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# TWO NEW ALGEBRAIC INEQUALITIES WITH $2n$ VARIABLES

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*Key words:* Algebraic inequality, Cauchy's inequality, Combinatorial identity, Algebraic identity.

*Abstract:* In this paper, by proving a combinatorial identity and an algebraic identity and by using Cauchy's inequality, two new algebraic inequalities involving  $2n$  positive variables are established.

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**Algebraic Inequalities with  $2n$   
Variables**

Xiao-Guang Chu, Cheng-En Zhang  
and Feng Qi

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## 1. Main results

When solving Question CIQ-103 in [2] and Question CIQ-142 in [5], the following two algebraic inequalities involving  $2n$  variables were posed.

**Theorem 1.1.** Let  $n \geq 2$  and  $x_i$  for  $1 \leq i \leq 2n$  be positive real numbers. Then

$$(1.1) \quad \sum_{i=1}^{2n} \frac{x_i^{2n-1}}{\sum_{k \neq i}^{2n} (x_i + x_k)^{2n-1}} \geq \frac{n}{2^{2n-2}(2n-1)}.$$

Equality in (1.1) holds if and only if  $x_i = x_j$  for all  $1 \leq i, j \leq 2n$ .

**Theorem 1.2.** Let  $n \geq 2$  and  $y_i$  for  $1 \leq i \leq 2n$  be positive real numbers. Then

$$(1.2) \quad \sum_{i=1}^{2n} \frac{y_i^2}{y_{i-1|2n} \sum_{k=i}^{i+n-2} y_{k|2n}} \geq \frac{2n}{n-1},$$

where  $m|2n$  means  $m \bmod 2n$  for all nonnegative integers  $m$ . Equality in (1.2) holds if and only if  $y_i = y_j$  for all  $1 \leq i, j \leq 2n$ .

The notation  $\sum_{k=i}^{i+n-2} y_{k|2n}$  in Theorem 1.2 could be illustrated with an example to clarify the meaning: If  $n = 5$  then  $\sum_{k=9}^{12} y_{k|10} = y_9 + y_{10} + y_1 + y_2$ .

In this article, by proving a combinatorial identity and an algebraic identity and by using Cauchy's inequality, these two algebraic inequalities (1.1) and (1.2) involving  $2n$  positive variables are proved.

Moreover, as a by-product of Theorem 1.1, the following inequality is deduced.

**Theorem 1.3.** For  $n \geq 2$  and  $1 \leq k \leq n-1$ ,

$$(1.3) \quad \sum_{p=1}^k p(p+1) \binom{2n}{k-p} < \frac{2^{2(n-1)} k(k+1)}{n}.$$

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## 2. Two Lemmas

In order to prove inequalities (1.1) and (1.2), the following two lemmas are necessary.

**Lemma 2.1.** *Let  $n$  and  $k$  be natural numbers such that  $n > k$ . Then*

$$(2.1) \quad \sum_{k=0}^{n-1} (n-k)^2 \binom{2n}{k} = 4^{n-1} n.$$

*Proof.* It is well known that

$$\binom{n}{k} = \binom{n}{n-k}, \quad k \binom{n}{k} = n \binom{n-1}{k-1},$$
$$k(k-1) \binom{n}{k} = n(n-1) \binom{n-2}{k-2}, \quad \sum_{i=0}^{2n} \binom{2n}{i} = 4^n.$$

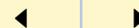
Then

$$\begin{aligned} & \sum_{k=0}^{n-1} (n-k)^2 \binom{2n}{k} \\ &= n^2 \sum_{k=0}^{n-1} \binom{2n}{k} - (2n-1) \sum_{k=0}^{n-1} k \binom{2n}{k} + \sum_{k=0}^{n-1} k(k-1) \binom{2n}{k} \\ &= n^2 \sum_{k=0}^{n-1} \binom{2n}{k} - 2n(2n-1) \sum_{k=1}^{n-1} \binom{2n-1}{k-1} + 2n(2n-1) \sum_{k=2}^{n-1} \binom{2n-2}{k-2} \end{aligned}$$



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$$\begin{aligned}
 &= n^2 \sum_{k=0}^{n-1} \binom{2n}{k} - 2n(2n-1) \sum_{k=0}^{n-2} \binom{2n-1}{k} + 2n(2n-1) \sum_{k=0}^{n-3} \binom{2n-2}{k} \\
 &= n^2 \frac{4^n - \binom{2n}{n}}{2} - 2n(2n-1) \frac{2^{2n-1} - 2 \binom{2n-1}{n-1}}{2} + 2n(2n-1) \frac{4^{n-1} - \binom{2n-2}{n-1} - 2 \binom{2n-2}{n-2}}{2} \\
 &= 4^{n-1}n + \frac{4n(2n-1) \binom{2n-1}{n} - n^2 \binom{2n}{n} - 2n(2n-1) \left[ \binom{2n-2}{n-1} + 2 \binom{2n-2}{n-2} \right]}{2} \\
 &= 4^{n-1}n + \left[ 2(2n-1)^2 - n(2n-1) - n(2n-1) - 2(2n-1)(n-1) \right] \binom{2n-2}{n-1} \\
 &= 4^{n-1}n.
 \end{aligned}$$

The proof of Lemma 2.1 is complete. □

**Lemma 2.2.** Let  $n \geq 2$  and  $y_i$  for  $1 \leq i \leq 2n$  be positive numbers. Denote  $x_i = y_i + y_{n+i}$  for  $1 \leq i \leq n$  and

$$(2.2) \quad A_n = \sum_{i=1}^{2n} y_i \sum_{k=i+1}^{n-1+i} y_{k|2n},$$

where  $m|2n$  means  $m \pmod{2n}$  for all nonnegative integers  $m$ . Then

$$(2.3) \quad A_n = \sum_{1 \leq i < j \leq n} x_i x_j.$$

*Proof.* Formula (2.2) can be written as

$$(2.4) \quad A_n = y_1(y_2 + \cdots + y_n) + y_2(y_3 + \cdots + y_{n+1}) + \cdots + y_{2n}(y_1 + \cdots + y_{n-1}).$$



From this, it is obtained readily that

$$A_n = \sum_{1 \leq i < j \leq 2n} y_i y_j - \sum_{i=1}^n y_i y_{n+i}$$

by induction on  $n$ . Since

$$\sum_{1 \leq i < j \leq n} x_i x_j = \sum_{1 \leq i < j \leq n} (y_i + y_{i+n})(y_j + y_{j+n}),$$

then

$$A_n = \sum_{1 \leq i < j \leq 2n} y_i y_j - \sum_{i=1}^n y_i y_{i+n} = \sum_{1 \leq i < j \leq n} (y_i + y_{i+n})(y_j + y_{j+n}) - \sum_{1 \leq i < j \leq n} x_i x_j,$$

which means that identity (2.3) holds. The proof of Lemma 2.2 is complete.  $\square$

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### 3. Proofs of the Main Results

*Proof of Theorem 1.1.* By Cauchy's inequality [1, 4], it follows that

$$(3.1) \quad \sum_{i=1}^{2n} \frac{x_i^{2n-1}}{\sum_{k \neq i}^{2n} (x_i + x_k)^{2n-1}} \sum_{i=1}^{2n} \sum_{j \neq i}^{2n} x_i (x_i + x_j)^{2n-1} \geq \left( \sum_{i=1}^{2n} x_i^n \right)^2.$$

Consequently, it suffices to show

$$\begin{aligned} (2n-1)4^{n-1} \left( \sum_{i=1}^{2n} x_i^n \right)^2 &\geq n \sum_{i=1}^{2n} \sum_{j \neq i}^{2n} x_i (x_i + x_j)^{2n-1} \\ \iff (2n-1)4^{n-1} \sum_{i=1}^{2n} x_i^{2n} + (2n-1)2^{2n-1} \sum_{1 \leq i < j \leq 2n} x_i^n x_j^n \\ &\geq n \sum_{k=0}^n \left[ \binom{2n-1}{k} + \binom{2n-1}{2n-k} \right] \sum_{i=1}^{2n} \sum_{j \neq i}^{2n} x_i^{2n-k} x_j^k \\ \iff (2n-1) (2^{2n-2} - n) \sum_{i=1}^{2n} x_i^{2n} \\ &\quad + \left[ (2n-1)2^{2n-1} - 2n \binom{2n-1}{n} \right] \sum_{1 \leq i < j \leq 2n} x_i^n x_j^n \\ &\geq n \sum_{k=1}^{n-1} \left[ \binom{2n-1}{k} + \binom{2n-1}{2n-k} \right] \sum_{i=1}^{2n} \sum_{j \neq i}^{2n} x_i^{2n-k} x_j^k. \end{aligned}$$





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Since  $\binom{n}{k} = \binom{n}{n-k}$  and  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ , the above inequality becomes

$$(3.2) \quad \left[ (2n-1)2^{2n-1} - n \binom{2n}{n} \right] \sum_{1 \leq i < j \leq 2n} x_i^n x_j^n \\ + (2n-1) (2^{2n-2} - n) \sum_{i=1}^{2n} x_i^{2n} - n \sum_{k=1}^{n-1} \binom{2n}{k} \sum_{i=1}^{2n} \sum_{j \neq i}^{2n} x_i^{2n-k} x_j^k \geq 0.$$

Utilization of  $\sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n}$  and  $\binom{2n}{0} = \binom{2n}{2n} = 1$  yields

$$2(2^{2n-2} - n) + (2n-1)2^{2n-1} - n \binom{2n}{n} - n \sum_{k=1, k \neq n}^{2n-1} \binom{2n}{k} \\ = 2^{2n}n - 2n - n \left[ \sum_{k=0}^{2n} \binom{2n}{k} - 2 \right] = 0.$$

Substituting this into (3.2) gives

$$(3.3) \quad \sum_{i=1, j=1, i \neq j}^{2n} \left\{ \sum_{q=0}^{n-1} \left[ 2^{2n-2} - n - n \sum_{k=1}^q \binom{2n}{k} \right] x_i^q x_j^q \sum_{k=0}^{2n-2q-2} x_i^{2n-2q-k-2} x_j^k \right\} \\ \times (x_i - x_j)^2 \geq 0,$$

where  $\sum_{k=1}^q \binom{2n}{k} = 0$  for  $q = 0$ . Employing (2.1) in the above inequality leads to

$$\sum_{p=0}^{n-1} (2n - 2p - 1) \left[ 2^{2n-2} - n \sum_{k=0}^p \binom{2n}{k} \right]$$



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$$\begin{aligned}
 &= 2^{2n-2}n^2 - n \left\{ \sum_{p=0}^{n-1} (2n - 2p - 1) \sum_{k=0}^p \binom{2n}{k} \right\} \\
 &= 2^{2n-2}n^2 - n \sum_{k=0}^{n-1} (n - k)^2 \binom{2n}{k} = 0.
 \end{aligned}$$

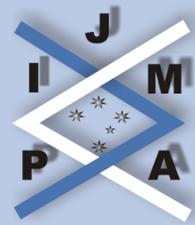
This implies that inequality (3.3) is equivalent to

$$\begin{aligned}
 (3.4) \quad &\sum_{i=1, j=1, i \neq j}^{2n} (x_i - x_j)^2 \left\{ \sum_{k=1}^n \left[ k 2^{2n-2} - n \sum_{q=0}^{k-1} (k - q) \binom{2n}{q} \right] x_i^{2n-k-1} x_j^{k-1} \right. \\
 &+ \left. \sum_{k=n+1}^{2n} \left[ (2n - k + 1) 2^{2n-2} - n \sum_{q=k}^{2n} (2n - q + 1) \binom{2n}{q - k} \right] x_i^{2n-k} x_j^{k-2} \right\} \geq 0,
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{i=1, j=1, i \neq j}^{2n} \left\{ \sum_{k=1}^{n-1} \left[ \frac{k(k+1)}{2} 2^{2n-2} - n \sum_{p=1}^k \frac{p(p+1)}{2} \binom{2n}{k-p} \right] \right. \\
 &\quad \left. \times x_i^{k-1} x_j^{k-1} \sum_{p=0}^{2n-2k-2} x_i^{2n-p-4} x_j^p \right\} (x_i - x_j)^4 \geq 0.
 \end{aligned}$$

In order to prove (3.4), it is sufficient to show

$$(3.5) \quad \frac{(n-1)[(n-1)+1]}{2} 2^{2n-2} - n \sum_{p=1}^{n-1} \frac{p(p+1)}{2} \binom{2n}{n-p-1} > 0.$$



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Considering (2.1), it is sufficient to show

$$(3.6) \quad \sum_{k=0}^{n-1} (n-k) \binom{2n}{k} > 2^{2n-2}.$$

By virtue of  $\binom{n}{k} = \binom{n}{n-k}$  and  $\sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n}$ , inequality (3.6) can be rearranged as

$$(3.7) \quad \sum_{k=0}^{n-1} (n-k) \binom{2n}{k} + \sum_{k=n+1}^{2n} (k-n) \binom{2n}{k} > 2^{2n-1},$$

$$\sum_{k=0}^{n-1} (2n-2k-1) \binom{2n}{k} + \sum_{k=n+1}^{2n} (2k-2n-1) \binom{2n}{k} > \binom{2n}{n}.$$

Since  $n \geq 2$  and  $\binom{2n}{n-1} + \binom{2n}{n+1} > \binom{2n}{n}$  is equivalent to  $2 > \frac{n+1}{n}$ , then inequalities (3.7), (3.6) and (3.5) are valid. The proof of Theorem 1.1 is complete.  $\square$

*Proof of Theorem 1.2.* By Lemma 2.2, it is easy to see that  $\sum_{i=1}^{2n} y_i = \sum_{i=1}^n x_i$ . From Cauchy's inequality [1, 4], it follows that

$$A_n \sum_{i=1}^{2n} \frac{y_i^2}{y_{i-1|2n} \sum_{k=i}^{i+n-2} y_{k|2n}} \geq \left( \sum_{i=1}^{2n} y_i \right)^2,$$

where  $A_n$  is defined by (2.2) or (2.3) in Lemma 2.2. Therefore, it is sufficient to



prove

$$\begin{aligned}
 (n-1) \left( \sum_{i=1}^{2n} y_i \right)^2 \geq 2nA_n &\iff (n-1) \left( \sum_{i=1}^n x_i \right)^2 \geq 2n \sum_{1 \leq i < j \leq n} x_i x_j \\
 &\iff (n-1) \sum_{i=1}^n x_i^2 \geq 2 \sum_{1 \leq i < j \leq 2n} x_i x_j \\
 &\iff \sum_{1 \leq i < j \leq 2n} (x_i - x_j)^2 \geq 0.
 \end{aligned}$$

The proof of Theorem 1.2 is complete. □

*Proof of Theorem 1.3.* Let

$$(3.8) \quad B_n = 2^{2(n-1)} k(k+1) - n \sum_{p=1}^k p(p+1) \binom{2n}{k-p}.$$

Then

$$\begin{aligned}
 (3.9) \quad B_{n+1} &= k(k+1)2^{2n-2}2^2 - (n+1) \sum_{p=1}^k p(p+1) \binom{2n+2}{k-p} \\
 &= 4B_n + \sum_{p=1}^k p(p+1) \left[ 4n \binom{2n}{k-p} - (n+1) \binom{2n+2}{k-p} \right] \\
 &\triangleq 4B_n + \sum_{p=1}^k p(p+1) C_{k-p}
 \end{aligned}$$

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and

$$\begin{aligned}
 C_q - C_{q+1} &= 4n \left[ \binom{2n}{q} - \binom{2n}{q+1} \right] - (n+1) \left[ \binom{2n+2}{q} - \binom{2n+2}{q+1} \right] \\
 &= 4n \binom{2n}{q} \left( 1 - \frac{2n-q}{q+1} \right) - (n+1) \binom{2n+2}{q} \left( 1 - \frac{2n+2-q}{q+1} \right) \\
 &= 4n \binom{2n}{q} \frac{2q-2n+1}{q+1} - (n+1) \binom{2n+2}{q} \frac{2q-2n-1}{q+1} \\
 &> \frac{2q-2n+1}{q+1} C_q
 \end{aligned}$$

for  $0 \leq q \leq k-1$ . Hence,

$$(3.10) \quad \frac{2n-q}{q+1} C_q > C_{q+1}.$$

From the above inequality and the facts that

$$(3.11) \quad C_n = \frac{2(2n-1)(n+1)}{n+2} \binom{2n}{n} > 0$$

and  $\frac{2n-q}{q+1} > 0$ , it follows easily that  $C_q > 0$ . Consequently, we have  $B_{n+1} > 4B_n$ , and then  $B_{k+2} > 4B_{k+1}$ . As a result, utilization of (3.5) gives

$$B_{k+1} > 0, \quad B_{k+2} > 0, \quad B_{k+3} > 0, \quad B_{k+4} > 0, \quad \dots, \quad B_{k+(n-k)} = B_n > 0.$$

The proof of inequality (1.3) is complete.  $\square$

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