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AN INEQUALITY IMPROVING THE SECOND HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES AND APPLICATIONS FOR SEMI-INNER PRODUCTS

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ABSTRACT. An inequality for convex functions defined on linear spaces is obtained which contains in a particular case a refinement for the second part of the celebrated Hermite-Hadamard inequality. Applications for semi-inner products on normed linear spaces are also provided.

Key words and phrases: Hermite-Hadamard integral inequality, Convex functions, Semi-Inner products.

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1. INTRODUCTION

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = (\nabla_{\pm} f[(1-s)a + sb])(b-a)$, $s \in (0,1)$
(ii) $g'_{+}(a,b)(0) = (\nabla_{+} f(a))(b-a)$
(iii) $g'_{-}(a,b)(1) = (\nabla_{-} f(b))(b-a)$

where $(\nabla_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives*, we recall that

$$\begin{aligned}(\nabla_{+} f(x))(y) &:= \lim_{h\to 0+}\left[\frac{f(x+hy)-f(x)}{h}\right],\\ (\nabla_{-} f(x))(y) &:= \lim_{k\to 0-}\left[\frac{f(x+ky)-f(x)}{k}\right],\quad x,y \in X.\end{aligned}$$

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The following inequality is the well known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left[(1-t)\,a+tb\right] dt \leq \frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b) : [0,1] \to \mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right) \leq \int_0^1 g(a,b)(t)\,dt \leq \frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [1].

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := (\nabla_+ f_0(y))(x) = \lim\limits_{t\to 0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i := (\nabla_- f_0(y))(x) = \lim\limits_{s\to 0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any $x,y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2]), assuming that $p,q \in \{s,i\}$ and $p \neq q$:

(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x \in X$;

(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x,y\rangle_p$ if $\alpha,\beta \geq 0$ and $x,y \in X$;

(aaa) $\left|\langle x,y\rangle_p\right| \leq \|x\|\,\|y\|$ for all $x,y \in X$;

(av) $\langle \alpha x+y, x\rangle_p = \alpha\langle x,x\rangle_p + \langle y,x\rangle_p$ if $x,y \in X$ and $\alpha \in \mathbb{R}$;

(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all $x,y \in X$;

(va) $\langle x+y,z\rangle_p \leq \|x\|\,\|z\| + \langle y,z\rangle_p$ for all $x,y,z \in X$;

(vaa) The mapping $\langle \cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);

(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0 \in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s = \langle y,x_0\rangle_i$ for all $y \in X$; in general $\langle y,x\rangle_i \leq \langle y,x\rangle_s$ for all $x,y \in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all $x,y \in X$.

Applying inequality (HH) for the convex function $f_0(x) = \frac{1}{2}\|x\|^2$, one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)\,x+ty\|^2\,dt \leq \frac{\|x\|^2+\|y\|^2}{2} \tag{1.1}$$

for any $x,y \in X$. The same (HH) inequality applied for $f_1(x) = \|x\|$, will give the following refinement of the triangle inequality:

$$\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)\,x+ty\|\,dt \leq \frac{\|x\|+\|y\|}{2}, \quad x,y \in X. \tag{1.2}$$

In this paper we point out an integral inequality for convex functions which is related to the first Hermite-Hadamard inequality in (HH) and investigate its applications for semi-inner products in normed linear spaces.

2. THE RESULTS

We start with the following lemma which is also of interest in itself.

Lemma 2.1. *Let $h : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then for any $\gamma \in [\alpha, \beta]$ one has the inequality*

$$\begin{aligned} (2.1) \qquad & \frac{1}{2}\left[(\beta-\gamma)^2 h'_+(\gamma) - (\gamma-\alpha)^2 h'_-(\gamma)\right] \\ & \leq (\gamma-\alpha) h(\alpha) + (\beta-\gamma) h(\beta) - \int_\alpha^\beta h(t)\, dt \\ & \leq \frac{1}{2}\left[(\beta-\gamma)^2 h'_-(\beta) - (\gamma-\alpha)^2 h'_+(\alpha)\right]. \end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $\gamma = \alpha$ or $\gamma = \beta$.

Proof. It is easy to see that for any locally absolutely continuous function $h : (\alpha, \beta) \to \mathbb{R}$, we have the identity

$$(2.2) \qquad \int_\alpha^\beta (t-\gamma) h'(t)\, dt = (\gamma-\alpha) h(\alpha) + (\beta-\gamma) h(\beta) - \int_\alpha^\beta h(t)\, dt$$

for any $\gamma \in (\alpha, \beta)$, where h' is the derivative of h which exists a.e. on (α, β).

Since h is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $\gamma \in (\alpha, \beta)$, we have the inequalities

$$(2.3) \qquad h'(t) \leq h'_-(\gamma) \text{ for a.e. } t \in [\alpha, \gamma]$$

and

$$(2.4) \qquad h'(t) \geq h'_+(\gamma) \text{ for a.e. } t \in [\gamma, \beta].$$

If we multiply (2.3) by $\gamma - t \geq 0$, $t \in [\alpha, \gamma]$ and integrate on $[\alpha, \gamma]$, we get

$$(2.5) \qquad \int_\alpha^\gamma (\gamma - t) h'(t)\, dt \leq \frac{1}{2}(\gamma-\alpha)^2 h'_-(\gamma)$$

and if we multiply (2.4) by $t - \gamma \geq 0$, $t \in [\gamma, \beta]$, and integrate on $[\gamma, \beta]$, we also have

$$(2.6) \qquad \int_\gamma^\beta (t-\gamma) h'(t)\, dt \geq \frac{1}{2}(\beta-\gamma)^2 h'_+(\gamma).$$

Now, if we subtract (2.5) from (2.6) and use the representation (2.2), we deduce the first inequality in (2.1).

If we assume that the first inequality (2.1) holds with a constant $C > 0$ instead of $\frac{1}{2}$, i.e.,

$$(2.7) \quad C\left[(\beta-\gamma)^2 h'_+(\gamma) - (\gamma-\alpha)^2 h'_-(\gamma)\right] \leq (\gamma-\alpha) h(\alpha) + (\beta-\gamma) h(\beta) - \int_\alpha^\beta h(t)\, dt$$

and take the convex function $h_0(t) := k\left|t - \frac{\alpha+\beta}{2}\right|$, $k > 0$, $t \in [\alpha, \beta]$, then

$$\begin{aligned} h'_{0+}\left(\frac{\alpha+\beta}{2}\right) &= k, \\ h'_{0-}\left(\frac{\alpha+\beta}{2}\right) &= -k, \end{aligned}$$

$$
\begin{aligned}
h_0(\alpha) &= \frac{k(\beta-\alpha)}{2} = h_0(\beta), \\
\int_\alpha^\beta h_0(t)\,dt &= \frac{1}{4}k(\beta-\alpha)^2,
\end{aligned}
$$

and the inequality (2.7) becomes, for $\gamma = \frac{\alpha+\beta}{2}$,

$$
C\left[\frac{1}{4}(\beta-\alpha)^2 k + \frac{1}{4}(\beta-\alpha)^2 k\right] \le \frac{1}{4}k(\beta-\alpha)^2,
$$

giving $C \le \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$ in the first inequality in (2.1).

If either $h'_+(\alpha) = -\infty$ or $h'_-(\beta) = -\infty$, then the second inequality in (2.1) holds true.

Assume that $h'_+(\alpha)$ and $h'_-(\beta)$ are finite. Since h is convex on $[\alpha,\beta]$, we have

$$
h'(t) \ge h'_+(\alpha) \text{ for a.e. } t \in [\alpha,\gamma] \quad (\gamma \text{ may be equal to } \beta) \tag{2.8}
$$

and

$$
h'(t) \le h'_-(\beta) \text{ for a.e. } t \in [\gamma,\beta] \quad (\gamma \text{ may be equal to } \alpha). \tag{2.9}
$$

If we multiply (2.8) by $\gamma - t \ge 0$, $t \in [\alpha,\gamma]$ and integrate on $[\alpha,\gamma]$, then we deduce

$$
\int_\alpha^\gamma (\gamma-t)\,h'(t)\,dt \ge \frac{1}{2}(\gamma-\alpha)^2 h'_+(\alpha) \tag{2.10}
$$

and if we multiply (2.9) by $t-\gamma \ge 0$, $t\in[\gamma,\beta]$, and integrate on $[\gamma,\beta]$, then we also have

$$
\int_\gamma^\beta (t-\gamma)\,h'(t)\,dt \le \frac{1}{2}(\beta-\gamma)^2 h'_-(\beta). \tag{2.11}
$$

Finally, if we subtract (2.10) from (2.11) and use the representation (2.2), we deduce the second part of (2.1).

Now, assume that the second inequality in (2.1) holds with a constant $D > 0$ instead of $\frac{1}{2}$, i.e.,

$$
\begin{aligned}
(\gamma-\alpha)f(\alpha) + (\beta-\gamma)f(\beta) - \int_\alpha^\beta f(t)\,dt \\
\ge D\left[(\beta-\gamma)^2 f'_-(\beta) - (\gamma-\alpha)^2 f'_+(\alpha)\right].
\end{aligned} \tag{2.12}
$$

If we consider the convex function h_0 given above, then we have $h'_-(\beta) = k$, $h'_+(\alpha) = -k$ and by (2.12) applied for h_0 and $x = \frac{\alpha+\beta}{2}$ we get

$$
\frac{1}{4}k(\beta-\alpha)^2 \le D\left[\frac{1}{4}k(\beta-\alpha)^2 + \frac{1}{4}k(\beta-\alpha)^2\right],
$$

giving $D \ge \frac{1}{2}$, and the sharpness of the constant $\frac{1}{2}$ is proved. □

Corollary 2.2. *With the assumptions of Lemma 2.1 and if $\gamma \in (\alpha,\beta)$ is a point of differentiability for h, then*

$$
\left(\frac{\alpha+\beta}{2} - \gamma\right)h'(\gamma) \le \left(\frac{\gamma-\alpha}{\beta-\alpha}\right)h(\alpha) + \left(\frac{\beta-\gamma}{\beta-\alpha}\right)h(\beta) - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt. \tag{2.13}
$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$
h\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt \le \frac{h(\alpha)+h(\beta)}{2}. \tag{2.14}
$$

The following corollary provides some bounds for the difference

$$\frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)\,dt.$$

Corollary 2.3. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequality*

$$\begin{aligned} 0 &\le \frac{1}{8}\left[h'_{+}\left(\frac{\alpha+\beta}{2}\right)-h'_{-}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ &\le \frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)\,dt \\ &\le \frac{1}{8}\left[h'_{-}(\beta)-h'_{+}(\alpha)\right](\beta-\alpha). \end{aligned} \tag{2.15}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

We are now able to state the corresponding result for convex functions defined on linear spaces.

Theorem 2.4. *Let X be a linear space, $a,b\in X$, $a\neq b$ and $f:[a,b]\subset X\to\mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned} &\frac{1}{2}\left[(1-s)^{2}\left(\bigtriangledown_{+}f\left[(1-s)a+sb\right]\right)(b-a)-s^{2}\left(\bigtriangledown_{-}f\left[(1-s)a+sb\right]\right)(b-a)\right] \\ &\qquad\le (1-s)f(a)+sf(b)-\int_{0}^{1}f\left[(1-t)a+tb\right]dt \\ &\qquad\le \frac{1}{2}\left[(1-s)^{2}\left(\bigtriangledown_{-}f(b)\right)(b-a)-s^{2}\left(\bigtriangledown_{+}f(a)\right)(b-a)\right]. \end{aligned} \tag{2.16}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $s=0$ or $s=1$.

Proof. Follows by Lemma 2.1 applied for the convex function

$$h(t)=g(a,b)(t)=f\left[(1-t)a+tb\right],\quad t\in[0,1],$$

and for the choices $\alpha=0$, $\beta=1$, and $\gamma=s$. □

Corollary 2.5. *If $f:[a,b]\to\mathbb{R}$ is as in Theorem 2.4 and Gâteaux differentiable in $c:=(1-\lambda)a+\lambda b$, $\lambda\in(0,1)$ along the direction $(b-a)$, then we have the inequality:*

$$\left(\frac{1}{2}-\lambda\right)\left(\bigtriangledown f(c)\right)(b-a)\le(1-\lambda)f(a)+\lambda f(b)-\int_{0}^{1}f\left[(1-t)a+tb\right]dt. \tag{2.17}$$

The following result related to the second Hermite-Hadamard inequality for functions defined on linear spaces also holds.

Corollary 2.6. *If f is as in Theorem 2.4, then*

$$\begin{aligned} &\frac{1}{8}\left[\bigtriangledown_{+}f\left(\frac{a+b}{2}\right)(b-a)-\bigtriangledown_{-}f\left(\frac{a+b}{2}\right)(b-a)\right] \\ &\qquad\le \frac{f(a)+f(b)}{2}-\int_{0}^{1}f\left[(1-t)a+tb\right]dt \\ &\qquad\le \frac{1}{8}\left[\left(\bigtriangledown_{-}f(b)\right)(b-a)-\left(\bigtriangledown_{+}f(a)\right)(b-a)\right]. \end{aligned} \tag{2.18}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Now, let $\Omega \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$.

If $F : \Omega \to \mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c} \in \Omega$ we have

$$\nabla F(\bar{c})(\bar{y}) = \sum_{i=1}^{n} \frac{\partial F(\bar{c})}{\partial x_i} \cdot y_i, \quad \bar{y} \in \mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i = 1, \ldots, n)$.

Using (2.16), we may state that

$$\begin{aligned} &\left(\frac{1}{2} - \lambda\right) \sum_{i=1}^{n} \frac{\partial F\left(\lambda \bar{a} + (1-\lambda)\bar{b}\right)}{\partial x_i} \cdot (b_i - a_i) \\ &\quad \leq (1-\lambda) F(\bar{a}) + \lambda F\left(\bar{b}\right) - \int_0^1 F\left[(1-t)\bar{a} + t\bar{b}\right] dt \\ &\quad \leq \frac{1}{2}\left[(1-\lambda)^2 \sum_{i=1}^{n} \frac{\partial F\left(\bar{b}\right)}{\partial x_i} \cdot (b_i - a_i) - \lambda^2 \sum_{i=1}^{n} \frac{\partial F(\bar{a})}{\partial x_i} \cdot (b_i - a_i)\right] \end{aligned} \tag{2.19}$$

for any $\bar{a}, \bar{b} \in \Omega$ and $\lambda \in (0,1)$.

In particular, for $\lambda = \frac{1}{2}$, we get

$$\begin{aligned} 0 &\leq \frac{F(\bar{a}) + F\left(\bar{b}\right)}{2} - \int_0^1 F\left[(1-t)\bar{a} + t\bar{b}\right] dt \\ &\leq \frac{1}{8} \sum_{i=1}^{n} \left(\frac{\partial F\left(\bar{b}\right)}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right) \cdot (b_i - a_i). \end{aligned} \tag{2.20}$$

In (2.20) the constant $\frac{1}{8}$ is sharp.

3. APPLICATIONS FOR SEMI-INNER PRODUCTS

Let $(X, \|\cdot\|)$ be a real normed linear space. We may state the following results for the semi-inner products $\langle \cdot, \cdot \rangle_i$ and $\langle \cdot, \cdot \rangle_s$.

Proposition 3.1. *For any $x, y \in X$ and $\sigma \in (0,1)$ we have the inequalities:*

$$\begin{aligned} &(1-\sigma)^2 \langle y - x, (1-\sigma)x + \sigma y \rangle_s - \sigma^2 \langle y - x, (1-\sigma)x + \sigma y \rangle_i \\ &\quad \leq (1-\sigma)\|x\|^2 + \sigma \|y\|^2 - \int_0^1 \|(1-t)x + ty\|^2 dt \\ &\quad \leq (1-\sigma)^2 \langle y - x, y \rangle_i - \sigma^2 \langle y - x, y \rangle_s. \end{aligned} \tag{3.1}$$

The second inequality in (3.1) also holds for $\sigma = 0$ or $\sigma = 1$.

The proof is obvious by Theorem 2.4 applied for the convex function $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$.

If the space is *smooth*, then we may put $[x, y] = \langle x, y \rangle_i = \langle x, y \rangle_s$ for each $x, y \in X$ and the first inequality in (3.1) becomes

$$(1 - 2\sigma)[y - x, (1-\sigma)x + \sigma y] \leq (1-\sigma)\|x\|^2 + \sigma\|y\|^2 - \int_0^1 \|(1-t)x + ty\|^2 dt. \tag{3.2}$$

An interesting particular case one can get from (3.1) is the one for $\sigma = \frac{1}{2}$,

$$0 \leq \frac{1}{8}\left[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i\right] \leq \frac{\|x\|^2+\|y\|^2}{2} - \int_0^1 \|(1-t)x+ty\|^2 dt \leq \frac{1}{4}\left[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s\right]. \tag{3.3}$$

The inequality (3.3) provides a refinement and a counterpart for the second inequality in (1.1).

If we consider now two linearly independent vectors $x, y \in X$ and apply Theorem 2.4 for $f(x) = \|x\|$, $x \in X$, then we get

Proposition 3.2. *For any linearly independent vectors $x, y \in X$ and $\sigma \in (0,1)$, one has the inequalities:*

$$\frac{1}{2}\left[(1-\sigma)^2 \frac{\langle y-x, (1-\sigma)x+\sigma y\rangle_s}{\|(1-\sigma)x+\sigma y\|} - \sigma^2 \frac{\langle y-x, (1-\sigma)x+\sigma y\rangle_i}{\|(1-\sigma)x+\sigma y\|}\right] \leq (1-\sigma)\|x\| + \sigma\|y\| - \int_0^1 \|(1-t)x+ty\| dt \leq \frac{1}{2}\left[(1-\sigma)^2 \frac{\langle y-x, y\rangle_i}{\|y\|} - \sigma^2 \frac{\langle y-x, x\rangle_s}{\|x\|}\right]. \tag{3.4}$$

The second inequality also holds for $\sigma = 0$ or $\sigma = 1$.

We note that if the space is smooth, then we have

$$\left(\frac{1}{2}-\sigma\right)\cdot\frac{[y-x, (1-\sigma)x+\sigma y]}{\|(1-\sigma)x+\sigma y\|} \leq (1-\sigma)\|x\| + \sigma\|y\| - \int_0^1 \|(1-t)x+ty\| dt \tag{3.5}$$

and for $\sigma = \frac{1}{2}$, (3.4) will give the simple inequality

$$0 \leq \frac{1}{8}\left[\left\langle y-x, \frac{\frac{x+y}{2}}{\left\|\frac{x+y}{2}\right\|}\right\rangle_s - \left\langle y-x, \frac{\frac{x+y}{2}}{\left\|\frac{x+y}{2}\right\|}\right\rangle_i\right] \leq \frac{\|x\|+\|y\|}{2} - \int_0^1 \|(1-t)x+ty\| dt \leq \frac{1}{8}\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right]. \tag{3.6}$$

The inequality (3.6) provides a refinement and a couterpart of the second inequality in (1.2).

Moreover, if we assume that $(H, \langle\cdot,\cdot\rangle)$ is an inner product space, then by (3.6) we get for any $x, y \in H$ with $\|x\| = \|y\| = 1$ that

$$0 \leq 1 - \int_0^1 \|(1-t)x+ty\| dt \leq \frac{1}{8}\|y-x\|^2. \tag{3.7}$$

The constant $\frac{1}{8}$ is sharp.

Indeed, if we choose $H = \mathbb{R}$, $\langle a, b\rangle = a \cdot b$, $x = -1$, $y = 1$, then we get equality in (3.7).

We give now some examples.

(1) Let $\ell^2(\mathbb{K})$, $\mathbb{K}=\mathbb{C},\mathbb{R}$; be the Hilbert space of sequences $x=(x_i)_{i\in\mathbb{N}}$ with $\sum_{i=0}^{\infty}|x_i|^2<\infty$. Then, by (3.7), we have the inequalities

$$0 \leq 1-\int_0^1\left(\sum_{i=0}^{\infty}|(1-t)x_i+ty_i|^2\right)^{\frac{1}{2}}dt \leq \frac{1}{8}\cdot\sum_{i=0}^{\infty}|y_i-x_i|^2, \tag{3.8}$$

for any $x,y\in\ell^2(\mathbb{K})$ provided $\sum_{i=0}^{\infty}|x_i|^2=\sum_{i=0}^{\infty}|y_i|^2=1.$

(2) Let μ be a positive measure, $L_2(\Omega)$ the Hilbert space of $\mu-$measurable functions on Ω with complex values that are $2-$integrable on Ω, i.e., $f\in L_2(\Omega)$ iff $\int_\Omega|f(t)|^2d\mu(t)<\infty$. Then, by (3.7), we have the inequalities

$$0 \leq 1-\int_0^1\left(\int_\Omega|(1-\lambda)f(t)+\lambda g(t)|^2d\mu(t)\right)^{\frac{1}{2}}d\lambda \leq \frac{1}{8}\cdot\int_\Omega|f(t)-g(t)|^2d\mu(t) \tag{3.9}$$

for any $f,g\in L_2(\Omega)$ provided $\int_\Omega|f(t)|^2d\mu(t)=\int_\Omega|g(t)|^2d\mu(t)=1.$

REFERENCES

[1] I. CIORANESCU, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[2] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: `http://rgmia.vu.edu.au/monographs`)