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A STUDY ON ALMOST INCREASING SEQUENCES

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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper by using an almost increasing sequence a general theorem on $\varphi - |C, \alpha|_k$ summability factors, which generalizes some known results, has been proved under weaker conditions.

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Contents

1	Introduction	3
2	Result	6
3	Proof of Theorem 2.1	8
4	Special Cases	12
	References	



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

Page 2 of 13

1. Introduction

Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n) . We denote by σ_n^α and t_n^α the n -th Cesàro means of order α , with $\alpha > -1$, of the sequences (s_n) and (na_n) , respectively, i.e.,

$$(1.1) \quad \sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} s_v$$

and

$$(1.2) \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v,$$

where

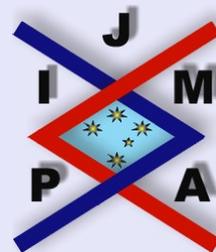
$$(1.3) \quad A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

The series $\sum a_n$ is said to be $|C, \alpha|_k$ summable for $k \geq 1$ and $\alpha > -1$, if (see [5])

$$(1.4) \quad \sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty.$$

and it is said to be $|C, \alpha; \beta|_k$ summable for $k \geq 1$, $\alpha > -1$ and $\beta \geq 0$, if (see [6])

$$(1.5) \quad \sum_{n=1}^{\infty} n^{\beta k + k - 1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\beta k - 1} |t_n^\alpha|^k < \infty.$$



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

Page 3 of 13

The series $\sum a_n$ is said to be φ - $|C, \alpha|_k$ summable for $k \geq 1$ and $\alpha > -1$, if (see [2])

$$(1.6) \quad \sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n^\alpha|^k < \infty.$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\varphi_n = n^{\beta+1-\frac{1}{k}}$) φ - $|C, \alpha|_k$ summability is the same as $|C, \alpha|_k$ (resp. $|C, \alpha; \beta|_k$) summability.

Bor [3] has proved the following theorem for φ - $|C, 1|_k$ summability factors of infinite series.

Theorem 1.1. *Let (X_n) be a positive non-decreasing sequence and let (λ_n) be a sequence such that*

$$(1.7) \quad |\lambda_n| X_n = O(1) \quad \text{as } n \rightarrow \infty$$

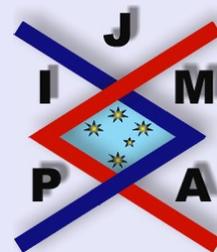
and

$$(1.8) \quad \sum_{v=1}^n v X_v |\Delta^2 \lambda_v| = O(1) \quad \text{as } n \rightarrow \infty.$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and

$$(1.9) \quad \sum_{v=1}^n v^{-k} |\varphi_v t_v|^k = O(X_n) \quad \text{as } n \rightarrow \infty,$$

then the series $\sum a_n \lambda_n$ is φ - $|C, 1|_k$ summable for $k \geq 1$.



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



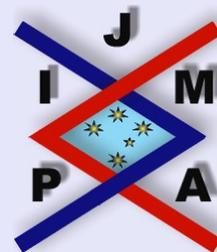
Go Back

Close

Quit

Page 4 of 13

The aim of this paper is to generalize Theorem 1.1 under weaker conditions for φ - $|C, \alpha|_k$ summability. For this we need the concept of almost increasing sequences. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is an almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of the theorem by replacing the increasing sequence with an almost increasing sequence.



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

Page 5 of 13

2. Result

Now, we shall prove the following:

Theorem 2.1. *Let (X_n) be an almost increasing sequence and the sequence (λ_n) such that conditions (1.7) – (1.8) of Theorem 1.1 are satisfied. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and if the sequence (w_n^α) , defined by (see [9])*

$$(2.1) \quad w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1 \end{cases}$$

satisfies the condition

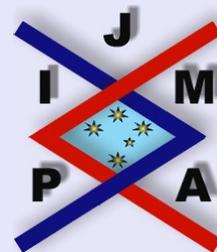
$$(2.2) \quad \sum_{n=1}^m n^{-k} (w_n^\alpha |\varphi_n|)^k = O(X_m) \quad \text{as } m \rightarrow \infty,$$

then the series $\sum a_n \lambda_n$ is $\varphi-|C, \alpha|_k$ summable for $k \geq 1$, $0 < \alpha \leq 1$ and $k\alpha + \epsilon > 1$.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. ([4]). *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$(2.3) \quad \left| \sum_{p=0}^v A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^m A_{m-p}^{\alpha-1} a_p \right|.$$



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

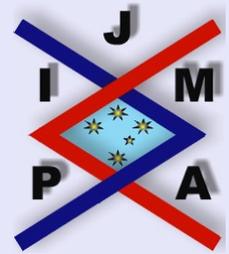
Page 6 of 13

Lemma 2.3. ([8]). If (X_n) is an almost increasing sequence and the conditions (1.7) and (1.8) of Theorem 1.1 are satisfied, then

$$(2.4) \quad \sum_{n=1}^m X_n |\Delta\lambda_n| = O(1)$$

and

$$(2.5) \quad mX_m |\Delta\lambda_m| = O(1), \quad m \rightarrow \infty.$$



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

Page 7 of 13

3. Proof of Theorem 2.1

Let (T_n^α) be the n -th (C, α) , with $0 < \alpha \leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} va_v \lambda_v.$$

Using Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} pa_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} va_v,$$

so that making use of Lemma 2.2, we have

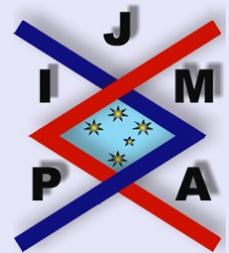
$$\begin{aligned} |T_n^\alpha| &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} |\Delta \lambda_v| \left| \sum_{p=1}^v A_{n-p}^{\alpha-1} pa_p \right| + \frac{|\lambda_n|}{A_n^\alpha} \left| \sum_{v=1}^n A_{n-v}^{\alpha-1} va_v \right| \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta \lambda_v| + |\lambda_n| w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.} \end{aligned}$$

Since

$$|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k \left(|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k \right),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n T_{n,r}^\alpha|^k < \infty \quad \text{for } r = 1, 2, \quad \text{by (1.6).}$$



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

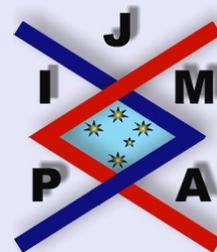
Close

Quit

Page 8 of 13

Now, when $k > 1$, applying Hölder's inequality with indices k and k' , where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
 & \sum_{n=2}^{m+1} n^{-k} |\varphi_n T_{n,1}^\alpha|^k \\
 & \leq \sum_{n=2}^{m+1} n^{-k} (A_n^\alpha)^{-k} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta \lambda_v| \right\}^k \\
 & = O(1) \sum_{n=2}^{m+1} n^{-k} n^{-\alpha k} |\varphi_n|^k \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (w_v^\alpha)^k |\Delta \lambda_v| \right\} \left\{ \sum_{v=1}^{n-1} |\Delta \lambda_v| \right\}^{k-1} \\
 & = O(1) \sum_{v=1}^m v^{\alpha k} (w_v^\alpha)^k |\Delta \lambda_v| \sum_{n=v+1}^{m+1} \frac{n^{-k} |\varphi_n|^k}{n^{\alpha k}} \\
 & = O(1) \sum_{v=1}^m v^{\alpha k} (w_v^\alpha)^k |\Delta \lambda_v| \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} |\varphi_n|^k}{n^{\alpha k + \epsilon}} \\
 & = O(1) \sum_{v=1}^m v^{\alpha k} (w_v^\alpha)^k |\Delta \lambda_v| v^{\epsilon-k} |\varphi_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k + \epsilon}} \\
 & = O(1) \sum_{v=1}^m v^{\alpha k} (w_v^\alpha)^k |\Delta \lambda_v| v^{\epsilon-k} |\varphi_v|^k \int_v^\infty \frac{dx}{x^{\alpha k + \epsilon}} \\
 & = O(1) \sum_{v=1}^m v |\Delta \lambda_v| v^{-k} (w_v^\alpha |\varphi_v|)^k \\
 & = O(1) \sum_{v=1}^{m-1} \Delta(v |\Delta \lambda_v|) \sum_{r=1}^v r^{-k} (w_r^\alpha |\varphi_r|)^k + O(1) m |\Delta \lambda_m| \sum_{v=1}^m v^{-k} (w_v^\alpha |\varphi_v|)^k
 \end{aligned}$$



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

Page 9 of 13

$$\begin{aligned}
&= O(1) \sum_{v=1}^{m-1} |\Delta(v|\Delta\lambda_v)| X_v + O(1)m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta^2\lambda_v| X_v + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_{v+1} + O(1)m |\Delta\lambda_m| X_m \\
&= O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.3.
Again, since $|\lambda_n| = O(1/X_n) = O(1)$, by (1.7), we have that

$$\begin{aligned}
&\sum_{n=1}^m n^{-k} |\varphi_n T_{n,2}^\alpha|^k \\
&= \sum_{n=1}^m |\lambda_n|^{k-1} |\lambda_n| n^{-k} (w_n^\alpha |\varphi_n|)^k \\
&= O(1) \sum_{n=1}^m |\lambda_n| n^{-k} (w_n^\alpha |\varphi_n|)^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n v^{-k} (w_v^\alpha |\varphi_v|)^k + O(1) |\lambda_m| \sum_{n=1}^m n^{-k} (w_n^\alpha |\varphi_n|)^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \rightarrow \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.3.



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

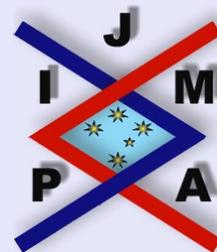
Quit

Page 10 of 13

Therefore, we get that

$$\sum_{n=1}^m n^{-k} |\varphi_n T_{n,r}^\alpha|^k = O(1) \quad \text{as } m \rightarrow \infty, \quad \text{for } r = 1, 2.$$

This completes the proof of the theorem.



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

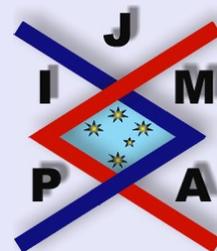
Close

Quit

Page 11 of 13

4. Special Cases

1. If we take (X_n) as a positive non-decreasing sequence, $\alpha = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ in Theorem 2.1, then we get Theorem 1.1.
2. If we take (X_n) as a positive non-decreasing sequence, $\alpha = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ in Theorem 2.1, then we get a result due to Mazhar [7] for $|C, 1|_k$ summability factors of infinite series.
3. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\epsilon = 1$ and $\varphi_n = n^{\beta+1-\frac{1}{k}}$), then we get a new result related to $|C, \alpha|_k$ (resp. $|C, \alpha; \beta|_k$) summability factors.



A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

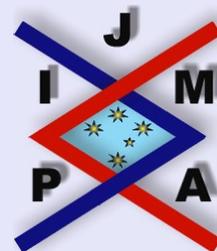
Close

Quit

Page 12 of 13

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A Study on Almost Increasing Sequences

Hüseyin Bor

Title Page

Contents



Go Back

Close

Quit

Page 13 of 13