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NOTE ON FENG QI'S INTEGRAL INEQUALITY

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ABSTRACT. We give a generalization of Feng Qi's result from [5] by showing that if a function $f \in \mathrm{C}^1([a,b])$ satisfies $f(a) \geq 0$ and $f'(x) \geq n(x-a)^{n-1}$ for $x \in [a,b]$ and a positive integer n then $\int_a^b [f(x)]^{n+2}dx \geq \left(\int_a^b f(x)dx\right)^{n+1}$ holds. This follows from our answer to Feng Qi's open problem.

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In the paper [5] Feng Qi proved the following proposition.

Proposition 1. *Suppose that* $f \in \mathrm{C}^n([a,b])$ *satisfies* $f^{(i)}(a) \geq 0$ *and* $f^{(n)}(x) \geq n!$ *for* $x \in [a,b]$, *where* $0 \leq i \leq n-1$ *and* $n \in \mathbb{N}$, *then*

$$\int_a^b [f(x)]^{n+2}dx \geq \left(\int_a^b f(x)dx\right)^{n+1}. \tag{1}$$

This motivated him to propose an open problem.

Problem 1. Under what conditions does the inequality

$$\int_a^b [f(x)]^{t}dx \geq \left(\int_a^b f(x)dx\right)^{t-1} \tag{2}$$

hold for $t > 1$?

In the joint paper [6], K.W. Yu and F. Qi obtained one answer to the above problem: inequality (2) is valid for all $f \in \mathrm{C}([a,b])$ such that $\int_a^b f(x)dx \geq (b-a)^{t-1}$ for given $t > 1$. Many authors considered different generalizations of Problem 1 (cf. [1, 2, 3, 4]).

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We will prove the following answer to Problem 1 which will imply a generalization of Proposition 1.

Theorem 2. *Suppose that* $f \in \mathrm{C}^1([a,b])$ *satisfies* $f(a) \geq 0$ *and* $f'(x) \geq (t-2)(x-a)^{t-3}$ *for* $x \in [a,b]$ *and* $t \geq 3$. *Then*

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x) dx \right)^{t-1} \tag{3}$$

holds. The equality holds only if $a = b$ *or* $f(x) = x - a$ *and* $t = 3$.

Proof. Let f be a function satisfying the conditions of Theorem 2. Function f is increasing because $f'(x) > 0$ for $x \in (a,b]$, so from $f(\xi) \leq f(x)$ for $\xi \in [a,x]$ we obtain

$$f(x)(x-a) \geq \int_a^x f(\xi) d\xi, \quad \text{for all } x \in [a,b]. \tag{4}$$

Now we define

$$F(x) \triangleq \int_a^x [f(\xi)]^t d\xi - \left(\int_a^x f(\xi) d\xi \right)^{t-1}.$$

Then $F(a) = 0$ and $F'(x) = f(x)G(x)$, where

$$G(x) = [f(x)]^{t-1} - (t-1) \left(\int_a^x f(\xi) d\xi \right)^{t-2}.$$

Clearly, $G(a) = [f(a)]^{t-1} \geq 0$ and

$$G'(x) = (t-1)f(x) \left([f(x)]^{t-3} f'(x) - (t-2) \left(\int_a^x f(\xi) d\xi \right)^{t-3} \right).$$

From the conditions of Theorem 2 and inequality (4) we have

$$[f(x)]^{t-3} f'(x) \geq (t-2)(f(x)(x-a))^{t-3} \geq (t-2) \left(\int_a^x f(\xi) d\xi \right)^{t-3}. \tag{5}$$

Thus $G'(x) \geq 0$, so with $G(a) \geq 0$ we get $G(x) \geq 0$. From $F(a) = 0$ and $F'(x) = f(x)G(x) \geq 0$ it follows that $F(x) \geq 0$ for all $x \in [a,b]$, particularly

$$F(b) = \int_a^b [f(\xi)]^t d\xi - \left(\int_a^b f(\xi) d\xi \right)^{t-1} \geq 0.$$

The equality in (3) holds only if $F'(x) = 0$ for all $x \in [a,b]$ which is equivalent to $f(a) = 0$ and $G'(x) = 0$ and according to (5), if $t > 3$, this is valid only for $f(a) = 0$, $f'(x) = (t-2)(x-a)^{t-3}$ and f constant on $[a,b]$. But the last two conditions cannot hold simultaneously if $b \neq a$. The other possibility for equality to hold is if $f(a) = 0$ and $t = 3$. In that case (5) implies that $f'(x) = 1$ on $[a,b]$ so $f(x) = x - a$. □

Corollary 3. *Suppose that* $f \in \mathrm{C}^1([a,b])$ *satisfies* $f(a) \geq 0$ *and* $f'(x) \geq n(x-a)^{n-1}$ *for* $x \in [a,b]$ *and a positive integer* n, *then*

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x) dx \right)^{n+1}.$$

Proof. Set $t = n + 2$ in Theorem 2. □

Remark 4. Now we show that Proposition 1 follows from Corollary 3. Let the function f satisfy the conditions of Proposition 1. Since $f^{(n)}(x) \geq n!$, successively integrating $n - 1$ times over $[a,x]$ we get $f'(x) \geq n(x-a)^{n-1}$, $x \in [a,b]$. Therefore the conditions of Corollary 3 are fulfilled.

REFERENCES

[1] L. BOUGOFFA, Notes on Qi type integral inequalities, *J. Inequal. Pure and Appl. Math.*, **4**(4) (2003), Art. 77. ONLINE [`http://jipam.vu.edu.au/article.php?sid=318`].

[2] V. CSISZÁR AND T.F. MÓRI, The convexity method of proving moment-type inequalities, *Statist. Probab. Lett.*, (2004), in press.

[3] S. MAZOUZI AND F. QI, On an open problem regarding an integral inequality, *J. Inequal. Pure and Appl. Math.*, **4**(2) (2003), Art. 31. ONLINE [`http://jipam.vu.edu.au/article.php?sid=269`].

[4] T.K. POGÁNY, On an open problem of F. Qi, *J. Inequal. Pure Appl. Math.*, **3**(4) (2002), Art. 54. ONLINE [`http://jipam.vu.edu.au/article.php?sid=206`].

[5] F. QI, Several integral inequalities, *J. Inequal. Pure and Appl. Math.*, **1**(2) (2000), Art. 19. ONLINE [`http://jipam.vu.edu.au/article.php?sid=113`].

[6] K.-W. YU AND F. QI, A short note on an integral inequality, *RGMIA Res. Rep. Coll.*, **4**(1) (2001), Art. 4, 23–25. ONLINE [`http://rgmia.vu.edu.au/v4n1.html`].