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EXTENSIONS OF HIONG'S INEQUALITY

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Abstract

In this paper, we treat the value distribution of $\phi f^{n-1}f^{(k)}$, where f is a transcendental meromorphic function, ϕ is a meromorphic function satisfying $T(r,\phi) = S(r,f)$, n and k are positive integers. We generalize some results of Hiong and Yu.

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Contents

1. Introduction

Let f be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \dots$$

(see Hayman [1], Yang [4]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set with finite measure.

In 1956, Hiong [3] proved the following inequality.

Theorem 1.1. *Let f be a non-constant meromorphic function; let a, b and c be three finite complex numbers such that $b \neq 0$, $c \neq 0$ and $b \neq c$; and let k be a positive integer. Then*

$$T(r,f) \leq N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f^{(k)}-b}\right) + N\left(r, \frac{1}{f^{(k)}-c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f).$$

Recently, Yu [5] extended Theorem 1.1 as follows.

Theorem 1.2. *Let f be a non-constant meromorphic function; and let b and c be two distinct nonzero finite complex numbers; and let n, k be two positive* *integers. If $\phi(\not\equiv 0)$ is a meromorphic function satisfying $T(r,\phi) = S(r,f)$, $n = 1$ or $n \geq k+3$, then*

$$T(r,f) \leq N\left(r,\frac{1}{f}\right) + \frac{1}{n}\left[N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-b}\right) + N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-c}\right)\right] - \frac{1}{n}\left[N(r,f) + N\left(r,\frac{1}{(\phi f^{n-1}f^{(k+1)})'}\right)\right] + S(r,f). \tag{1.1}$$

If f is entire, then (1.1) is valid for all positive integers $n(\neq 2)$.

In [5], the author expected that (1.1) is also valid for $n = 2$ if f is entire.

In this note, we prove that (1.1) is valid for all positive integers n even if f is meromorphic.

Theorem 1.3. *Let f be a non-constant meromorphic function; and let b and c be two distinct nonzero finite complex numbers; and let n, k be two positive integers. If $\phi(\not\equiv 0)$ is a meromorphic function satisfying $T(r,\phi) = S(r,f)$, then*

$$T(r,f) \leq N\left(r,\frac{1}{f}\right) + \frac{1}{n}\left[N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-b}\right) + N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-c}\right)\right] - N(r,f) - \frac{1}{n}\left[(k-1)\overline{N}(r,f) + N\left(r,\frac{1}{(\phi f^{n-1}f^{(k+1)})'}\right)\right] + S(r,f). \tag{1.2}$$

In [6], the author proved

Theorem 1.4. *Let f be a transcendental meromorphic function; and let n be a positive integer. Then either $f^n f' - a$ or $f^n f' + a$ has infinitely many zeros, where $a(\not\equiv 0)$ is a meromorphic function satisfying $T(r,a) = S(r,f)$.*

In this note, we will prove

Theorem 1.5. *Let f be a transcendental meromorphic function; and let n be a positive integer. Then either $f^n f' - a$ or $f^n f' - b$ has infinitely many zeros, where $a(\not\equiv 0)$ and $b(\not\equiv 0)$ are two meromorphic functions satisfying $T(r,a) = S(r,f)$ and $T(r,b) = S(r,f)$.*

2. Proof of Theorems

For the proofs of Theorem 1.3 and 1.5, we require the following lemmas.

Lemma 2.1. *[2]. If f is a transcendental meromorphic function and $K > 1$, then there exists a set $M(K)$ of upper logarithmic density at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K-1)) \exp(e(1-K))\}$$

such that for every positive integer k,

$$\limsup_{r\to\infty, r\notin M(K)} \frac{T(r,f)}{T(r,f^{(k)})} \leq 3eK. \tag{2.1}$$

Lemma 2.2. *If f is a transcendental meromorphic function and $\phi(\not\equiv 0)$ is a meromorphic function satisfying $T(r,\phi) = S(r,f)$. Then $\phi f^{n-1} f^{(k)} \not\equiv$ constant for every positive integer n.*

Proof. Suppose that $\phi f^{n-1} f^{(k)} \equiv$ constant. If $n = 1$, then $\phi f^{(k)} \equiv$ constant. Therefore,
$T(r, f^{(k)}) = S(r,f)$, which implies that

$$\limsup_{r\to\infty, r\notin M(K)} \frac{T(r,f)}{T(r,f^{(k)})} = \infty.$$

This is contradiction to Lemma 2.1.

If $n \geq 2$, then $T(r, f^{n-1}f^{(k)}) = S(r, f)$. On the other hand,

$$\begin{aligned} nT(r,f) &\leq T(r, f^{n-1}f^{(k)}) + T\left(r, \frac{f}{f^{(k)}}\right) + S(r,f) \\ &\leq T(r, f^{n-1}f^{(k)}) + T\left(r, \frac{f^{(k)}}{f}\right) + S(r,f) \\ &\leq T(r, f^{n-1}f^{(k)}) + N\left(r, \frac{f^{(k)}}{f}\right) + S(r,f) \\ &\leq T(r, f^{n-1}f^{(k)}) + N(r, \frac{1}{f}) + N(r, f^{n-1}f^{(k)}) + S(r,f) \\ &\leq 2T(r, f^{n-1}f^{(k)}) + T(r,f) + S(r,f). \end{aligned}$$

Hence $T(r,f) \leq \frac{2}{n-1}T(r, f^{n-1}f^{(k)}) + S(r,f)$, Therefore, $T(r,f) = S(r,f)$, which is a contradiction. Which completes the proof of this lemma. □

Lemma 2.3. *[1]. If f is a meromorphic function, and a_1, a_2, a_3 are distinct meromorphic functions satisfying $T(r, a_j) = S(r, f)$ for $j = 1, 2, 3$. Then*

$$T(r,f) \leq \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r,f).$$

Proof of Theorem 1.3. By Lemma 2.2, we have $\phi f^{n-1}f^{(k)} \not\equiv$ constant if n and k are positive integers. By (4.17) of [1], we have

$$m\left(r, \frac{1}{f^n}\right) + m\left(r, \frac{1}{\phi f^{n-1}f^{(k)} - b}\right) + m\left(r, \frac{1}{\phi f^{n-1}f^{(k)} - c}\right) \tag{2.2}$$

$$
\begin{aligned}
&\le m\left(r, \frac{1}{\phi f^{n-1} f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) \\
&\quad + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) + S(r, f) \\
&\le m\left(r, \frac{1}{\phi f^{n-1} f^{(k)}}\right) + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) \\
&\quad + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) + S(r, f) \\
&\le m\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f) \\
&\le T(r, (\phi f^{n-1} f^{(k)})') - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f) \\
&\le T(r, \phi f^{n-1} f^{(k)}) + \overline{N}(r, f) - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f)
\end{aligned}
$$

By (2.2), we have

$$
\begin{aligned}
&T(r, f^n) + T(r, \phi f^{n-1} f^{(k)}) \\
&\quad \le N\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) \\
&\qquad + \overline{N}(r, f) - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f).
\end{aligned}
$$

Therefore,

$$\begin{aligned} nT(r,f) \le{}& nN\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-b}\right) + N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-c}\right) \\ &+ \overline{N}(r,f) - N\left(r,\frac{1}{(\phi f^{n-1}f^{(k)})'}\right) - N(r,f^{n-1}f^{(k)}) + S(r,f) \\ \le{}& nN\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-b}\right) + N\left(r,\frac{1}{\phi f^{n-1}f^{(k)}-c}\right) \\ &- nN(r,f) - (k-1)\overline{N}(r,f) - N\left(r,\frac{1}{(\phi f^{n-1}f^{(k)})'}\right) + S(r,f), \end{aligned}$$

thus we get (1.2). This completes the proof of Theorem 1.3. □

Proof of Theorem 1.5. By Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} 2T(r,f) &= T\left(r, ff'\cdot\frac{f}{f'}\right) \\ &\le T(r,ff') + T\left(r,\frac{f}{f'}\right) + S(r,f) \\ &\le T(r,ff') + T\left(r,\frac{f'}{f}\right) + S(r,f) \\ &\le T(r,ff') + N\left(r,\frac{f'}{f}\right) + S(r,f) \\ &= T(r,ff') + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f) \end{aligned}$$

$$\leq T(r, ff') + T(r, f) + \frac{1}{3}N(r, ff') + S(r, f).$$

Thus we get

$$T(r, f) \leq \frac{4}{3}T(r, ff') + S(r, f).$$

Hence we get $T(r, a) = S(r, ff')$ and $T(r, b) = S(r, ff')$.

By Lemma 2.3, we have

$$\begin{aligned} T(r, ff') &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{ff' - a}\right) + \overline{N}\left(r, \frac{1}{ff' - b}\right) + S(r, ff') \\ &\leq \frac{1}{3}N(r, ff') + \overline{N}\left(r, \frac{1}{ff' - a}\right) + \overline{N}\left(r, \frac{1}{ff' - b}\right) + S(r, ff'). \end{aligned}$$

Hence we get

$$T(r, f) \leq \frac{3}{2}\left[\overline{N}\left(r, \frac{1}{ff' - a}\right) + \overline{N}\left(r, \frac{1}{ff' - b}\right)\right] + S(r, ff').$$

Thus we know that either $ff' - a$ or $ff' - b$ has infinitely many zeros. □

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