x)|dx \\ &\le ||f - g||_p ||1||_q \\ &= \left(\sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx \right)^{\frac{1}{p}} \end{aligned} \tag{2.1}$$

by Hölder's inequality as $f,\ g \in BV^{(p)}[0,1]$ and $BV^{(p)}[0,1] \subset L^p[0,1]$.

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Hence,

$$
\begin{aligned}
|\hat{f}(n)|^p &\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} |f(x) - f(a_{i-1})|^p dx. \\
&\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} (V(f, p, [a_{i-1}, a_i]))^p dx \\
&= \sum_{i=1}^{2^k} (V(f, p, [a_{i-1}, a_i]))^p \left(\frac{1}{2^k}\right) \\
&\leq \left(\frac{1}{2^k}\right) (V(f, p, [0, 1]))^p \\
&\leq \left(\frac{2}{n}\right) (V(f, p, [0, 1]))^p,
\end{aligned}
$$

which completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Let $c > 0$. Using Jensen's inequality and proceeding as in Theorem 2.1, we get

$$
\begin{aligned}
\phi\left(c \int_0^1 |f(x) - g(x)| dx\right) &\leq \int_0^1 \phi(c|f(x) - g(x)|) dx \\
&= \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} \phi(c|f(x) - f(a_{i-1})|) dx \\
&\leq \sum_{i=1}^{2^k} \int_{a_{i-1}}^{a_i} V(cf, \phi, [a_{i-1}, a_i]) dx
\end{aligned}
$$

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$$= \sum_{i=1}^{2^k} V(cf, \phi, [a_{i-1}, a_i]) \left(\frac{1}{2^k}\right)$$
$$\leq \left(\frac{2}{n}\right) V(cf, \phi, [0,1]).$$

Since ϕ is convex and $\phi(0) = 0$, for sufficiently small $c \in (0,1)$, $V(cf, \phi, [0,1]) < 1/2$.

This completes the proof of Theorem 2.2 in view of (2.1). □

Remark 1. If $\phi(x) = x^p$, $p \geq 1$, then the class ϕBV coincides with the class $BV^{(p)}$ and Theorem 2.2 with Theorem 2.1.

Remark 2. Note that in the proof of Theorems 2.1 and 2.2, we have used the fact that if $a = a_0 < a_1 < \cdots < a_n = b$, then

$$\sum_{i=1}^{n} (V(f, p, [a_{i-1}, a_i]))^p \leq (V(f, p, [a, b]))^p$$

and

$$\sum_{i=1}^{n} V(f, \phi, [a_{i-1}, a_i]) \leq V(f, \phi, [a, b]),$$

for any $n \geq 2$ (see [2, 1.17, p. 15]). Such inequalities for functions of the class $\Lambda BV^{(p)}$ $(p \geq 1)$ (resp., $\phi\Lambda BV$), which contain $BV^{(p)}$ (resp., ϕBV) properly, do not hold true.

In fact, the following proposition shows that the validity of such inequalities for the class $\Lambda BV^{(p)}$ (resp., $\phi\Lambda BV$) virtually reduces the class to $BV^{(p)}$ (resp., ϕBV). Hence we prove Theorem 2.3 and Theorem 2.4 applying a technique different from the Taibleson-like technique [6] which we have applied in proving Theorem 2.1 and Theorem 2.2.

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Proposition 2.6. *Let $f \in \phi\Lambda BV[a,b]$. If there is a constant C such that*

$$\sum_{i=1}^{n} V_\Lambda(f,\phi,[a_{i-1},a_i]) \le C V_\Lambda(f,\phi,[a,b])),$$

for any sequence of points $\{a_i\}_{i=0}^n$ with $a = a_0 < a_1 < \cdots < a_n = b$, then $f \in \phi BV[a,b]$.

Proof. For any partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$, we have

$$\begin{aligned}\sum_{i=1}^{n} \phi(|f(x_i) - f(x_{i-1})|) &= \lambda_1 \sum_{i=1}^{n} \frac{\phi(|f(x_i) - f(x_{i-1})|)}{\lambda_1} \\ &\le \lambda_1 \sum_{i=1}^{n} V_\Lambda(f,\phi,[x_{i-1},x_i]) \\ &\le \lambda_1 C V_\Lambda(f,\phi,[a,b]),\end{aligned}$$

which shows that $f \in \phi BV[a,b]$. □

Remark 3. $\phi(x) = x^p$ $(p \ge 1)$ in this proposition will give an analogous result for $\Lambda BV^{(p)}$.

To prove Theorem 2.3 and Theorem 2.4, we need the following lemma.

Lemma 2.7. *For any $n \in \mathbb{N}$, $|\hat{f}(n)| \le \omega_p(1/n; f)$, where $\omega_p(\delta; f)$ $(\delta > 0,\ p \ge 1)$ denotes the integral modulus of continuity of order p of f given by*

$$\omega_p(\delta; f) = \sup_{|h| \le \delta} \left(\int_0^1 |f(x+h) - f(x)|^p dx \right)^{\frac{1}{p}}.$$

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Proof. The inequality [1, Theorem IV, p. 382] $|\hat{f}(n)| \le \omega_1(1/n; f)$ and the fact that $\omega_1(1/n; f) \le \omega_p(1/n; f)$ for $p \ge 1$ immediately proves the lemma. □

Proof of Theorem 2.3. For any $n \in \mathbb{N}$, put $\theta_n = \sum_{j=1}^{n} 1/\lambda_j$. Let $f \in \Lambda BV^{(p)}[0, 1]$. For $0 < h \le 1/n$, put $k = [1/h]$. Then for a given $x \in \mathbb{R}$, all the points $x + jh,\ j = 0, 1, \dots, k$ lie in the interval $[x, x + 1]$ of length 1 and

$$\int_0^1 |f(x) - f(x+h)|^p dx = \int_0^1 |f_j(x)|^p dx, \qquad j = 1, 2, \dots, k,$$

where $f_j(x) = f(x + (j-1)h) - f(x + jh)$, for all $j = 1, 2, \dots, k$. Since the left hand side of this equation is independent of j, multiplying both sides by $1/(\lambda_j \theta_k)$ and summing over $j = 1, 2, \dots, k$, we get

$$\begin{aligned}
\int_0^1 |f(x) - f(x+h)|^p dx &\le \left(\frac{1}{\theta_k}\right) \int_0^1 \sum_{j=1}^{k} \left(\frac{|f_j(x)|^p}{\lambda_j}\right) dx \\
&\le \frac{(V_\Lambda(f, p, [0, 1]))^p}{\theta_k} \\
&\le \frac{(V_\Lambda(f, p, [0, 1]))^p}{\theta_n},
\end{aligned}$$

because $\{\lambda_j\}$ is non-decreasing and $0 < h \le 1/n$. The case $-1/n \le h < 0$ is similar and we get using Lemma 2.7,

$$|\hat{f}(n)|^p \le (\omega_p(1/n; f))^p \le \frac{(V_\Lambda(f, p, [0, 1]))^p}{\theta_n}.$$

This proves Theorem 2.3. □

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Proof of Theorem 2.4. Let $f \in \phi\Lambda BV[0,1]$. Then for h, k and $f_j(x)$ as in the proof of Theorem 2.3 and for $c > 0$ by Jensen's inequality,

$$\phi\left(c\int_0^1 |f(x)-f(x+h)|)dx\right) \le \int_0^1 \phi(c|f(x)-f(x+h)|)dx$$
$$= \int_0^1 \phi(c|f_j(x)|)dx, \quad j = 1,2,\dots,k.$$

Multiplying both sides by $1/(\lambda_j\theta_k)$ and summing over $j = 1,2,\dots,k$, we get

$$\phi\left(c\int_0^1 |f(x)-f(x+h)|)dx\right) \le \left(\frac{1}{\theta_k}\right)\int_0^1 \sum_{j=1}^{k}\left(\frac{\phi(c|f_j(x)|)}{\lambda_j}\right)dx$$
$$\le \frac{V_\Lambda(cf,\phi,[0,1])}{\theta_k}$$
$$\le \frac{V_\Lambda(cf,\phi,[0,1])}{\theta_n}.$$

Since ϕ is convex and $\phi(0) = 0$, $\phi(\alpha x) \le \alpha\phi(x)$ for $0 < \alpha < 1$. So we may choose c sufficiently small so that $V_\Lambda(cf,\phi,[0,1]) \le 1$. But then we have

$$\int_0^1 |f(x)-f(x+h)|)dx \le \frac{1}{c}\phi^{-1}\left(\frac{1}{\theta_n}\right).$$

Thus it follows in view of Lemma 2.7 that

$$|\hat{f}(n)| \le \omega_1(1/n;f) \le \frac{1}{c}\phi^{-1}\left(\frac{1}{\theta_n}\right),$$

which proves Theorem 2.4. □

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Proof of Theorem 2.5. It is known [3] that if ΓBV contains ΛBV properly with $\Gamma = \{\gamma_n\}$ then $\theta_n \neq O(\rho_n)$, where $\rho_n = \sum_{j=1}^{n} \frac{1}{\gamma_j}$ for each n. Also, if $c_0 = 0$, $c_{n+1} = 1$ and $c_1 < c_2 < \cdots < c_n$ denote all the n points of $(0,1)$ where the function φ_n changes its sign in $(0,1)$, $n_0 \in \mathbb{N}$ is such that $\rho_n \geq \frac{1}{2}$ for all $n \geq n_0$ and $E = \{n \in \mathbb{N} : n \geq n_0 \text{ is even}\}$, then for each $n \in E$, for the function

$$f_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{4\rho_n} \chi_{[c_{k-1}, c_k)}$$

extended 1-periodically on $\mathbb{R}$,

$$V_\Gamma(f_n, [0,1]) = \sum_{k=1}^{n+1} \frac{|f_n(c_k) - f_n(c_{k-1})|}{\gamma_k} = \sum_{k=1}^{n} \frac{1}{\gamma_k} \cdot \frac{1}{2\rho_n} = \frac{1}{2}$$

because

$$f_n(c_{n+1}) = f_n(1) = f_n(0) = \frac{1}{4\rho_n} = f_n(c_n)$$

as $\varphi_n \equiv 1$ on $[c_0, c_1)$. Hence $||f_n|| = \frac{1}{4\rho_n} + \frac{1}{2} \leq 1$ for each $n \in E$ in the Banach space $\Gamma BV[0,1]$ with $||f|| = |f(0)| + V_\Gamma(f, [0,1])$. Observe that for $f \in \Gamma BV[0,1]$

$$||f||_1 \leq \int_0^1 \left(\frac{|f(x) - f(0)|}{\gamma_1} \gamma_1 + |f(0)| \right) dx \leq C||f||, \qquad C = \max\{1, \gamma_1\},$$

and hence, for each $n \in \mathbb{N}$ the linear map $T_n : \Gamma BV[0,1] \to \mathbb{R}$ defined by $T_n(f) = \theta_n \hat{f}(n)$ is bounded as

$$|T_n(f)| = \theta_n |\hat{f}(n)| \leq \theta_n ||f||_1 \leq \theta_n C||f||, \qquad \forall f \in \Gamma BV[0,1].$$

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Next, for each $n \in E$ since $f_n \cdot \varphi_n = \frac{1}{4\rho_n}$ on $[0,1)$, we see that

$$T_n(f_n) = \theta_n \hat{f}_n(n) = \theta_n \int_0^1 f_n(x)\varphi_n(x)dx = \frac{1}{4}\left(\frac{\theta_n}{\rho_n}\right) \neq O(1)$$

and hence

$$\sup\{||T_n|| : n \in \mathbb{N}\} \geq \sup\{||T_n|| : n \in E\} \geq \sup\{|T_n(f_n)| : n \in E\} = \infty.$$

Therefore, an application of the Banach-Steinhaus theorem gives an $f \in \Gamma BV[0,1]$ such that $\sup\{|T_n(f)| : n \in \mathbb{N}\} = \infty$. It follows that $\theta_n \hat{f}(n) = T_n(f) \neq O(1)$ and hence the theorem is proved. □

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