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ON THE INTERPOINT DISTANCE SUM INEQUALITY

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ABSTRACT. Let n points be arbitrarily placed in $B(D)$, a disk in $\mathbb{R}^2$ having diameter D. Denote by l_{ij} the Euclidean distance between point i and j. In this paper, we show

$$\sum_{i=1}^{n}\left(\min_{j\neq i} l_{ij}^2\right) \leq \frac{D^2}{0.3972}.$$

We then extend the result to $\mathbb{R}^3$.

Key words and phrases: Combinatorial geometry, Distance geometry, Interpoint distance sum inequality, Optimization.

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1. INTRODUCTION

To estimate upper bounds on the maximum number of simultaneously successful wireless transmissions and the maximum achievable per-node end-to-end throughput under the general network scenario, Arpacioglu and Haas [1] introduced the following interesting inequalities. For the sake of clarity in presentation, we use the notation $argmin_{j\in J}\{S_j\}$ to denote the index of the smallest point in the set $\{S_j\}$ ($j \in J$). If there are several smallest elements, we take the first one.

Theorem 1.1 ([1])**.** *Let $B(D)$ be a disk in $\mathbb{R}^2$ having diameter D. Let n points be arbitrarily placed in $B(D)$. Suppose each point is indexed by a distinct integer between 1 and n. Let l_{ij} be the Euclidean distance between points i and j. Define the mth closest point to point i, a_{im}, and the Euclidean distance between point i and the mth closest point to point i, u_{im}, as follows:*

$$a_{i1} := \underset{\substack{j\in\{1,2,\ldots,n\},\\ j\neq i}}{argmin}\{l_{ij}\}, \quad 1\leq i\leq n,$$

$$a_{im} := \underset{\substack{j\in\{1,2,\ldots,n\},\\ j\notin\{i\}\cup\{a_{ik}\}_{k=1}^{m-1}}}{argmin}\{l_{ij}\}, \quad 1\leq i\leq n,\ 2\leq m\leq n-1,$$

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$$u_{im} := l_{ia_{im}}, \quad 1 \le i \le n,\ 1 \le m \le n-1.$$

Then

$$\sum_{i=1}^{n} u_{im}^2 \le \frac{mD^2}{c_2}, \quad 1 \le m \le n-1, \tag{1.1}$$

where

$$c_2 := \frac{2}{3} - \frac{\sqrt{3}}{2\pi} \approx 0.3910.$$

We observed [2] that the interpoint distance sum inequality (1.1) can be simply yet significantly strengthened.

Proposition 1.2. *Define* $B(D), D, n, l_{ij}, a_{im}, u_{im}, c_2$ *as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{im}^2 \le \frac{mD^2}{c_2}, \quad 1 \le m < c_2 n, \tag{1.2}$$

$$\sum_{i=1}^{n} u_{im}^2 \le nD^2, \quad c_2 n < m \le n-1. \tag{1.3}$$

The proof follows from (1.1) and the fact that $u_{im} \le D$.

As a direct application, we improved [2] the upper bounds on the maximum number of simultaneously successful wireless transmissions and the maximum achievable per-node end-to-end throughput under the same general network scenario as in Arpacioglu and Haas [1].

2. MAIN RESULT

In this section, we show that the interpoint distance sum inequality (1.1) when $m = 1$ can be further improved.

Theorem 2.1. *Define* $B(D), D, n, l_{ij}, a_{im}, u_{im}, c_2$ *as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{i1}^2 \le \frac{D^2}{0.3972}.$$

Proof. The case $n = 2$ is trivial to verify since $m = 1$ and $u_{im} \le D$. So we assume $n \ge 3$. The proof is based on that of Theorem 1.1 [1]. Denote the disk of diameter x and center i by $B_i(x)$. Define the following sets of disks

$$R_m := \{B_i(u_{im}) : 1 \le i \le n\}, \quad 1 \le m \le n-1.$$

First consider the disks in R_1. As shown in [1], all disks in R_1 are non-overlapping, i.e., the distance between the centers of any two disks is smaller than the sum of the radii of the two disks.

Denote by $A(X)$ the area of a region X. We try to find a lower bound on $f_{im} := A(B(D) \cap B_i(u_{im}))/A(B_i(u_{im}))$ for every $1 \le i \le n$ and $1 \le m \le n-1$. Pick any point S from the boundary of $B(D)$ and consider the overlap ratio

$$f_{im}^S := \frac{A(B(D) \cap B_S(u_{im}))}{A(B_S(u_{im}))}, \qquad 1 \le i \le n,\ 1 \le m \le n-1.$$

Using Figure 2.1, one can obtain the geometrical computation formula: $f_{im}^S = f(y)|_{y=\frac{u_{im}}{D}}$, where

$$f(y) := \frac{1}{\pi}\left(1 - \frac{2}{y^2}\right)\arccos\left(\frac{y}{2}\right) + \frac{1}{y^2} - \frac{1}{\pi}\sqrt{\frac{1}{y^2} - \frac{1}{4}}. \tag{2.1}$$

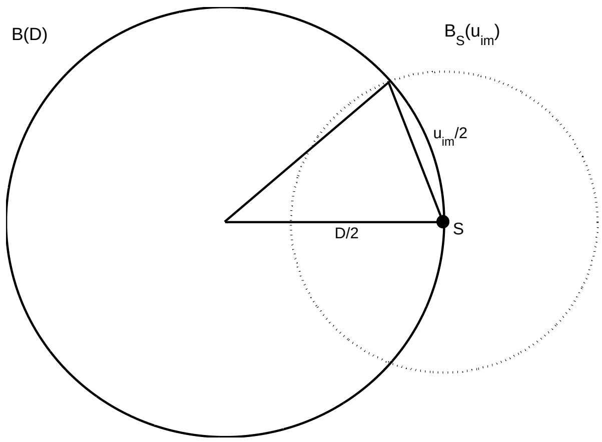

Figure 2.1: Computation of the overlap ratio between $B(D)$ and $B_s(u_{im})$.

Actually $f(y)$ is a decreasing function of y. We have $f_{im}^S \geq f(1)$ due to $u_{im} \leq D$. Also $f_{im} \geq f_{im}^S$. Setting $c_2 := f(1)$, we obtain the following lower bound on f_{im} for every $1 \leq i \leq n$ and $1 \leq m \leq n-1$,

$$f_{im} \geq c_2, \quad \text{where } c_2 = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} \approx 0.3910.$$

Therefore the area of the parts of the disks in R_m that lie in $B(D)$ is at least $c_2 A(B(D))$. Hence, for every $1 \leq i \leq n$ and $1 \leq m \leq n-1$,

$$A(B_i(u_{im}) \cap B(D)) \geq c_2 A(B_i(u_{im})). \tag{2.2}$$

For a given value m, adding the n inequalities in (2.2), we obtain

$$\sum_{i=1}^{n} A(B_i(u_{im}) \cap B(D)) \geq c_2 \sum_{i=1}^{n} A(B_i(u_{im})), \quad \forall\, 1 \leq m \leq n-1. \tag{2.3}$$

Since all disks in R_1 are non-overlapping, we have

$$\sum_{i=1}^{n} A(B_i(u_{im}) \cap B(D)) \leq A(B(D)). \tag{2.4}$$

Inequalities (2.3) and (2.4) imply

$$A(B(D)) \geq c_2 \sum_{i=1}^{n} A(B_i(u_{im})).$$

Notice that $A(B(D)) = \pi D^2/4$ and $A(B_i(u_{i1})) = \pi u_{i1}^2/4$. Therefore,

$$\sum_{i=1}^{n} u_{i1}^2 \leq \frac{D^2}{c_2}. \tag{2.5}$$

Also, it is easy to see that $f(y)$, defined in (2.1), is a concave function. Then $f(y)$ has a linear underestimation, denoted by

$$l(y) := c_2 + k - ky,$$

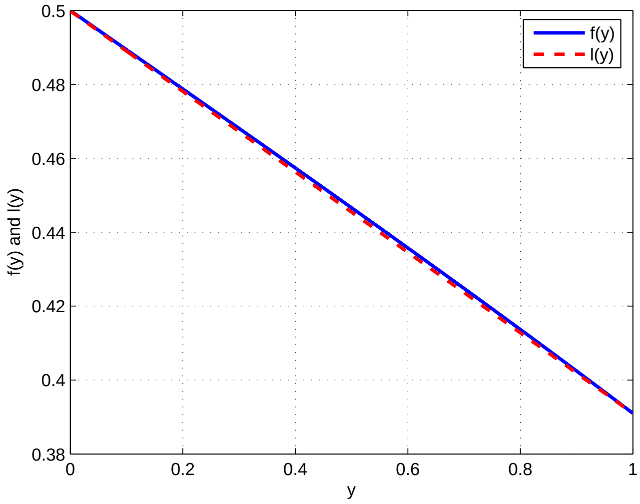

Figure 2.2: Variations of $f(y)$ and $l(y)$.

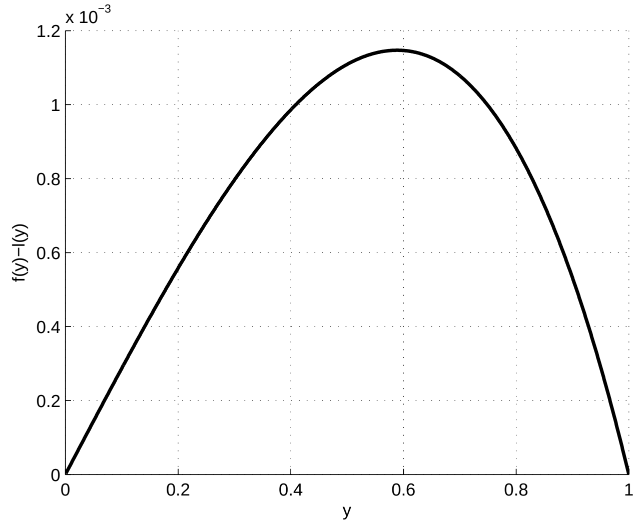

Figure 2.3: Variation of $f(y) - l(y)$.

where

$$k := \frac{f(0) - f(1)}{1 - 0} = \lim_{y \to 0} f(y) - f(1) = 0.5 - c_2 \approx 0.1090.$$

Figure 2.2 shows the variation of $f(y)$ and $l(y)$, respectively. Figure 2.3 shows the variation of $f(y) - l(y)$ with respect to y.

Now we have

$$f_{im} \geq f_{im}^S = f\left(\frac{u_{im}}{D}\right) \geq c_2 + k - k\frac{u_{im}}{D}.$$

Therefore, for every $1 \leq i \leq n$ and $1 \leq m \leq n - 1$,

$$A(B_i(u_{im}) \cap B(D)) \geq (c_2 + k)A(B_i(u_{im})) - k\frac{u_{im}}{D}A(B_i(u_{im})). \tag{2.6}$$

Adding all the n inequalities in (2.6) for a given m, we obtain

$$\sum_{i=1}^{n} A(B_i(u_{im}) \cap B(D)) \geq (c_2 + k) \sum_{i=1}^{n} A(B_i(u_{im})) - \frac{k}{D} \sum_{i=1}^{n} u_{im} A(B_i(u_{im})), \quad \forall 1 \leq m \leq n-1.$$

Using (2.4) and the facts $A(B(D)) = \pi D^2/4$ and $A(B_i(u_{i1})) = \pi u_{i1}^2/4$, we obtain

$$D^2 \geq (c_2 + k) \sum_{i=1}^{n} u_{i1}^2 - \frac{k}{D} \sum_{i=1}^{n} u_{i1}^3. \tag{2.7}$$

Now consider the following optimization problem ($n \geq 3$):

$$\max \sum_{i=1}^{n} u_{i1}^3 \tag{2.8}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} u_{i1}^2 \leq \frac{D^2}{c_2} \tag{2.9}$$

$$0 \leq u_{i1} \leq D, \quad i = 1, \ldots, n. \tag{2.10}$$

The objective function (2.8) is strictly convex and the feasible region defined by (2.9) – (2.10) is also convex. Since $n \geq 3$ and $2 < \frac{1}{c_2} < 3$, the inequality (2.9) holds at any of the optimal solutions. Therefore the optimal solutions of (2.8) – (2.10) must occur at the vertices of the set

$$\left\{ (u_{i1}) : \sum_{i=1}^{n} u_{i1}^2 = \frac{D^2}{c_2},\ 0 \leq u_{i1} \leq D, i = 1, \ldots, n \right\}.$$

Any (u_{i1}) with two components lying strictly between 0 and D cannot be a vertex. Therefore every optimal solution of (2.8) – (2.10) has $\left\lfloor \frac{1}{c_2} \right\rfloor$ components with the value D, one component with the value $\sqrt{\frac{1}{c_2} - \left\lfloor \frac{1}{c_2} \right\rfloor} D$ and the others are zeros, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Then the optimal objective value is

$$\left\lfloor \frac{1}{c_2} \right\rfloor D^3 + \left(\frac{1}{c_2} - \left\lfloor \frac{1}{c_2} \right\rfloor \right)^{\frac{3}{2}} D^3.$$

In other words, we have proved for valid u_{i1} that

$$\sum_{i=1}^{n} u_{i1}^3 \leq \left\lfloor \frac{1}{c_2} \right\rfloor D^3 + \left(\frac{1}{c_2} - \left\lfloor \frac{1}{c_2} \right\rfloor \right)^{\frac{3}{2}} D^3.$$

Now (2.7) becomes

$$D^2 \geq c_2 \sum_{i=1}^{n} u_{i1}^2 + k \left(\sum_{i=1}^{n} u_{i1}^2 - \left(\left\lfloor \frac{1}{c_2} \right\rfloor + \left(\frac{1}{c_2} - \left\lfloor \frac{1}{c_2} \right\rfloor \right)^{\frac{3}{2}} \right) D^2 \right). \tag{2.11}$$

Then we have

$$\sum_{i=1}^{n} u_{i1}^2 \leq \frac{D^2 \left(1 + k \left(\left\lfloor \frac{1}{c_2} \right\rfloor + \left(\frac{1}{c_2} - \left\lfloor \frac{1}{c_2} \right\rfloor \right)^{\frac{3}{2}} \right) \right)}{c_2 \left(1 + k \frac{1}{c_2} \right)}.$$

Comparing with (2.5), we actually obtain a new c_2^+:

$$c_2^+ = \frac{c_2\left(1 + k\frac{1}{c_2}\right)}{1 + k\left(\left\lfloor\frac{1}{c_2}\right\rfloor + \left(\frac{1}{c_2} - \left\lfloor\frac{1}{c_2}\right\rfloor\right)^{\frac{3}{2}}\right)} \approx 0.3957 \tag{2.12}$$

such that

$$\sum_{i=1}^{n} u_{i1}^2 \le \frac{D^2}{c_2^+}.$$

Iteratively repeating the same approach, we obtain a sequence $\{c^{(i)}\}$ ($i = 1, 2, \dots$), where $c^{(0)} = c_2$, $c^{(1)} = c_2^+$ and

$$c^{(i+1)} = \frac{0.5}{1 + k\left(\left\lfloor\frac{1}{c^{(i)}}\right\rfloor + \left(\frac{1}{c^{(i)}} - \left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{3}{2}}\right)}. \tag{2.13}$$

Clearly, we can conclude that $c^{(i)} < \frac{1}{2}$ for all i since the denominator above is greater than 1. Secondly, we prove that $c^{(i)} > \frac{1}{3}$ for all i by mathematical induction. We have shown that $c^{(0)} > \frac{1}{3}$ and $c^{(1)} > \frac{1}{3}$. Now assume $c^{(i)} > \frac{1}{3}$. Since

$$\left\lfloor\frac{1}{c^{(i)}}\right\rfloor + \left(\frac{1}{c^{(i)}} - \left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{3}{2}} \le \left\lfloor\frac{1}{c^{(i)}}\right\rfloor + \left(\frac{1}{c^{(i)}} - \left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right) = \frac{1}{c^{(i)}},$$

we have

$$c^{(i+1)} = \frac{0.5}{1 + k\left(\left\lfloor\frac{1}{c^{(i)}}\right\rfloor + \left(\frac{1}{c^{(i)}} - \left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{3}{2}}\right)} \ge \frac{0.5}{1 + \frac{k}{c^{(i)}}} > \frac{0.5}{1 + 3k} > \frac{1}{3}.$$

To sum up, we obtain $\frac{1}{3} < c^{(i)} < \frac{1}{2}$, which implies that $\left\lfloor\frac{1}{c^{(i)}}\right\rfloor = 2$. Therefore, the iterative formula of $c^{(i+1)}$ (2.13) becomes

$$c^{(i+1)} = \frac{0.5}{1 + k\left(2 + \left(\frac{1}{c^{(i)}} - 2\right)^{\frac{3}{2}}\right)}.$$

It is easy to verify that the sequence $\{c^{(i)}\}$ is monotone increasing with a limit value 0.3972. □

3. EXTENSION

Theorem 3.1. *Let $B(D)$ be a sphere in $\mathbb{R}^3$ having diameter D. Let n points be arbitrarily placed in $B(D)$. l_{ij}, a_{im}, u_{im} are similarly defined as in Theorem 1.1. Then*

$$\sum_{i=1}^{n} u_{i1}^3 \le \frac{D^3}{0.3168}, \tag{3.1}$$

$$\sum_{i=1}^{n} u_{im}^3 \le \frac{mD^3}{c_3}, \quad 2 \le m < c_3 n, \tag{3.2}$$

$$\sum_{i=1}^{n} u_{im}^3 \le nD^3, \quad c_3 n < m \le n - 1, \tag{3.3}$$

where $c_3 = 0.3125$.

Proof. To begin with, we prove the first inequality (3.1). The case $n = 2$ is trivial since $m = 1$ and $u_{im} \leq D$. So we assume that $n \geq 3$. The proof is based on that of Theorem 1.1 [1]. Denote the sphere of diameter x and center i by $B_i(x)$. Define the following sets of spheres

$$R_m := \{B_i(u_{im}) : 1 \leq i \leq n\}, \quad 1 \leq m \leq n-1.$$

First consider the spheres in R_1. As shown in [1], all spheres in R_1 are non-overlapping, i.e., the distance between the centers of any two spheres is smaller than the sum of the radii of the two spheres.

Denote by $A(X)$ the volume of a region X. We try to find a lower bound on $f_{im} := V(B(D) \cap B_i(u_{im}))/V(B_i(u_{im}))$ for every $1 \leq i \leq n$ and $1 \leq m \leq n-1$. Pick any point S from the boundary of $B(D)$ and consider the overlap ratio

$$f_{im}^S := \frac{V(B(D) \cap B_S(u_{im}))}{V(B_S(u_{im}))}, \quad 1 \leq i \leq n,\ 1 \leq m \leq n-1. \tag{3.4}$$

Using a 3-dimensional version of Figure 2.1, one can obtain the geometrical computation formula: $f_{im}^S = f(y)|_{y=\frac{u_{im}}{D}}$, where

$$f(y) := \frac{1}{2} - \frac{3y}{16}.$$

Actually $f(y)$ is a decreasing function of y. We have $f_{im}^S \geq f(1)$ due to $u_{im} \leq D$. Also $f_{im} \geq f_{im}^S$. Setting $c_3 := f(1)$, we obtain the following lower bound on f_{im} for every $1 \leq i \leq n$ and $1 \leq m \leq n-1$,

$$f_{im} \geq c_3, \quad \text{where} \quad c_3 = \frac{5}{16} = 0.3125.$$

Therefore the area of the parts of the disks in R_m that lie in $B(D)$ is at least $c_3 A(B(D))$. Hence, for every $1 \leq i \leq n$ and $1 \leq m \leq n-1$,

$$V(B_i(u_{im}) \cap B(D)) \geq c_3 V(B_i(u_{im})). \tag{3.5}$$

For a given value m, adding the n inequalities in (3.5), we obtain

$$\sum_{i=1}^{n} V(B_i(u_{im}) \cap B(D)) \geq c_3 \sum_{i=1}^{n} V(B_i(u_{im})), \quad \forall 1 \leq m \leq n-1. \tag{3.6}$$

Since all spheres in R_1 are non-overlapping, we have

$$\sum_{i=1}^{n} V(B_i(u_{im}) \cap B(D)) \leq V(B(D)). \tag{3.7}$$

Inequalities (3.6) and (3.7) imply

$$V(B(D)) \geq c_3 \sum_{i=1}^{n} V(B_i(u_{im})).$$

Notice that $V(B(D)) = \pi D^3/6$ and $V(B_i(u_{i1})) = \pi u_{i1}^3/6$. Therefore,

$$\sum_{i=1}^{n} u_{i1}^3 \leq \frac{D^3}{c_3}. \tag{3.8}$$

Defining $k = \frac{3}{16} = 0.1875$, we have

$$f_{im} \geq f_{im}^S = f\left(\frac{u_{im}}{D}\right) \geq c_3 + k - k\frac{u_{im}}{D}.$$

Therefore, for every $1 \leq i \leq n$ and $1 \leq m \leq n-1$,

$$V(B_i(u_{im}) \cap B(D)) \geq (c_3+k)V(B_i(u_{im})) - k\frac{u_{im}}{D}V(B_i(u_{im})). \tag{3.9}$$

Adding the n inequalities in (3.9) for a given m, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} V(B_i(u_{im}) \cap B(D)) \\ &\qquad \geq (c_3+k)\sum_{i=1}^{n} V(B_i(u_{im})) - \frac{k}{D}\sum_{i=1}^{n} u_{im}V(B_i(u_{im})), \quad \forall 1 \leq m \leq n-1. \end{aligned} \tag{3.10}$$

Using (3.7) and the facts $V(B(D)) = \pi D^3/6$ and $V(B_i(u_{i1})) = \pi u_{i1}^3/6$, we have

$$D^3 \geq (c_3+k)\sum_{i=1}^{n} u_{i1}^3 - \frac{k}{D}\sum_{i=1}^{n} u_{i1}^4. \tag{3.11}$$

Now consider the following optimization problems ($n \geq 3$):

$$\max \sum_{i=1}^{n} u_{i1}^4 \tag{3.12}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} u_{i1}^3 \leq \frac{D^3}{c_3} \tag{3.13}$$

$$0 \leq u_{i1} \leq D, \quad i = 1, \dots, n. \tag{3.14}$$

The objective function (3.12) is strictly convex and the feasible region defined by (3.13) – (3.14) is also convex. Since $n \geq 3$ and $2 < \frac{1}{c_3} < 3$, the inequality (3.13) holds at any of the optimal solutions. Therefore the optimal solutions of (3.12) – (3.14) must occur at vertices of the set

$$\left\{ (u_{i1}) : \sum_{i=1}^{n} u_{i1}^3 = \frac{D^3}{c_3},\ 0 \leq u_{i1} \leq D, i = 1, \dots, n \right\}.$$

Any (u_{i1}) with two components lying strictly between 0 and D cannot be a vertex. Therefore every optimal solution of (3.12) – (3.14) has $\left\lfloor \frac{1}{c_3} \right\rfloor$ components with the value D, one component with the value $\sqrt{\frac{1}{c_3} - \left\lfloor \frac{1}{c_3} \right\rfloor} D$ and the others are zeros, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Then the optimal objective value is

$$\left\lfloor \frac{1}{c_3} \right\rfloor D^4 + \left(\frac{1}{c_3} - \left\lfloor \frac{1}{c_3} \right\rfloor \right)^{\frac{4}{3}} D^4.$$

In other words, we have proved for valid u_{i1} that

$$\sum_{i=1}^{n} u_{i1}^4 \leq \left\lfloor \frac{1}{c_3} \right\rfloor D^4 + \left(\frac{1}{c_3} - \left\lfloor \frac{1}{c_3} \right\rfloor \right)^{\frac{4}{3}} D^4.$$

Now (3.11) becomes

$$D^3 \geq c_3 \sum_{i=1}^{n} u_{i1}^3 + k\left(\sum_{i=1}^{n} u_{i1}^3 - \left(\left\lfloor \frac{1}{c_3} \right\rfloor + \left(\frac{1}{c_3} - \left\lfloor \frac{1}{c_3} \right\rfloor \right)^{\frac{4}{3}} \right) D^3 \right). \tag{3.15}$$

Then we have

$$\sum_{i=1}^{n} u_{i1}^3 \leq \frac{D^3\left(1+k\left(\left\lfloor\frac{1}{c_3}\right\rfloor+\left(\frac{1}{c_3}-\left\lfloor\frac{1}{c_3}\right\rfloor\right)^{\frac{4}{3}}\right)\right)}{c_3(1+k\frac{1}{c_3})}.$$

Comparing with (3.8), we actually obtain a new c_3^+:

$$c_3^+ = \frac{c_3\left(1+k\frac{1}{c_3}\right)}{1+k\left(\left\lfloor\frac{1}{c_3}\right\rfloor+\left(\frac{1}{c_3}-\left\lfloor\frac{1}{c_3}\right\rfloor\right)^{\frac{4}{3}}\right)} \approx 0.3156 \tag{3.16}$$

such that

$$\sum_{i=1}^{n} u_{i1}^3 \leq \frac{D^3}{c_3^+}.$$

Iteratively repeating the same approach, we obtain a sequence $\{c^{(i)}\}$ ($i = 1, 2, \dots$), where $c^{(0)} = c_3$, $c^{(1)} = c_3^+$ and

$$c^{(i+1)} = \frac{0.5}{1+k\left(\left\lfloor\frac{1}{c^{(i)}}\right\rfloor+\left(\frac{1}{c^{(i)}}-\left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{4}{3}}\right)}. \tag{3.17}$$

First we conclude that $c^{(i)} < \frac{1}{3}$ for all i. We prove this by mathematical induction. We have $c^{(0)} = 0.3125 < \frac{1}{3}$. Now assume that $c^{(i)} < \frac{1}{3}$, which also implies $\left\lfloor\frac{1}{c^{(i)}}\right\rfloor \geq 3$. Then based on (3.17), we have

$$\begin{aligned} c^{(i+1)} &= \frac{0.5}{1+k\left(\left\lfloor\frac{1}{c^{(i)}}\right\rfloor+\left(\frac{1}{c^{(i)}}-\left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{4}{3}}\right)} \\ &\leq \frac{0.5}{1+k\left\lfloor\frac{1}{c^{(i)}}\right\rfloor} \leq \frac{0.5}{1+3k} < \frac{1}{3}. \end{aligned}$$

Secondly, we prove

$$c^{(i)} > \frac{1}{4}$$

for all i by mathematical induction. We have shown $c^{(0)} > \frac{1}{4}$. Now assume $c^{(i)} > \frac{1}{4}$. Since

$$\left\lfloor\frac{1}{c^{(i)}}\right\rfloor+\left(\frac{1}{c^{(i)}}-\left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{4}{3}} \leq \left\lfloor\frac{1}{c^{(i)}}\right\rfloor+\left(\frac{1}{c^{(i)}}-\left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right) = \frac{1}{c^{(i)}},$$

we have

$$c^{(i+1)} = \frac{0.5}{1+k\left(\left\lfloor\frac{1}{c^{(i)}}\right\rfloor+\left(\frac{1}{c^{(i)}}-\left\lfloor\frac{1}{c^{(i)}}\right\rfloor\right)^{\frac{4}{3}}\right)} \geq \frac{0.5}{1+\frac{k}{c^{(i)}}} > \frac{0.5}{1+4k} > \frac{1}{4}.$$

To sum up, we obtain $\frac{1}{4} < c^{(i)} < \frac{1}{3}$, which implies that $\left\lfloor\frac{1}{c^{(i)}}\right\rfloor = 3$. Therefore, the iterative formula (2.13) of $c^{(i+1)}$ becomes

$$c^{(i+1)} = \frac{0.5}{1+k\left(2+\left(\frac{1}{c^{(i)}}-3\right)^{\frac{4}{3}}\right)}.$$

It is easy to verify that the sequence $\{c^{(i)}\}$ is monotone increasing with a limit value 0.3168.

Next, consider the spheres in R_m for every $2 \leq m \leq n-1$. In this case, there can be overlaps between some pairs of spheres in R_m. However, as shown in [1], any arbitrarily chosen point within $B(D)$ can belong to at most m overlapping spheres from R_m. Then for every $2 \leq m \leq n-1$, we have

$$\sum_{i=1}^{n} V(B_i(u_{im}) \cap B(D)) \leq mV(B(D)).$$

It follows that

$$mD^3 \geq c_3 \sum_{i=1}^{n} u_{i1}^3.$$

The last inequality (3.3) directly follows from the fact $u_{im} \leq D$. □

REFERENCES

[1] O. ARPACIOGLU AND Z.J. HAAS, On the scalability and capacity of planar wireless networks with omnidirectional antennas, *Wirel. Commun. Mob. Comput.*, **4** (2004), 263–279.

[2] Y. XIA AND H.Y. LIU, Improving upper bound on the capacity of planar wireless networks with omnidirectional antennas, in Baozong Yuan and Xiaofang Tang (Eds.) *Proceedings of the IET 2nd International Conference on Wireless, Mobile & Multimedia Networks*, (2008), 191–194.