## THE BEST CONSTANT FOR AN ALGEBRAIC INEQUALITY

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Abstract: We determine the best constant  $\lambda$  for the inequality  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \geq 1$ 

 $\frac{\lambda}{1+16(\lambda-16)xyzt};$  where x,y,z,t>0; x+y+z+t=1. We also consider an analogous inequality with three variables. As a corollary we establish a re-

finement of Euler's inequality.



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### 1. Introduction

Recently the following inequality was proved [1, 2]:

(1.1) 
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge \frac{25}{1 + 48xyz},$$

where x, y, z > 0; x + y + z = 1. This inequality is the special case of the inequality

(1.2) 
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge \frac{\lambda}{1 + 3(\lambda - 9)xyz},$$

where  $\lambda > 0$ . Substituting in this inequality  $x = y = \frac{1}{4}$ ,  $z = \frac{1}{2}$  we obtain  $0 < \lambda \le 25$ . So  $\lambda = 25$  is the best constant for the inequality (1.2). As an immediate application one has the following geometric inequality [3]:

(1.3) 
$$\frac{R}{r} \ge 2 + \lambda \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a+b+c)^2},$$

where R and r are respectively the circumradius and inradius, and a,b,c are sides of a triangle, and  $\lambda \leq 8$ . Substituting a=b=3, c=2 and the corresponding values  $R=\frac{9\sqrt{2}}{8}$  and  $r=\frac{1}{\sqrt{2}}$  in (1.3) we obtain  $\lambda \leq 8$ . So  $\lambda=8$  is the best constant for the inequality (1.3), which is a refinement of Euler's inequality.

It is interesting to compare (1.3) with other known estimates of  $\frac{R}{r}$ . For example, it is well known that:

(1.4) 
$$\frac{R}{r} \ge \frac{(a+b)(b+c)(c+a)}{4abc}.$$

For the triangle with sides a=b=3, c=2 inequality (1.3) is stronger than (1.4) even for  $\lambda=3$ . But for  $\lambda=2$  inequality (1.4) is stronger than (1.3) for arbitrary



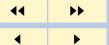
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triangles. This follows from the algebraic inequality:

(1.5) 
$$\frac{(x+y)(y+z)(z+x)}{8xyz} \ge \frac{3(x^2+y^2+z^2)}{(x+y+z)^2},$$

where x, y, z > 0, which is in turn equivalent to (1.1).

The main aim of the present article is to determine the best constant for the following analogue of the inequality (1.2):

(1.6) 
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge \frac{\lambda}{1 + 16(\lambda - 16)xyzt},$$

where x, y, z, t > 0; x + y + z + t = 1.

It is known that the best constant for the inequality

(1.7) 
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \le \lambda + \frac{16 - \lambda}{256xyzt},$$

where x, y, z, t > 0; x + y + z + t = 1, is  $\lambda = \frac{176}{27}$  (see e.g. [4, Corollary 2.13]). In [4] the problem on the determination of the best constants for inequalities similar to (1.7), with n variables was also completely studied:

$$\sum_{i=1}^{n} \frac{1}{x_i} \le \lambda + \frac{n^2 - \lambda}{n^n \prod_{i=1}^{n} x_i},$$

where  $x_1, x_2, \ldots, x_i > 0$ ,  $\sum_{i=1}^n x_i = 1$ . The best constant for this inequality is  $\lambda = n^2 - \frac{n^n}{(n-1)^{n-1}}$ . In particular if n = 3 then the strongest inequality is

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{9}{4} + \frac{1}{4xyz},$$



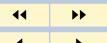
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where  $x,y,z>0,\,x+y+z=1$ , which is in turn equivalent to the geometric inequality

$$p^2 \ge 16Rr - 5r^2,$$

where p is the semiperimeter of a triangle. But this inequality follows directly from the formula for the distance between the incenter I and the centroid G of a triangle:

$$|IG|^2 = \frac{1}{9}(p^2 + 5r^2 - 16Rr).$$

For some recent results see [4], [6] - [8] and especially [9].



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## 2. Prelimary Results

The results presented in this section aim to demonstrate the main ideas of the proof of Theorem 3.1 in a more simpler problem. Corollaries have an independent interest.

**Theorem 2.1.** Let x, y, z > 0 and x + y + z = 1. Then the inequality (1.1) is true.

*Proof.* We shall prove the equivalent inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 48(xy + yz + zx) \ge 25.$$

Without loss of generality we may suppose that  $x+y \leq \frac{1}{\sqrt[3]{3}}$ . Indeed, if  $x+y > \frac{1}{\sqrt[3]{3}}$ ,  $y+z > \frac{1}{\sqrt[3]{3}}$ ,  $z+x > \frac{1}{\sqrt[3]{3}}$  then by summing these inequalities we obtain  $2 > \frac{3}{\sqrt[3]{3}}$ , which is false.

Let

$$f(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 48(xy + yz + zx).$$

We shall prove that

$$f(x, y, z) \ge f\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right) \ge 25.$$

The first inequality in this chain obtains, after simplifications, the form  $\frac{1}{12} \ge xy(x+y)$ , which is the consequence of  $x+y \le \frac{1}{\sqrt[3]{3}}$ . Denoting  $\frac{x+y}{2} = \ell$   $(z=1-2\ell)$  in the second inequality of the chain, after some simplification, we obtain

$$144\ell^4 - 168\ell^3 + 73\ell^2 - 14\ell + 1 \ge 0 \iff (3\ell - 1)^2(4\ell - 1)^2 \ge 0.$$



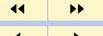
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**Corollary 2.2.** Let x, y, z > 0. Then inequality (1.5) holds true.

*Proof.* Inequality (1.5) is homogeneous in its variables x, y, z. We may suppose, without loss of generality, that x + y + z = 1, after which the inequality obtains the following form:

$$\frac{xy+yz+zx-xyz}{xyz} \ge 24(1-2(xy+yz+zx))$$

$$\iff \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 \ge 24 - 48(xy+yz+zx).$$

By Theorem 2.1 the last inequality is true.

**Corollary 2.3.** For an arbitrary triangle the following inequality is true:

$$\frac{R}{r} \ge 2 + 8 \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{(a+b+c)^2}.$$

*Proof.* Using known formulas

$$S = \sqrt{p(p-a)(p-b)(p-c)}, \quad R = \frac{abc}{4S}, \quad r = \frac{S}{p},$$

where S and p are respectively, the area and semiperimeter of a triangle, we transform the inequality to

$$\frac{2abc}{(a+b-c)(b+c-a)(c+a-b)} \ge \frac{18(a^2+b^2+c^2)-12(ab+bc+ca)}{(a+b+c)^2}.$$

Using substitutions a = x + y, b = y + z, c = z + x, where x, y, z are positive numbers by the triangle inequality, we transform the last inequality to

$$\frac{(x+y)(y+z)(z+x)}{xyz} \ge \frac{24(x^2+y^2+z^2)}{(x+y+z)^2},$$

which follows from Corollary 2.2.



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### 3. Main Result

**Theorem 3.1.** The greatest value of the parameter  $\lambda$ , for which the inequality

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge \frac{\lambda}{1 + 16(\lambda - 16)xyzt},$$

where x, y, z, t > 0, x + y + z + t = 1, is true is

$$\lambda = \frac{582\sqrt{97} - 2054}{121}.$$

*Proof.* Substituting in the inequality (1.6) the values

$$x = y = z = \frac{5 + \sqrt{97}}{72}, \ t = \frac{19 - \sqrt{97}}{24},$$

we obtain,

$$\lambda \le \lambda_0 = \frac{582\sqrt{97} - 2054}{121}.$$

We shall prove that inequality (1.6) holds for  $\lambda = \lambda_0$ .

Without loss of generality we may suppose that  $x \le y \le z \le t$ . We define sequences  $\{x_n\}, \{y_n\}, \{z_n\} \ (n \ge 0)$  by the equalities

$$x_0 = x$$
,  $y_0 = y$ ,  $z_0 = z$ ,  $x_{2k+1} = \frac{1}{2}(x_{2k} + y_{2k})$ ,  $y_{2k+1} = \frac{1}{2}(x_{2k} + y_{2k})$ ,  $z_{2k+1} = z_{2k}$ ,

and

$$x_{2k+2} = x_{2k+1}, \quad y_{2k+2} = \frac{1}{2}(y_{2k+1} + z_{2k+1}), \quad z_{2k+2} = \frac{1}{2}(y_{2k+1} + z_{2k+1}),$$



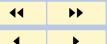
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where  $k \geq 0$ . From these equalities we obtain,

$$y_n = \frac{x+y+z}{3} + \frac{2z-x-y}{3} \left(-\frac{1}{2}\right)^n$$

where n > 1. Then we have,

$$\lim y_n = \frac{x+y+z}{3}.$$

Since  $x_{2k} = x_{2k-1} = y_{2k-1}$  and  $z_{2k+1} = z_{2k} = y_{2k}$  for k > 0, then we have also,

(3.1) 
$$\lim x_n = \lim z_n = \lim y_n = \frac{x + y + z}{3}.$$

We note also that,

(3.2) 
$$(x+y)^3(z+t) \le \frac{1}{\lambda_0 - 16}.$$

Indeed, on the contrary we have,

$$(x+y)(z+t)^3 \ge (x+y)^3(z+t) > \frac{1}{\lambda_0 - 16}$$

from which we obtain,

$$(x+y)^2(z+t)^2 > \frac{1}{\lambda_0 - 16}.$$

Then

$$\frac{1}{16} = \left(\frac{(x+y) + (z+t)}{2}\right)^4 \ge (x+y)^2(z+t)^2 > \frac{1}{\lambda_0 - 16},$$



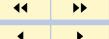
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which is false, because  $\lambda_0 < 32$ . Therefore the inequality (3.2) is true. In the same manner, we can prove that

(3.3) 
$$(x+z)^3(y+t) \le \frac{1}{\lambda_0 - 16}.$$

Let

$$f(x,y,z,t) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} + 16(\lambda_0 - 16)(xyz + xyt + xzt + yzt).$$

Firstly we prove that

(3.4) 
$$f(x, y, z, t) = f(x_0, y_0, z_0, t) \ge f(x_1, y_1, z_1, t)$$

$$\iff \frac{1}{x} + \frac{1}{y} + 16(\lambda_0 - 16)xy(z + t)$$

$$\ge \frac{4}{x + y} + 16(\lambda_0 - 16)\left(\frac{x + y}{2}\right)^2(z + t)$$

$$\iff \frac{1}{\lambda_0 - 16} \ge 4xy(x + y)(z + t),$$

which follows from (3.2).

Since for arbitrary  $n \geq 0$  the inequality  $x_n \leq y_n \leq z_n \leq t$  is true then in the same manner we can prove that

$$(3.5) f(x_{2k}, y_{2k}, z_{2k}, t) \ge f(x_{2k+1}, y_{2k+1}, z_{2k+1}, t),$$

where k > 0.

We shall now prove that

$$(3.6) f(x_{2k+1}, y_{2k+1}, z_{2k+1}, t) \ge f(x_{2k+2}, y_{2k+2}, z_{2k+2}, t),$$



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where  $k \ge 0$ . Denote  $x' = x_{2k+1}, \ y' = y_{2k+1}, \ z' = z_{2k+1}, \ t' = t$ . By analogy with (3.3) we may write,

$$(x'+z')^3(y'+t') \le \frac{1}{\lambda_0 - 16}.$$

Since x' = y' then we can write the last inequality in this form:

$$(3.7) (y'+z')^3(x'+t') \le \frac{1}{\lambda_0 - 16}.$$

Similar to (3.4), simplifying (3.6) we obtain,

$$\frac{1}{\lambda_0 - 16} \ge 4y'z'(y' + z')(x' + t'),$$

which follows from (3.7).

By (3.4) - (3.6) we have,

(3.8) 
$$f(x, y, z, t) \ge f(x_n, y_n, z_n, t)$$

for  $n \ge 0$ . Denote  $\ell = \frac{x+y+z}{3}$  then  $t = 1 - 3\ell$ . Since f(x,y,z,t) is a continuous function for x,y,z,t>0, then tending n to  $\infty$  in (3.8), we obtain, by (3.1),

(3.9) 
$$f(x, y, z, t) \ge \lim_{n \to \infty} f(x_n, y_n, z_n, t) = f(\ell, \ell, \ell, 1 - 3\ell).$$

Thus it remains to show that

$$(3.10) f(\ell,\ell,\ell,1-3\ell) \ge \lambda_0.$$

After elementary but lengthy computations we transform (3.10) into

$$(4\ell - 1)^2 ((\lambda_0 - 16)\ell(3\ell - 1)(8\ell + 1) + 3) \ge 0,$$

where  $0 < \ell < \frac{1}{3}$ . It suffices to show that

$$\lambda_0 - 16 \le \frac{-3}{\ell(3\ell - 1)(8\ell + 1)} = g(\ell),$$



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for  $0 < \ell < \frac{1}{3}$ . The function  $g(\ell)$  obtains its minimum value at the point

$$\ell = \ell_0 = \frac{5 + \sqrt{97}}{72} \in \left(0, \frac{1}{3}\right),$$

at which  $g(\ell_0) = \lambda_0 - 16$ . Consequently, the last inequality is true.

From (3.9) and (3.10) it follows that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t} \ge \frac{\lambda_0}{1 + 16(\lambda_0 - 16)xyzt},$$

and the equality holds only for quadruples  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ ,  $(\ell_0, \ell_0, \ell_0, 1 - 3\ell_0)$  and 3 other permutations of the last.

The proof of Theorem 3.1 is complete.

*Remark* 1. An interesting problem for further exploration would be to determine the best constant  $\lambda$  for the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i} \ge \frac{\lambda}{1 + n^{n-2}(\lambda - n^2) \prod_{i=1}^{n} x_i},$$

where  $x_1, x_2, \ldots, x_n > 0$ ,  $\sum_{i=1}^n x_i = 1$ , for n > 4. It seems very likely that the number

$$\lambda = \frac{12933567 - 93093\sqrt{22535}}{4135801}\alpha + \frac{17887113 + 560211\sqrt{22535}}{996728041}\alpha^2 - \frac{288017}{17161},$$

where  $\alpha = \sqrt[3]{8119 + 48\sqrt{22535}}$ , is the best constant in the case n = 5. For greater values of n, it is reasonable to find an asymptotic formula of the best constant.



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