Journal of Inequalities in Pure and Applied Mathematics

ESTIMATES FOR THE $\partial-$NEUMANN OPERATOR ON STRONGLY PSEUDO-CONVEX DOMAIN WITH LIPSCHITZ BOUNDARY

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ISSN (electronic): 1443-5756
101-04

volume 5, issue 3, article 70, 2004.

Received 29 February, 2004; accepted 19 April, 2004.

Communicated by: J. Sándor

Abstract

On a bounded strongly pseudo-convex domain X in $\mathbb{C}^n$ with a Lipschitz boundary, we prove that the $\bar{\partial}-$Neumann operator N can be extended as a bounded operator from Sobolev $(-1/2)-$spaces to the Sobolev $(1/2)-$spaces. In particular, N is compact operator on Sobolev $(-1/2)-$spaces.

2000 Mathematics Subject Classification: Primary 35N15; Secondary 32W05.
Key words: Sobolev estimate, Neumann problem, Lipschitz domains.

Contents

1. Introduction

Let X be a bounded pseudo-convex domain in $\mathbb{C}^n$ with the standard Hermitian metric. The $\bar{\partial}-$Neumann operator N is the (bounded) inverse of the (unbounded) Laplace-Beltrami operator $\Box$. The $\bar{\partial}-$Neumann problem has been studied extensively when the domain X has smooth boundaries (see [12], [1], [3], [18], [19], [21], and [22]). Dahlberg [6] and Jerison and Kenig [17] established the work on the Dirichlet and classical Neumann problem on Lipschitz domains. The compactness of N on Lipschitz pseudo-convex domains is studied in Henkin and Iordan [14]. Let $W^s_{(p,q)}(X)$ be the Hilbert spaces of $(p,q)-$forms with $W^s(X)-$coefficients. Henkin, Iordan, and Kohn in [15] and Michel and Shaw in [23] showed that N is bounded from $L^2_{(p,q)}(X)$ to $W^{1/2}_{(p,q)}(X)$ on domains with piecewise smooth strongly pseudo-convex boundary by two different methods. Also Michel and Shaw in [24] proved that N is bounded on $W^{1/2}_{(p,q)}(X)$ when the domain is only bounded pseudo-convex Lipschitz with a plurisubharmonic defining function. Other results in this direction belong to Bonami and Charpentier [4], Straube [26], Engliš [10], and Ehsani [7], [8], and [9]. In fact, the main aim of this work is to establish the following:

Theorem 1.1. *Let $X \subset\subset \mathbb{C}^n$ be a bounded strongly pseudo-convex domain with Lipschitz boundary. For each $0 \le p \le n$, $1 \le q \le n-1$, the $\bar{\partial}-$Neumann operator*

$$N : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q)}(X)$$

satisfies the following estimate: for any $\varphi \in L^2_{(p,q)}(X)$, there exists a constant $c > 0$ such that

$$\|N\varphi\|_{1/2(X)} \le c\|\varphi\|_{-1/2(X)}, \tag{1.1}$$

where $c = c(X)$ *is independent of* φ*; i.e.,* N *can be extended as a bounded operator from* $W^{-1/2}_{(p,q)}(X)$ *into* $W^{1/2}_{(p,q)}(X)$*. In particular,* N *is a compact operator on* $L^2_{(p,q)}(X)$ *and* $W^{-1/2}_{(p,q)}(X)$*.*

2. Notations and the $\bar{\partial}-$Neumann Problem

We will use the standard notation of Hörmander [16] for differential forms. Let X be a bounded domain of $\mathbb{C}^n$. We express a $(p,q)-$form φ on X as follows:

$$\varphi = \sum_{I,J} \varphi_{IJ} dz^I \wedge d\bar{z}^J,$$

where I and J are strictly increasing multi-indices with lengths p and q, respectively. We denote by $\Lambda_{(p,q)}(X)$ the space of differential forms of class C^∞ and of type (p,q) on X. Let

$$\Lambda_{(p,q)}(\bar{X}) = \{\varphi|_{\bar{X}};\varphi \in \Lambda_{(p,q)}(\mathbb{C}^n)\},$$

be the subspace of $\Lambda_{(p,q)}(X)$ whose elements can be extended smoothly up to the boundary ∂X of X. For $\varphi, \psi \in \Lambda_{(p,q)}(\bar{X})$, the inner product and norm are defined as usual by

$$\langle \varphi, \psi \rangle = \sum_{I,J} \int_X \varphi_{IJ} \overline{\psi}_{IJ} dv, \quad \text{and} \quad \|\varphi\|^2 = \int_X |\varphi|^2 dv,$$

where dv is the Lebesgue measure. Let $\Lambda_{0,(p,q)}(X)$ be the subspace of $\Lambda_{(p,q)}(\bar{X})$ whose elements have compact support disjoint from ∂X.

The operator $\bar{\partial} : \Lambda_{(p,q-1)}(X) \longrightarrow \Lambda_{(p,q)}(X)$ is defined by

$$\bar{\partial}\varphi = \sum_k \sum_{IJ} \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J.$$

The formal adjoint operator δ of $\bar\partial$ is defined by :

$$\langle \delta\varphi, \psi\rangle = \langle \varphi, \bar\partial\psi\rangle$$

for any $\varphi \in \Lambda_{(p,q)}(X)$ and $\psi \in \Lambda_{0,(p,q-1)}(X)$. It is easily seen that $\bar\partial$ is a closed, linear, densely defined operator, and $\bar\partial$ forms a complex, i.e., $\bar\partial^2 = 0$. We denote by $L^2_{(p,q)}(X)$ the Hilbert space of all (p,q) forms with square integrable coefficients. We denote again by $\bar\partial : L^2_{(p,q-1)}(X) \longrightarrow L^2_{(p,q)}(X)$ the maximal extension of the original $\bar\partial$. Then $\bar\partial$ is a closed, linear, densely defined operator, and forms a complex, i.e., $\bar\partial^2 = 0$. Therefore, the adjoint operator $\bar\partial^\star : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q-1)}(X)$ of $\bar\partial$ is also a closed, linear, defined operator. We denote the domain and the range of $\bar\partial$ in $L^2_{(p,q)}(X)$ by $\mathrm{Dom}_{(p,q)}(\bar\partial)$ and $\mathrm{Range}_{(p,q)}(\bar\partial)$ respectively.

We define the Laplace-Beltrami operator

$$\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q)}(X)$$

on

$$\mathrm{Dom}_{(p,q)}(\Box) = \{\varphi \in \mathrm{Dom}_{(p,q)}(\bar\partial) \cap \mathrm{Dom}_{(p,q)}(\bar\partial^\star); \bar\partial\varphi \in \mathrm{Dom}_{(p,q+1)}(\bar\partial^\star)$$

and $\bar\partial^\star\varphi \in \mathrm{Dom}_{(p,q-1)}(\bar\partial)\}$.

Let

$$\mathrm{Ker}_{(p,q)}(\Box) = \{\varphi \in \mathrm{Dom}_{(p,q)}(\bar\partial) \cap \mathrm{Dom}_{(p,q)}(\bar\partial^\star); \quad \bar\partial\varphi = 0 \quad \text{and} \quad \bar\partial^\star\varphi = 0\}.$$

Definition 2.1. *A domain $X \subset\subset \mathbb{C}^n$ is said to be strongly pseudo-convex with $C^\infty-$boundary if there exist an open neighborhood U of the boundary ∂X of X and a C^∞ function $\lambda : U \longrightarrow \Re$ having the following properties:*

(i) $X \cap U = \{z \in U; \lambda(z) < 0\}$.

(ii) $\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \lambda(z)}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \geq L(z)|\eta|^2; z \in U$, $\eta = (\eta^1, \dots, \eta^n) \in \mathbb{C}^n$ and $L(z) > 0$.

(iii) The gradient $\nabla \lambda(z) = \left(\frac{\partial \lambda(z)}{\partial x^1}, \frac{\partial \lambda(z)}{\partial y^1}, \dots, \frac{\partial \lambda(z)}{\partial x^n}, \frac{\partial \lambda(z)}{\partial y^n} \right) \neq 0$

for $z = (z^1, \dots, z^n) \in U$; $z^\alpha = x^\alpha + iy^\alpha$.

Let $f : \Re^{2n-1} \longrightarrow \Re$ be a function that satisfies the Lipschitz condition

$$|f(x) - f(x')| \leq T|x - x'| \quad \text{for all} \quad x, x' \in \Re^{2n-1}. \tag{2.1}$$

The smallest T in which (2.1) holds is called the bound of the Lipschitz constant. By choosing finitely many balls $\{V_j\}$ covering ∂X, the Lipschitz constant for a Lipschitz domain is the smallest T such that the Lipschitz constant is bounded in every ball $\{V_j\}$.

Definition 2.2. *A bounded domain X in $\mathbb{C}^n$ is called a strongly pseudo-convex domain with Lipschitz boundary ∂X if there exists a Lipschitz defining function ϱ in a neighborhood of $\bar{X}$ such that the following condition holds:*

(i) Locally near every point of the boundary ∂X, after a smooth change of coordinates, ∂X is the graph of a Lipschitz function.

(ii) There exists a constant $c_1 > 0$ such that,

$$\sum_{\alpha,\beta}^{n} \frac{\partial^2 \varrho}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \geq c_1 |\eta|^2, \qquad \eta = (\eta^1, \dots, \eta^n) \in \mathbb{C}^n, \tag{2.2}$$

where (2.2) is defined in the distribution sense.

Let $W^s(X)$, $s \geq 0$, be defined as the space of all $u|_X$ such that $u \in W^s(\mathbb{C}^n)$. We define the norm of $W^s(X)$ by

$$\|u\|_{s(X)} = \inf\{\|v\|_{s(\mathbb{C}^n)}, v \in W^s(\mathbb{C}^n), v|_X = u\}.$$

We use $W^s_{(p,q)}(X)$ to denote Hilbert spaces of $(p,q)-$forms with $W^s(X)$ coefficients and their norms are denoted by $\| \quad \|_{s(X)}$. Let $W^s_0(X)$ be the completion of $C^\infty_0(X)-$functions under the $W^s(X)-$norm. Restricting to a small neighborhood U near a boundary point, we shall choose special boundary coordinates $t_1, \ldots, t_{2n-1}, \lambda$ such that $t_1, \ldots, t_{2n-1}$ restricted to ∂X are coordinates for ∂X. Let $D_{t_j} = \partial/\partial t_j$, $j = 1, \ldots, 2n-1$, and $D_\lambda = \partial/\partial\lambda$. Thus D_{t_j}'s are the tangential derivatives on ∂X, and D_λ is the normal derivative. For a multi-index $\beta = (\beta_1, \ldots, \beta_{2n-1})$, where each β_j is a nonnegative integer, D^β_t denotes the product of D_{t_j}'s with order $|\beta| = \beta_1 + \cdots + \beta_{2n-1}$, i.e., $D^\beta_t = D^{\beta_1}_{t_1} \cdots D^{\beta_{2n-1}}_{t_{2n-1}}$. For any $\phi \in C^\infty_0(\bar{X})$ with compact support in U, we define the tangential Fourier transform for ϕ in a special boundary chart by

$$\widetilde{\phi}(\nu, \lambda) = \int_{R^{2n-1}} e^{-i\langle t,\nu\rangle}\phi(t, \lambda)dt,$$

where $\nu = (\nu_1, \ldots, \nu_{2n-1})$ and $\langle t, \nu\rangle = t_1\nu_1 + \cdots + t_{2n-1}\nu_{2n-1}$. We define the tangential Sobolev norms $|||\cdot|||_s$ by

$$|||\phi|||_s = \int_{R^{2n-1}} \int_{-\infty}^{0} (1 + |\nu|^2)^s |\widetilde{\phi}(\nu, \lambda)| d\lambda d\nu.$$

We recall the L^2 existence theorem for the $\bar\partial-$Neumann operator on any bounded pseudo-convex domain $X \subset \mathbb{C}^n$. Following Hörmander L^2- estimates for $\bar\partial$ on any bounded pseudoconvex domains, one can prove that $\Box$ has closed range and $\mathrm{Ker}_{(p,q)}(\Box) = \{0\}$. The $\bar\partial-$Neumann operator N is the inverse of $\Box$. In fact, one can prove

Proposition 2.1 (Hörmander [16]). *Let X be a bounded pseudo-convex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n$ and $1 \leq q \leq n$, there exists a bounded linear operator*

$$N : L^2_{(p,q)}(X) \longrightarrow L^2_{(p,q)}(X)$$

such that we have the following:

(i) Range$_{(p,q)}(N)$ $\subset$Dom$_{(p,q)}(\Box)$ and $\Box N = N\Box = I$ on Dom$_{(p,q)}(\Box)$.

(ii) For any $\varphi \in L^2_{(p,q)}(X)$, $\varphi = \bar\partial\bar\partial^\star N\varphi + \bar\partial^\star\bar\partial N\varphi$.

(iii) If δ is the diameter of X,we have the following estimates:

$$\|N\varphi\| \leq \frac{e\delta^2}{q}\|\varphi\|$$

$$\|\bar\partial N\varphi\| \leq \sqrt{\frac{e\delta^2}{q}}\|\varphi\|$$

$$\|\bar\partial^\star N\varphi\| \leq \sqrt{\frac{e\delta^2}{q}}\|\varphi\|$$

for any $\varphi \in L^2_{(p,q)}(X)$.

For a detailed proof of this proposition see Shaw [25], Proposition 2.3, and Chen and Shaw [5], Theorem 4.4.1.

Theorem 2.2 (Rellich Lemma). *Let X be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary. If $s > t \geq 0$, the inclusion $W^s(X) \hookrightarrow W^t(X)$ is compact.*

3. Proof of the Main Theorem

To prove the main theorem we first obtain the following estimates on each smooth subdomain. As Lemma 2.1 in Michel and Shaw [23], we prove the following lemma:

Lemma 3.1. *Let $X \subset\subset \mathbb{C}^n$ be a bounded strongly pseudo-convex domain with Lipschitz boundary. Then, there exists an exhaustion $\{X_\mu\}$ of X with the following conditions:*

(i) $\{X_\mu\}$ is an increasing sequence of relatively compact subsets of X and $\cup_\mu X_\mu = X$.

(ii) Each $\{X_\mu\}$ has a C^∞ plurisubharmonic defining Lipschitz function λ_μ such that

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \lambda_\mu(z)}{\partial z^\alpha \partial \bar{z}^\beta} \eta^\alpha \bar{\eta}^\beta \geq c_1 |\eta|^2$$

for $z \in \partial X_\mu$ and $\eta \in \mathbb{C}^n$, where $c_1 > 0$ is a constant independent of μ.

(iii) There exist positive constants c_2, c_3 such that $c_2 \leq |\nabla \lambda_\mu| \leq c_3$ on ∂X_μ, where c_2, c_3 are independent of μ.

Proof. Let $\aleph = \{z \in X | -\delta_0 < \varrho(z) < 0\}$, where $\delta_0 > 0$ is sufficiently small. Thus, there exists a constant $c_1 > 0$ such that the function $\sigma_0(z) = \varrho(z) - c_1|z|^2$ is a plurisubharmonic on $\aleph$. Let δ_μ be a decreasing sequence such that $\delta_\mu \searrow 0$, and we define $X_{\delta_\mu} = \{z \in X | \varrho(z) < -\delta_\mu\}$. Then $\{X_{\delta_\mu}\}$ is a sequence of relatively compact subsets of X with union equal to X. Let $\Psi \in C_0^\infty(\mathbb{C}^n)$ be a function depending only on $|z_1|, \ldots, |z_n|$ and such that

(i) $\Psi \geq 0$.

(ii) $\Psi = 0$ when $|z| > 1$.

(iii) $\int \Psi d\lambda = 1$, where $d\lambda$ is the Lebesgue measure.

We define $\Psi_\varepsilon(z) = \frac{1}{\varepsilon^{2n}}\Psi(\frac{z}{\varepsilon})$ for $\varepsilon > 0$.
For each $z \in X_{\delta_\mu}$,, $0 < \varepsilon < \delta_\mu$, we define

$$\varrho_\varepsilon(z) = \varrho \star \Psi_\varepsilon(z) = \int \varrho(z - \varepsilon\zeta)\Psi(\zeta)d\lambda(\zeta).$$

Then $\varrho_\varepsilon \in C^\infty(X_{\delta_\mu})$ and $\varrho_\varepsilon \searrow \varrho$ on X_{δ_μ} when $\varepsilon \searrow 0$. Since

$$\begin{aligned}
\frac{\partial^2 \varrho_\varepsilon(z)}{\partial z^\alpha \partial \bar{z}^\beta} &= \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}\{(\sigma_0(z) + c_1|z|^2) \star \Psi_\varepsilon(z)\} \\
&= \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}\{\sigma_0(z) \star \Psi_\varepsilon(z) + c_1|z|^2 \star \Psi_\varepsilon(z)\} \\
&= \frac{\partial^2 \sigma_0(z)}{\partial z^\alpha \partial \bar{z}^\beta} \star \Psi_\varepsilon(z) + c_1 \frac{\partial^2 |z|^2}{\partial z^\alpha \partial \bar{z}^\beta} \star \Psi_\varepsilon(z)
\end{aligned}$$

for $z \in X_{\delta_\mu} \cap \aleph$, and $\eta \in \mathbb{C}^n$ it follows that

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \varrho_\varepsilon(z)}{\partial z^\alpha \partial \bar{z}^\beta}\eta^\alpha \bar{\eta}^\beta \\
&\qquad = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \sigma_0(z)}{\partial z^\alpha \partial \bar{z}^\beta}\eta^\alpha \bar{\eta}^\beta \star \Psi_\varepsilon(z) + c_1 \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 |z|^2}{\partial z^\alpha \partial \bar{z}^\beta}\eta^\alpha \bar{\eta}^\beta \star \Psi_\varepsilon(z) \\
&\qquad \geq c_1|\eta|^2.
\end{aligned}$$

Each $\varrho_{\varepsilon_\mu}$ is well defined if $0 < \varepsilon_\mu < \delta_{\mu+1}$ for $z \in X_{\delta_{\mu+1}}$. Let $c_3 = \sup_{\bar{X}} |\nabla \varrho|$, then for ε_μ sufficiently small, we have $\varrho(z) < \varrho_{\varepsilon_\mu}(z) < \varrho(z) + c_3\varepsilon_\mu$ on $X_{\delta_{\mu+1}}$. For each μ, we choose $\varepsilon_\mu = \frac{1}{2c_3}(\delta_{\mu-1} - \delta_\mu)$ and $\zeta_\mu \in (\delta_{\mu+1}, \delta_\mu)$. We define $X_\mu = \{z \in \mathbb{C}^n | \quad \varrho_{\varepsilon_\mu} < -\zeta_\mu\}$. Since $\varrho(z) < \varrho_{\varepsilon_\mu}(z) < -\zeta_\mu < -\delta_{\mu+1}$, we have that $X_\mu \subset X_{\delta_{\mu+1}}$. Also, if $z \in X_{\delta_{\mu-1}}$, then $\varrho_{\varepsilon_\mu}(z) < \varrho(z) + c_3\varepsilon_\mu < -\delta_\mu < -\zeta_\mu$. Thus we have

$$X_{\delta_{\mu+1}} \supset X_\mu \supset X_{\delta_{\mu-1}}$$

and (i) is satisfied. Then the function $\lambda_\mu = \varrho_{\varepsilon_\mu} + \zeta_\mu$ satisfies (ii). Now, we prove (iii). First, since a Lipschitz function is almost everywhere differentiable (see Evans and Gariepy [11] for a proof of this fact), the gradient of a Lipschitz function exists almost everywhere and we have $|\nabla \varrho| \leq c_3$ a.e. in $\bar{X}$ and $|\nabla \lambda_\mu| \leq c_3$ on ∂X_μ. Secondly, we show that $|\nabla \lambda_\mu|$ is uniformly bounded from below. To do that, since ∂X is Lipschitz from our assumption, then there exists a finite covering $\{V_j\}_{1\leq j\leq m}$ of ∂X such that $\bar{V}_j \subset \bar{U}_j$ for $1 \leq j \leq m$, a finite set of unit vectors $\{\chi_j\}_{1\leq j\leq m}$ and $c_2 > 0$ such that the inner product $(\nabla \varrho, \chi_j) \geq c_2 > 0$ a.e. for $z \in V_j$, $1 \leq j \leq m$. Since this is preserved by convolution, (iii) is proved. Moreover, we have $\nabla \lambda_\mu \neq 0$ in a small neighborhood of ∂X_μ. Thus ∂X_μ is smooth. Then, the proof is complete. $\square$

We use a subscript μ to indicate operators on X_μ.

Proposition 3.2. *Let $\{X_\mu\}$ be the same as in Lemma 3.1. There exists a constant $c_4 > 0$, such that for any $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \mathrm{Dom}_{(p,q)}(\bar{\partial}^\star_\mu)$, $0 \leq p \leq n$, $1 \leq q \leq n-1$,*

$$\|\varphi\|^2_{1/2(X_\mu)} \leq c_4 \left(\left\|\bar{\partial}\varphi\right\|^2_{X_\mu} + \left\|\bar{\partial}^\star_\mu \varphi\right\|^2_{X_\mu} \right), \tag{3.1}$$

where c_4 is independent of φ and μ. If $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap$ Dom$_{(p,q)}(\Box_\mu)$, then

$$\|\varphi\|^2_{1/2(X_\mu)} \leq c_4 \|\Box_\mu \varphi\|^2_{X_\mu}, \tag{3.2}$$

where c_4 is independent of φ and μ.

Proof. Since $|\nabla \lambda_\mu| \neq 0$ on a neighborhood W of ∂X_μ, then the function $\eta_\mu = \lambda_\mu / |\nabla \lambda_\mu|$ is defined on W. We extend η_μ to be negative smoothly inside X_μ. Then η_μ is a defining function in a neighborhood of $\bar{X}_\mu$ such that $\eta_\mu < 0$ on X_μ, $\eta_\mu = 0$ on ∂X_μ and $|\nabla \eta_\mu| = 1$ on W. Then, by simple calculation as in Lemma 2.2 in Michel and Shaw [23] and by using the identity of Morrey-Kohn-Hörmander which was proved in Chen and Shaw [5], Proposition 4.3.1, and from (ii)and (iii) in Lemma 3.1, it follows that there exists a constant $c_5 > 0$ such that for any $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \mathrm{Dom}_{(p,q)}(\bar{\partial}^\star_\mu)$,

$$\sum_{I,J} \sum_k \left| \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} \right|^2 + \int_{\partial X_\mu} |\varphi|^2 ds_\mu \leq c_5 \left(\|\bar{\partial}\varphi\|^2_{X_\mu} + \|\bar{\partial}^\star_\mu \varphi\|^2_{X_\mu} \right). \tag{3.3}$$

Let $z \in \partial X_\mu$ and U be a special boundary chart containing z. From Kohn [20], Proposition 3.10 and Chen and Shaw [5], Lemma 5.2.2, the tangential Sobolev norm $\sum_{j=1}^n |||D^j \varphi|||_{\epsilon-1}$, and the ordinary Sobolev norm $||\varphi||_\epsilon$ are equivalent for $\varphi \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^\star)$ where the support of φ lies in $U \cap \bar{X}_\mu$, $D^j \varphi = \partial \varphi / \partial x_j$, $(j = 1, 2, \ldots, 2n)$, and $\epsilon > 0$. Then, from Folland and Kohn [12], Theorems 2.4.4 and 2.4.5, it follows that there exists a neighborhood $V \subset U$ of z and a positive constant c_6 such that

$$\|\varphi\|^2_{1/2(X_\mu)} \leq c_6 \left(\sum_{I,J} \sum_k \left| \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} \right|^2 + \|\varphi\|^2_{X_\mu} + \int_{\partial X_\mu} |\varphi|^2 ds_\mu \right), \tag{3.4}$$

for $\varphi \in \Lambda_{0,(p,q)}(V \cap \bar{X}_\mu)$. Since X_μ is a Lipschitz domain, then c_6 depends only on the Lipschitz constant. Also from Lemma 3.1, if $\{X_\mu\}_{\mu=1}^{\infty}$ is uniformly Lipschitz, then the constant c_6 can be chosen to depend only on the Lipschitz constant of ∂X_μ, which is independent of μ. Now cover ∂X_μ by finitely many charts $\{V_i\}_{i=1}^{m}$ such that this conclusion holds on each chart, and choose V_0 so that $X_\mu - \cup_1^m V_i \subset V_0 \subset \bar{V}_0 \subset X_\mu$. Then, the estimate (3.4) holds for all $\varphi \in \Lambda_{0,(p,q)}(V_0)$. Using a partition of unity subordinate to $\{V_i\}_0^m$, the estimate (3.4) now reads

$$\|\varphi\|^2_{1/2(X_\mu)} \leq c_6 \left(\sum_{I,J} \sum_k \left| \frac{\partial \varphi_{IJ}}{\partial \bar{z}^k} \right|^2 + \|\varphi\|^2_{X_\mu} + \int_{\partial X_\mu} |\varphi|^2 ds_\mu \right), \tag{3.5}$$

for any $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \mathrm{Dom}_{(p,q)}(\bar{\partial}^\star_\mu)$. It follows from Proposition 2.1 that

$$\|\varphi\|^2_{X_\mu} \leq \frac{e\delta^2}{q} \left(\|\bar{\partial}\varphi\|^2_{X_\mu} + \|\bar{\partial}^\star \varphi\|^2_{X_\mu} \right).$$

Therefore, by taking $c_4 = c_6 \left(\frac{e\delta^2}{q} + c_5 \right)$, and by using (3.3) and (3.5) inequality (3.1) is proved. Also, since

$$\|\bar{\partial}\varphi\|^2_{X_\mu} + \|\bar{\partial}^\star_\mu \varphi\|^2_{X_\mu} \leq \|\Box_\mu \varphi\|_{X_\mu} \|\varphi\|_{X_\mu},$$

when $\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap \mathrm{Dom}_{(p,q)}(\Box_\mu)$. Then, (3.2) is proved also. $\square$

Proof of Theorem 1.1. We shall apply the Michel and Shaw technique in [23] with the suitable modifications required. Let $\{X_\mu\}$ be the same as in Lemma 3.1 and N_μ denote the $\bar{\partial}-$Neumann operator on $L^2_{(p,q)}(X_\mu)$. Since X is a strongly pseudo-convex domain with Lipschitz boundary, then by using Lemma 3.1, it can be approximated by domains with smooth boundary which are uniformly Lipschitz. Then, X_μ is a Lipschitz domain, and so $C^\infty(\bar{X}_\mu)$ is dense in $W^s(X_\mu)$ in the $W^s(X_\mu)-$norm. Then, to prove this theorem, it suffices to prove (1.1) for any $\varphi \in \Lambda_{(p,q)}(\bar{X})$. By using the boundary regularity for N_μ which was established by Kohn [19], we have $N_\mu\varphi \in \Lambda_{(p,q)}(\bar{X}_\mu) \cap$ Dom$_{(p,q)}(\Box_\mu)$. The $\bar{\partial}-$Neumann operator N is the inverse of the operator $\Box$. By using (iii) in Proposition 2.1, we have

$$\|N_\mu\varphi\|_{X_\mu} \leq \frac{e\delta^2}{q}\|\varphi\|_{X_\mu} \leq \frac{e\delta^2}{q}\|\varphi\|_X,$$

and

$$\|\bar{\partial}N_\mu\varphi\|_{X_\mu} + \|\bar{\partial}^\star_\mu N_\mu\varphi\|_{X_\mu} \leq 2\sqrt{\frac{e\delta^2}{q}}\|\varphi\|_{X_\mu} \leq 2\sqrt{\frac{e\delta^2}{q}}\|\varphi\|_X.$$

Then, there is no loss of generality if we assume that

$$N_\mu\varphi = 0 \quad in \quad X\backslash X_\mu.$$

Then there is a subsequence of $N_\mu\varphi$, still denoted by $N_\mu\varphi$, converging weakly to some element $\psi \in L^2_{(p,q)}(X)$ and $\bar{\partial}\psi \in L^2_{(p,q+1)}(X)$. This implies that $\psi \in$ Dom$_{(p,q)}(\bar{\partial})$. Now, we show that $\psi \in$ Dom$_{(p,q)}(\bar{\partial}^\star_\mu)$ as follows: for any $u \in$ Dom$_{(p,q-1)}(\bar{\partial}) \cap L^2_{(p,q-1)}(X)$,

$$\begin{aligned}
\left|\left\langle\psi, \bar{\partial}u\right\rangle_X\right| &= \lim_{\mu\longrightarrow\infty}\left|\left\langle N_\mu\varphi, \bar{\partial}u\right\rangle_{X_\mu}\right| \\
&= \lim_{\mu\longrightarrow\infty}\left|\left\langle\bar{\partial}^\star_\mu N_\mu\varphi, u\right\rangle_{X_\mu}\right| \\
&\leq \|\varphi\|_X\|u\|_X.
\end{aligned}$$

Thus $\psi \in \mathrm{Dom}_{(p,q)}(\bar{\partial}^{\star}_{\mu})$. Also, we show that $\bar{\partial}\psi \in \mathrm{Dom}_{(p,q+1)}(\bar{\partial}^{\star}_{\mu})$ and $\bar{\partial}^{\star}_{\mu}\psi \in \mathrm{Dom}_{(p,q-1)}(\bar{\partial})$ as follows: by using (ii) in Proposition 2.1, we have

$$\left\|\bar{\partial}\bar{\partial}^{\star}_{\mu}N_{\mu}\varphi\right\|^2_{X_\mu} + \left\|\bar{\partial}^{\star}_{\mu}\bar{\partial}N_{\mu}\varphi\right\|^2_{X_\mu} = \|\varphi\|^2_{X_\mu} \le \|\varphi\|^2_{X}. \tag{3.6}$$

Thus $\bar{\partial}\bar{\partial}^{\star}_{\mu}\psi$ is the L^2 weak limit of some subsequence of $\bar{\partial}\bar{\partial}^{\star}_{\mu}N_{\mu}\varphi$ and $\bar{\partial}^{\star}_{\mu}\psi \in \mathrm{Dom}_{(p,q-1)}(\bar{\partial})$. By using (3.6), we have, for any $v \in \mathrm{Dom}_{(p,q)}(\bar{\partial}) \cap L^2_{(p,q)}(X)$,

$$\begin{aligned}\left|\langle\bar{\partial}\psi, \bar{\partial}v\rangle_X\right| &= \lim_{\mu\longrightarrow\infty}\left|\langle\bar{\partial}N_{\mu}\varphi, \bar{\partial}v\rangle_{X_\mu}\right| \\ &= \lim_{\mu\longrightarrow\infty}\left|\langle\bar{\partial}^{\star}_{\mu}\bar{\partial}N_{\mu}\varphi, v\rangle_{X_\mu}\right| \\ &\le \|\varphi\|_X\|v\|_X.\end{aligned}$$

Thus $\bar{\partial}\psi \in \mathrm{Dom}_{(p,q+1)}(\bar{\partial}^{\star}_{\mu})$ and $\bar{\partial}^{\star}_{\mu}\bar{\partial}\psi$ is the weak limit of a subsequence of $\bar{\partial}^{\star}_{\mu}\bar{\partial}N_{\mu}\varphi$. This implies that $\psi \in \mathrm{Dom}_{(p,q)}(\Box_\mu)$ and $\Box_\mu\psi = \varphi$. Since N is one to one on $L^2_{(p,q)}(X)$, then we conclude that $\psi = N\varphi$. Since X_μ is a Lipschitz domain. Hence $\Lambda(\bar{X})$ are dense in $W^s(X)$ in $W^s(X)-$norm. If $s \le 1/2$, we can show that $\Lambda_0(\bar{X})$ are dense in $W^s(X)$ as in Theorem 1.4.2.4 in Grisvard [13]. Thus

$$W^{1/2}(X) = W^{1/2}_0(X).$$

It follows from the Generalized Schwartz inequality (see Proposition (A.1.1) in Folland and Kohn [12]) that

$$\left|\langle h, f\rangle_{X_\mu}\right| \le \|h\|_{1/2(X_\mu)}\|f\|_{-1/2(X_\mu)},$$

for any $h \in W^{1/2}_{(p,q)}(X_\mu)$ and $f \in W^{-1/2}_{(p,q)}(X_\mu)$. By using (3.1), there exists a constant $c_4 > 0$ such that for any $\varphi \in L^2_{(p,q)}(\bar{X}_\mu) \cap \text{Dom}_{(p,q)}(\Box_\mu)$, $0 \le p \le n$ and $1 \le q \le n$,

$$\begin{aligned} \|\varphi\|^2_{1/2(X_\mu)} &\le c_4(\|\bar{\partial}\varphi\|^2_{X_\mu} + \|\bar{\partial}^\star_\mu\varphi\|^2_{X_\mu}) \\ &= c_4 \langle \varphi, \Box_\mu \varphi \rangle_{X_\mu} \\ &\le c_4 \|\varphi\|_{1/2(X_\mu)} \|\Box_\mu \varphi\|_{-1/2(X_\mu)}, \end{aligned} \tag{3.7}$$

where c_4 is independent of φ and μ. Substituting $N_\mu\varphi$ into (3.7), we have

$$\|N_\mu\varphi\|_{1/2(X_\mu)} \le c_4 \|\Box_\mu N_\mu \varphi\|_{-1/2(X_\mu)} = c_4 \|\varphi\|_{-1/2(X_\mu)}, \tag{3.8}$$

where c_4 is independent of φ and μ. By using the extension operator on Euclidean space (see Theorem 1.4.3.1 in Grisvard [13]), it follows that for any Lipschitz domain $X_\mu \subset \mathbb{C}^n$,

$$R_\mu : W^{1/2}(X_\mu) \longrightarrow W^{1/2}(\mathbb{C}^n)$$

such that for each $\varphi \in W^{1/2}(X_\mu)$, $R_\mu\varphi = \varphi$ on X_μ and

$$\|R_\mu\varphi\|_{1/2(\mathbb{C}^n)} \le c_5 \|\varphi\|_{1/2(X_\mu)}, \tag{3.9}$$

for some positive constant c_5. The constant c_5 in (3.9) can be chosen independent of μ since extension exists for any Lipschitz domain (see Theorem 1.4.3.1 in Grisvard [13]). By applying R_μ to $N_\mu\varphi$ component-wise, we have, by using (3.8) and (3.9), that

$$\|R_\mu N_\mu \varphi\|_{1/2(X)} \le \|R_\mu N_\mu \varphi\|_{1/2(\mathbb{C}^n)} \le c_5 \|N_\mu\varphi\|_{1/2(X_\mu)} \le c \|\varphi\|_{-1/2(X_\mu)},$$

where $c > 0$ is independent of μ. Since $W^{1/2}_{(p,q)}(X)$ is a Hilbert space, then from the weak compactness for Hilbert spaces, there exists a subsequence of $R_\mu N_\mu \varphi$ which converges weakly in $W^{1/2}_{(p,q)}(X)$. Since $R_\mu N_\mu \varphi$ converges weakly to $N\varphi$ in $L^2_{(p,q)}(X)$, we conclude that $N\varphi \in W^{1/2}_{(p,q)}(X)$ and

$$\|N\varphi\|_{1/2(X)} \leq \lim_{\mu \longrightarrow \infty} \|R_\mu N_\mu \varphi\|_{1/2(X_\mu)} \leq c\|\varphi\|_{-1/2(X)}.$$

Thus, N can be extended as a bounded operator from $W^{-1/2}_{(p,q)}(X)$ to $W^{1/2}_{(p,q)}(X)$.

To prove that N is compact, we note that for any bounded domain X with Lipschitz boundary there exists a continuous linear operator

$$R : W^{1/2}(X) \longrightarrow W^{1/2}(\mathbb{C}^n)$$

such that $R\phi|_X = \phi$. Also, we note that the inclusion map

$$W^{1/2}(X) \longrightarrow L^2(X) = W^0(X)$$

is compact. Thus, by using the Rellich Lemma for $\mathbb{C}^n$, we conclude that

$$W^{1/2}(X) \hookrightarrow W^{-1/2}(X)$$

is compact and this proves that N is compact on $W^{-1/2}_{(p,q)}(X)$ and $L^2_{(p,q)}(X)$. $\square$

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