

Journal of Inequalities in Pure and Applied Mathematics

EXTENSIONS OF HIONG'S INEQUALITY

MING-LIANG FANG AND DEGUI YANG

Department of Mathematics,
Nanjing Normal University,
Nanjing, 210097,
People's Republic of China

EMail: mlfang@pine.njnu.edu.cn

College of Sciences,
South China Agricultural University,
Guangzhou, 510642,
People's Republic of China

EMail: yangde@macs.biu.ac.il

©2000 Victoria University
ISSN (electronic): 1443-5756
079-02



volume 3, issue 5, article 76,
2002.

*Received 30 June, 2002;
accepted 12 July, 2002.*

Communicated by: H.M. Srivastava

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper, we treat the value distribution of $\phi f^{n-1} f^{(k)}$, where f is a transcendental meromorphic function, ϕ is a meromorphic function satisfying $T(r, \phi) = S(r, f)$, n and k are positive integers. We generalize some results of Hiong and Yu.

2000 Mathematics Subject Classification: Primary 30D35, 30A10

Key words: Inequality, Value distribution, Meromorphic function.

Contents

1	Introduction	3
2	Proof of Theorems	6
	References	



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

Close

Quit

Page 2 of 11

1. Introduction

Let f be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r, f), m(r, f), N(r, f), \overline{N}(r, f), \dots$$

(see Hayman [1], Yang [4]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \rightarrow +\infty$, possibly outside of a set with finite measure.

In 1956, Hiong [3] proved the following inequality.

Theorem 1.1. *Let f be a non-constant meromorphic function; let a, b and c be three finite complex numbers such that $b \neq 0, c \neq 0$ and $b \neq c$; and let k be a positive integer. Then*

$$T(r, f) \leq N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f^{(k)}-b}\right) + N\left(r, \frac{1}{f^{(k)}-c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).$$

Recently, Yu [5] extended Theorem 1.1 as follows.

Theorem 1.2. *Let f be a non-constant meromorphic function; and let b and c be two distinct nonzero finite complex numbers; and let n, k be two positive*



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

Close

Quit

Page 3 of 11

integers. If $\phi (\neq 0)$ is a meromorphic function satisfying $T(r, \phi) = S(r, f)$, $n = 1$ or $n \geq k + 3$, then

$$(1.1) \quad T(r, f) \leq N\left(r, \frac{1}{f}\right) + \frac{1}{n} \left[N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) \right] - \frac{1}{n} \left[N(r, f) + N\left(r, \frac{1}{(\phi f^{n-1} f^{(k+1)})'}\right) \right] + S(r, f).$$

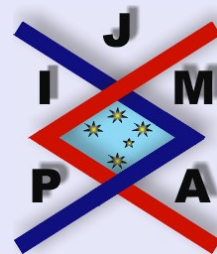
If f is entire, then (1.1) is valid for all positive integers $n (\neq 2)$.

In [5], the author expected that (1.1) is also valid for $n = 2$ if f is entire.

In this note, we prove that (1.1) is valid for all positive integers n even if f is meromorphic.

Theorem 1.3. Let f be a non-constant meromorphic function; and let b and c be two distinct nonzero finite complex numbers; and let n, k be two positive integers. If $\phi (\neq 0)$ is a meromorphic function satisfying $T(r, \phi) = S(r, f)$, then

$$(1.2) \quad T(r, f) \leq N\left(r, \frac{1}{f}\right) + \frac{1}{n} \left[N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) \right] - N(r, f) - \frac{1}{n} \left[(k-1)\bar{N}(r, f) + N\left(r, \frac{1}{(\phi f^{n-1} f^{(k+1)})'}\right) \right] + S(r, f).$$



Title Page

Contents



Go Back

Close

Quit

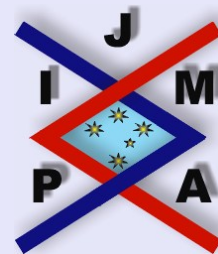
Page 4 of 11

In [6], the author proved

Theorem 1.4. *Let f be a transcendental meromorphic function; and let n be a positive integer. Then either $f^n f' - a$ or $f^n f' + a$ has infinitely many zeros, where $a (\neq 0)$ is a meromorphic function satisfying $T(r, a) = S(r, f)$.*

In this note, we will prove

Theorem 1.5. *Let f be a transcendental meromorphic function; and let n be a positive integer. Then either $f^n f' - a$ or $f^n f' - b$ has infinitely many zeros, where $a (\neq 0)$ and $b (\neq 0)$ are two meromorphic functions satisfying $T(r, a) = S(r, f)$ and $T(r, b) = S(r, f)$.*



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

Close

Quit

Page 5 of 11

2. Proof of Theorems

For the proofs of Theorem 1.3 and 1.5, we require the following lemmas.

Lemma 2.1. [2]. *If f is a transcendental meromorphic function and $K > 1$, then there exists a set $M(K)$ of upper logarithmic density at most*

$$\delta(K) = \min\{(2e^{K-1} - 1)^{-1}, (1 + e(K - 1)) \exp(e(1 - K))\}$$

such that for every positive integer k ,

$$(2.1) \quad \limsup_{r \rightarrow \infty, r \notin M(K)} \frac{T(r, f)}{T(r, f^{(k)})} \leq 3eK.$$

Lemma 2.2. *If f is a transcendental meromorphic function and $\phi (\not\equiv 0)$ is a meromorphic function satisfying $T(r, \phi) = S(r, f)$. Then $\phi f^{n-1} f^{(k)} \not\equiv \text{constant}$ for every positive integer n .*

Proof. Suppose that $\phi f^{n-1} f^{(k)} \equiv \text{constant}$. If $n = 1$, then $\phi f^{(k)} \equiv \text{constant}$. Therefore,

$T(r, f^{(k)}) = S(r, f)$, which implies that

$$\limsup_{r \rightarrow \infty, r \notin M(K)} \frac{T(r, f)}{T(r, f^{(k)})} = \infty.$$

This is contradiction to Lemma 2.1.



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

Close

Quit

Page 6 of 11

If $n \geq 2$, then $T(r, f^{n-1}f^{(k)}) = S(r, f)$. On the other hand,

$$\begin{aligned}
 nT(r, f) &\leq T(r, f^{n-1}f^{(k)}) + T\left(r, \frac{f}{f^{(k)}}\right) + S(r, f) \\
 &\leq T(r, f^{n-1}f^{(k)}) + T\left(r, \frac{f^{(k)}}{f}\right) + S(r, f) \\
 &\leq T(r, f^{n-1}f^{(k)}) + N\left(r, \frac{f^{(k)}}{f}\right) + S(r, f) \\
 &\leq T(r, f^{n-1}f^{(k)}) + N\left(r, \frac{1}{f}\right) + N(r, f^{n-1}f^{(k)}) + S(r, f) \\
 &\leq 2T(r, f^{n-1}f^{(k)}) + T(r, f) + S(r, f).
 \end{aligned}$$

Hence $T(r, f) \leq \frac{2}{n-1}T(r, f^{n-1}f^{(k)}) + S(r, f)$, Therefore, $T(r, f) = S(r, f)$, which is a contradiction. Which completes the proof of this lemma. \square

Lemma 2.3. [1]. If f is a meromorphic function, and a_1, a_2, a_3 are distinct meromorphic functions satisfying $T(r, a_j) = S(r, f)$ for $j = 1, 2, 3$. Then

$$T(r, f) \leq \sum_{j=1}^3 \bar{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f).$$

Proof of Theorem 1.3. By Lemma 2.2, we have $\phi f^{n-1}f^{(k)} \neq \text{constant}$ if n and k are positive integers. By (4.17) of [1], we have

$$(2.2) \quad m\left(r, \frac{1}{f^n}\right) + m\left(r, \frac{1}{\phi f^{n-1}f^{(k)} - b}\right) + m\left(r, \frac{1}{\phi f^{n-1}f^{(k)} - c}\right)$$



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents

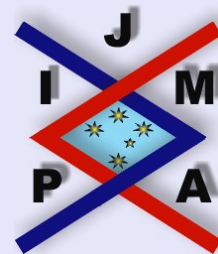


Go Back

Close

Quit

Page 7 of 11



Title Page

Contents



Go Back

Close

Quit

Page 8 of 11

$$\begin{aligned}
 &\leq m\left(r, \frac{1}{\phi f^{n-1} f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) \\
 &\quad + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) + S(r, f) \\
 &\leq m\left(r, \frac{1}{\phi f^{n-1} f^{(k)}}\right) + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) \\
 &\quad + m\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) + S(r, f) \\
 &\leq m\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f) \\
 &\leq T(r, (\phi f^{n-1} f^{(k)})') - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f) \\
 &\leq T(r, \phi f^{n-1} f^{(k)}) + \bar{N}(r, f) - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f)
 \end{aligned}$$

By (2.2), we have

$$\begin{aligned}
 &T(r, f^n) + T(r, \phi f^{n-1} f^{(k)}) \\
 &\leq N\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) \\
 &\quad + \bar{N}(r, f) - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 nT(r, f) &\leq nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) \\
 &\quad + \bar{N}(r, f) - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) - N(r, f^{n-1} f^{(k)}) + S(r, f) \\
 &\leq nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - b}\right) + N\left(r, \frac{1}{\phi f^{n-1} f^{(k)} - c}\right) \\
 &\quad - nN(r, f) - (k-1)\bar{N}(r, f) - N\left(r, \frac{1}{(\phi f^{n-1} f^{(k)})'}\right) + S(r, f),
 \end{aligned}$$

thus we get (1.2). This completes the proof of Theorem 1.3. \square

Proof of Theorem 1.5. By Nevanlinna's first fundamental theorem, we have

$$\begin{aligned}
 2T(r, f) &= T\left(r, ff' \cdot \frac{f}{f'}\right) \\
 &\leq T(r, ff') + T\left(r, \frac{f}{f'}\right) + S(r, f) \\
 &\leq T(r, ff') + T\left(r, \frac{f'}{f}\right) + S(r, f) \\
 &\leq T(r, ff') + N\left(r, \frac{f'}{f}\right) + S(r, f) \\
 &= T(r, ff') + \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + S(r, f)
 \end{aligned}$$



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

Close

Quit

Page 9 of 11

$$\leq T(r, ff') + T(r, f) + \frac{1}{3}N(r, ff') + S(r, f).$$

Thus we get

$$T(r, f) \leq \frac{4}{3}T(r, ff') + S(r, f).$$

Hence we get $T(r, a) = S(r, ff')$ and $T(r, b) = S(r, ff')$.

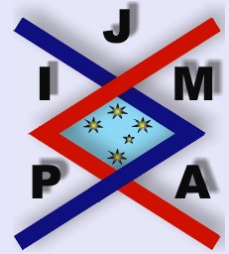
By Lemma 2.3, we have

$$\begin{aligned} T(r, ff') &\leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{ff' - a}\right) + \bar{N}\left(r, \frac{1}{ff' - b}\right) + S(r, ff') \\ &\leq \frac{1}{3}N(r, ff') + \bar{N}\left(r, \frac{1}{ff' - a}\right) + \bar{N}\left(r, \frac{1}{ff' - b}\right) + S(r, ff'). \end{aligned}$$

Hence we get

$$T(r, f) \leq \frac{3}{2} \left[\bar{N}\left(r, \frac{1}{ff' - a}\right) + \bar{N}\left(r, \frac{1}{ff' - b}\right) \right] + S(r, ff').$$

Thus we know that either $ff' - a$ or $ff' - b$ has infinitely many zeros. \square



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

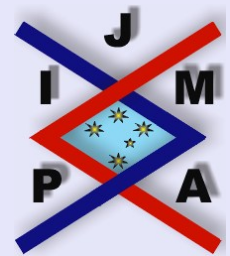
Close

Quit

Page 10 of 11

References

- [1] W.K. HAYMAN, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
- [2] W.K. HAYMAN AND J. MILES, On the growth of a meromorphic function and its derivatives, *Complex Variables*, **12**(1989), 245–260.
- [3] K.L. HIONG, Sur la limitation de $T(r, f)$ sans intervention des pôles, *Bull. Sci. Math.*, **80** (1956), 175–190.
- [4] L. YANG, *Value Distribution Theory*, Springer-Verlag, Berlin, 1993.
- [5] K.W. YU, On the distribution of $\phi(z)f^{n-1}(z)f^{(k)}(z)$, *J. of Ineq. Pure and Appl. Math.*, **3**(1) (2002), Article 8. [ONLINE: http://jipam.vu.edu.au/v3n1/037_01.html]
- [6] K.W. YU, A note on the product of a meromorphic function and its derivative, *Kodai Math. J.*, **24**(3) (2001), 339–343.



Extensions of Hiong's Inequality

Ming-liang Fang and Degui Yang

Title Page

Contents



Go Back

Close

Quit

Page 11 of 11