Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 3, Article 90, 2006

GENERALIZATIONS OF THE TRAPEZOID INEQUALITIES BASED ON A NEW MEAN VALUE THEOREM FOR THE REMAINDER IN TAYLOR'S FORMULA

A. I. KECHRINIOTIS AND N. D. ASSIMAKIS

DEPARTMENT OF ELECTRONICS
TECHNOLOGICAL EDUCATIONAL INSTITUTE OF LAMIA, GREECE
kechrin@teilam.gr

DEPARTMENT OF ELECTRONICS
TECHNOLOGICAL EDUCATIONAL INSTITUTE OF LAMIA, GREECE
AND
DEPARTMENT OF INFORMATICS WITH APPLICATIONS TO BIOMEDICINE
UNIVERSITY OF CENTRAL GREECE
GREECE
assimakis@teilam.gr

Received 01 April, 2005; accepted 10 May, 2006
Communicated by P. Cerone

ABSTRACT. Generalizations of the classical and perturbed trapezoid inequalities are developed using a new mean value theorem for the remainder in Taylor's formula. The resulting inequalities for N-times differentiable mappings are sharp.

Key words and phrases: Classical trapezoid inequality, Perturbed trapezoid inequality, Mean value theorem, Generalizations.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

In the literature on numerical integration, see for example [12], [13], the following estimation is well known as the trapezoid inequality:

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{12}\sup_{x\in(a,b)}|f''(x)|,$$

where the mapping $f:[a,b]\to\mathbb{R}$ is twice differentiable on the interval (a,b), with the second derivative bounded on (a,b).

ISSN (electronic): 1443-5756

We thank Prof. P. Cerone for his constructive and helpful suggestions.
101-05

In [3] N. Barnett and S. Dragomir proved an inequality for $n-$time differentiable functions which for $n=1$ takes the following form:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{b-a}{8}(\Gamma-\gamma),$$

where $f:[a,b]\to\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ such that $-\infty<\gamma\le f'(x)\le\Gamma<\infty,\forall x\in(a,b)$. In [15] N. Ujević reproved the above result via a generalization of Ostrowski's inequality.

For more results on the trapezoid inequality and their applications we refer to [4], [9], [11], [12].

In [10] S. Dragomir et al. obtained the following perturbed trapezoid inequality involving the Grüss inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(b)+f(a)}{2}+\frac{(f'(b)-f'(a))(b-a)}{12}\right| \le\frac{1}{32}(\Gamma_2-\gamma_2)(b-a)^2,$$

where f is twice differentiable on the interval (a,b), with the second derivative bounded on (a,b), and $\gamma_2:=\inf_{x\in(a,b)}f''(x)$, $\Gamma_2=:\sup_{x\in(a,b)}f''(x)$. In [6] P. Cerone and S. Dragomir improved the above inequality replacing the constant $\frac{1}{32}$ by $\frac{1}{24\sqrt{5}}$ and in [8] X. Cheng and J. Sun replaced the constant $\frac{1}{24\sqrt{5}}$ by $\frac{1}{36\sqrt{3}}$. For more results concerning the perturbed trapezoid inequality we refer to the papers of N. Barnett and S. Dragomir [1], [2], as well as, to the paper of N. Ujević [14].

In [5] P. Cerone and S. Dragomir obtained some general three-point integral inequalities for $n-$times differentiable functions, involving two functions $\alpha,\beta:[a,b]\to[a,b]$ such that $\alpha(x)\le x$ and $\beta(x)\ge x$ for all $x\in[a,b]$. As special cases (for $\alpha(x):=x$, $\beta(x):=x$) trapezoid type inequalities for $n-$times differentiable functions result. For more trapezoid-type inequalities involving $n-$times differentiable functions we refer to [6], [7], [16].

In this paper we state a mean value Theorem for the remainder in Taylor's formula. We then develop a sharp general integral inequality for $n-$times differentiable mappings involving a real parameter. Three generalizations of the classical trapezoid inequality and two generalizations of the perturbed trapezoid inequality are obtained. The resulting inequalities for $n-$times differentiable mappings are sharp.

2. MEAN VALUE THEOREM

For convenience we set

$$R_n(f;a,b):=f(b)-\sum_{i=0}^{n}\frac{(b-a)^i}{i!}f^{(i)}(a).$$

We prove the following mean value Theorem for the remainder in Taylor's formula:

Theorem 2.1. *Let $f,\ g\in C^n[a,b]$ such that $f^{(n+1)},g^{(n+1)}$ are integrable and bounded on (a,b). Assume that $g^{(n+1)}(x)>0$ for all $x\in(a,b)$. Then for any $t\in[a,b]$ and any positive valued mappings $\alpha,\beta:[a,b]\to\mathbb{R}$, the following estimation holds:*

$$m\le\frac{\alpha(t)R_n(f;t,b)+(-1)^{n+1}\beta(t)R_n(f;t,a)}{\alpha(t)R_n(g;t,b)+(-1)^{n+1}\beta(t)R_n(g;t,a)}\le M, \tag{2.1}$$

where $m:=\inf_{x\in(a,b)}\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$, $M:=\sup_{x\in(a,b)}\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$.

Proof. Since $g^{(n+1)}$, $\alpha,\ \beta$ are positive valued functions on (a,b), we clearly have that for all $t\in[a,b]$ the following inequality holds:

$$\alpha(t)\int_t^b (b-x)^n g^{(n+1)}(x)\,dx+\beta(t)\int_a^t (x-a)^n g^{(n+1)}(x)\,dx>0,$$

which, by using the Taylor's formula with an integral remainder, can be rewritten in the following form:

$$\alpha(t)\,R_n(g;t,b)+(-1)^{n+1}\beta(t)\,R_n(g;t,a)>0. \tag{2.2}$$

Moreover, we have

$$\alpha(t)\int_t^b (b-x)^n g^{(n+1)}(x)\left(\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}-m\right)dx \\ +\beta(t)\int_a^t (x-a)^n g^{(n+1)}(x)\left(\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}-m\right)\geq 0,$$

or equivalently

$$\alpha(t)\int_t^b (b-x)^n f^{(n+1)}(x)\,dx+(-1)^{n+1}\beta(t)\int_t^a (a-x)^n f^{(n+1)}(x)\,dx \\ \geq m\left(\alpha(t)\int_t^b (b-x)^n g^{(n+1)}(x)\,dx+(-1)^{n+1}\beta(t)\int_t^a (a-x)^n g^{(n+1)}(x)\,dx\right). \tag{2.3}$$

Using the Taylor's formula with an integral remainder, (2.3) can be rewritten in the following form:

$$\alpha(t)\,R_n(f;t,b)+(-1)^{n+1}\beta(t)\,R_n(f;t,a) \\ \geq m\left(\alpha(t)\,R_n(g;t,b)+(-1)^{n+1}\beta(t)\,R_n(g;t,a)\right). \tag{2.4}$$

Dividing (2.4) by (2.2) we get

$$m\leq\frac{\alpha(t)\,R_n(f;t,b)+(-1)^{n+1}\beta(t)\,R_n(f;t,a)}{\alpha(t)\,R_n(g;t,b)+(-1)^{n+1}\beta(t)\,R_n(g;t,a)}. \tag{2.5}$$

On the other hand, we have

$$\alpha(t)\int_t^b (b-x)^n g^{(n+1)}(x)\left(M-\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}\right)dx \\ +\beta(t)\int_a^t (x-a)^n g^{(n+1)}(x)\left(M-\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}\right)\geq 0.$$

or equivalently

$$\alpha(t)\,R_n(f;t,b)+(-1)^{n+1}\beta(t)\,R_n(f;t,a) \\ \leq M\left(\alpha(t)\,R_n(g;t,b)+(-1)^{n+1}\beta(t)\,R_n(g;t,a)\right). \tag{2.6}$$

Dividing (2.6) by (2.2) we get

$$\frac{\alpha(t)\,R_n(f;t,b)+(-1)^{n+1}\beta(t)\,R_n(f;t,a)}{\alpha(t)\,R_n(g;t,b)+(-1)^{n+1}\beta(t)\,R_n(g;t,a)}\leq M. \tag{2.7}$$

Combining (2.5) with (2.7) we get (2.1). $\square$

Theorem 2.2. *Let $f,\ g \in C^n[a,b]$ such that $f^{(n+1)}, g^{(n+1)}$ are integrable and bounded on (a,b). Assume that $g^{(n+1)}(x) > 0$ for all $x \in (a,b)$. Then for any $t \in [a,b]$ and any integrable and positive valuated mappings $\alpha, \beta : [a,b] \to \mathbb{R}_+$, the following estimation holds:*

$$m \leq \frac{\int_a^b \left(\alpha(t) R_n(f;t,b) + (-1)^{n+1} \beta(t) R_n(f;t,a)\right) dt}{\int_a^b \left(\alpha(t) R_n(g;t,b) + (-1)^{n+1} \beta(t) R_n(g;t,a)\right) dt} \leq M, \tag{2.8}$$

where m, M are as in Theorem 2.1.

Proof. Integrating (2.2), (2.4), (2.6) in Theorem 2.1 over $[a,b]$ we get

$$\int_a^b \left(\alpha(t) R_n(g;t,b) + (-1)^{n+1} \beta(t) R_n(g;t,a)\right) dt > 0, \tag{2.9}$$

and

$$\begin{aligned} m \int_a^b & \left(\alpha(t) R_n(g;t,b) + (-1)^{n+1} \beta(t) R_n(g;t,a)\right) dt \\ & \leq \int_a^b \left(\alpha(t) R_n(f;t,b) + (-1)^{n+1} \beta(t) R_n(f;t,a)\right) dt \\ & \leq M \int_a^b \left(\alpha(t) R_n(g;t,b) + (-1)^{n+1} \beta(t) R_n(g;t,a)\right) dt. \end{aligned} \tag{2.10}$$

Dividing (2.10) by (2.9) we get (2.8). □

3. GENERAL INTEGRAL INEQUALITIES

For convenience we denote

$$\gamma_n(f) := \inf_{x\in(a,b)} f^{(n)}(x), \qquad \Gamma_n(f) := \sup_{x\in(a,b)} f^{(n)}(x).$$

For our purpose we shall use Theorems 2.1 and 2.2, as well as, an identity:

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n)}$ is integrable on $[a,b]$. Then for any positive number ρ the following identity holds:*

$$\begin{aligned} \frac{1}{(b-a)} \int_a^b & \left(\rho R_n(f;x,b) + (-1)^{n+1} R_n(f;x,a)\right) dx \\ &= -\frac{(n+1)\left(\rho + (-1)^{n+1}\right)}{b-a} \int_a^b f(x)\,dx + \rho f(b) + (-1)^{n+1} f(a) \\ &\quad + \sum_{k=0}^{n-1} (n-k) \frac{(-1)^{n+k+1} f^{(k)}(b) + \rho f^{(k)}(a)}{(k+1)!} (b-a)^{k+1}. \end{aligned} \tag{3.1}$$

Proof. Using the analytical form of the remainder in Taylor's formula we have

$$\begin{aligned} \frac{1}{(b-a)} \int_a^b & \left(\rho R_n(f;x,b) + (-1)^{n+1} R_n(f;x,a)\right) dx \\ &= \rho f(b) + (-1)^{n+1} f(a) - \frac{1}{(b-a)} \sum_{k=0}^{n} \int_a^b \frac{\rho(b-x)^k + (-1)^{n+1}(a-x)^k}{k!} f^{(k)}(x)\,dx \\ &= \rho f(b) + (-1)^{n+1} f(a) - \frac{1}{(b-a)} \sum_{k=0}^{n} I_k, \end{aligned} \tag{3.2}$$

where

$$I_k := \int_a^b \frac{\rho (b-x)^k + (-1)^{n+1} (a-x)^k}{k!} f^{(k)} (x)\, dx, \ (k = 0, 1, ..., n).$$

For $k \geq 1$, using integration by parts we obtain

$$I_k - I_{k-1} = -\frac{(-1)^{n+k} f^{(k-1)} (b) + \rho f^{(k-1)} (a)}{k!} (b-a)^k . \tag{3.3}$$

Further, the following identity holds:

$$\sum_{k=0}^{n} I_k = (n+1) I_0 + \sum_{k=1}^{n} (n+1-k) (I_k - I_{k-1}) . \tag{3.4}$$

Combining (3.2) with (3.4) and (3.3) we get

$$\begin{aligned} \frac{1}{(b-a)} \int_a^b & \left(\rho R_n (f; x, b) + (-1)^{n+1} R_n (f; x, a)\right) dt \\ &= \rho f (b) + (-1)^{n+1} f (a) - \frac{(n+1) \left(\rho + (-1)^{n+1}\right)}{b-a} \int_a^b f (x)\, dx \\ &\quad + \sum_{k=1}^{n} (n+1-k) \frac{(-1)^{n+k} f^{(k-1)} (b) + \rho f^{(k-1)} (a)}{k!} (b-a)^k . \end{aligned} \tag{3.5}$$

Replacing k by $k+1$ in (3.5), we get (3.1). □

Theorem 3.2. *Let $f \in C^n [a, b]$ such that $f^{(n+1)}$ is integrable and bounded on (a, b). Then for any positive number ρ the following estimation holds:*

$$\begin{aligned} &\frac{(1+\rho) (b-a)^{n+1}}{(n+2)! (n+1)} \gamma_{n+1} (f) \\ &\quad \leq -\frac{\rho + (-1)^{n+1}}{(b-a)} \int_a^b f (x)\, dx + \frac{\rho f (b) + (-1)^{n+1} f (a)}{(n+1)} \\ &\qquad + \sum_{k=0}^{n-1} \frac{(n-k)}{(n+1)} \frac{(-1)^{n+k+1} f^{(k)} (b) + \rho f^{(k)} (a)}{(k+1)!} (b-a)^k \\ &\quad \leq \frac{(1+\rho) (b-a)^{n+1}}{(n+2)! (n+1)} \Gamma_{n+1} (f) , \end{aligned} \tag{3.6}$$

The inequalities in (3.6) are sharp.

Proof. Choosing $g (x) = x^{n+1}$, $\alpha (x) = \rho$, $\beta (x) = 1$ in (2.1) in Theorem 2.1, and then using the identity $R_n (g; a, x) = (x-a)^{n+1}$ we get

$$\begin{aligned} &\frac{\rho (b-t)^{n+1} + (-1)^{n+1} (a-t)^{n+1}}{(n+1)!} \gamma_{n+1} (f) \\ &\quad \leq \rho R_n (f; t, b) + (-1)^{n+1} R_n (f; t, a) \\ &\quad \leq \frac{\rho (b-t)^{n+1} + (-1)^{n+1} (a-t)^{n+1}}{(n+1)!} \Gamma_{n+1} (f) , \end{aligned} \tag{3.7}$$

for all $t \in [a,b]$. Integrating (3.7) with respect to t from a to b we have

$$(1+\rho)\frac{(b-a)^{n+1}}{(n+2)!}\gamma_{n+1}(f) \leq \frac{1}{b-a}\int_a^b \left(\rho R_n(f;t,b) + (-1)^{n+1} R_n(f;t,a)\right) dt \leq (1+\rho)\frac{(b-a)^{n+1}}{(n+2)!}\Gamma_{n+1}(f). \tag{3.8}$$

Setting (3.1) (Lemma 3.1) in (3.8) and dividing the resulting estimation by $(n+1)$, we get (3.6). Moreover, choosing $f(x) = x^{n+1}$ in (3.6), the equality holds. Therefore the inequalities in (3.6) are sharp. □

Remark 3.3. Applying Theorem 3.2 for $n = 1$ we get immediately the classical trapezoid inequality:

$$\frac{(b-a)^2}{12}\gamma_2(f) \leq \frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{(b-a)^2}{12}\Gamma_2(f), \tag{3.9}$$

where $f:[a,b] \to \mathbb{R}$ is continuously differentiable on $[a,b]$ and twice differentiable on (a,b), with the second derivative f'' integrable and bounded on (a,b) .

Remark 3.4. Theorem 3.2 for $n = 2$ becomes the following form:

$$\frac{(1+\rho)(b-a)^3}{72}\gamma_3(f) \leq \frac{1-\rho}{(b-a)}\int_a^b f(x)\,dx + \frac{(2\rho-1)f(a)-(2-\rho)f(b)}{3} + \frac{f'(b)+\rho f'(a)}{6}(b-a) \leq \frac{(1+\rho)(b-a)^3}{72}\Gamma_3(f),$$

where $\rho \in \mathbb{R}_+$, $f \in C^2[a,b]$ and such that f''' is bounded and integrable on (a,b) .

Theorem 3.5. *Let f, g be two mappings as in Theorem 2.2. Then for any $\rho \in \mathbb{R}_+$ the following estimation holds:*

$$m \leq \frac{I_n(f;\rho,a,b)}{I_n(g;\rho,a,b)} \leq M, \tag{3.10}$$

where $m := \inf_{x\in(a,b)} \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$, $M := \sup_{x\in(a,b)} \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$, *and*

$$I_n(f;\rho,a,b) := -\frac{\rho+(-1)^{n+1}}{(b-a)}\int_a^b f(x)\,dx + \frac{\rho f(b) + (-1)^{n+1} f(a)}{(n+1)} + \sum_{k=0}^{n-1}\frac{(n-k)}{(n+1)}\frac{(-1)^{n+k+1} f^{(k)}(b) + \rho f^{(k)}(a)}{(k+1)!}(b-a)^k.$$

Proof. Setting $\alpha(x) = \rho$, $\beta(x) = 1$ in (2.1) of Theorem 2.1, and using the identity (3.1) in Lemma 3.1 we get (3.9) . □

4. GENERALIZED CLASSICAL TRAPEZOID INEQUALITIES

Using the inequality (3.6) in Theorem 3.2 we obtain two generalizations of the classical trapezoid inequality, which will be used in the last section. Moreover, combining both generalizations we obtain a third generalization of the classical trapezoid inequality.

Theorem 4.1. *Let $f \in C^n[a,b]$ such that $f^{(n+1)}$ is integrable and bounded on (a,b). Suppose n is odd. Then the following estimation holds:*

$$\begin{aligned}(4.1)\qquad &\frac{1}{(n+2)!\,(n+1)}(b-a)^{n+1}\gamma_{n+1}(f)\\ &\le -\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2}\\ &\quad + \sum_{k=1}^{n-1}\frac{(n-k)}{2(n+1)}\frac{(b-a)^k\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)}{(k+1)!}\\ &\le \frac{1}{(n+2)!\,(n+1)}(b-a)^{n+1}\Gamma_{n+1}(f).\end{aligned}$$

The inequalities in (4.1) are sharp.

Proof. From (3.6) in Theorem 3.2 by $\rho = 1$, obviously we get (4.1). □

Theorem 4.2. *Let $f \in C^n[a,b]$ such that $f^{(n+1)}$ is integrable and bounded on (a,b). Suppose n is odd. Then we have*

$$\begin{aligned}(4.2)\qquad &\frac{2(b-a)^{n+1}}{n(n+3)!}\gamma_{n+1}(f)\\ &\le -\frac{1}{b-a}\int_a^b f(x)\,dx + \sum_{k=0}^{n-1}\frac{(n-k)}{n}\cdot\frac{\left((-1)^k f^{(k)}(b)+f^{(k)}(a)\right)(b-a)^k}{(k+2)!}\\ &\le \frac{2(b-a)^{n+1}}{n(n+3)!}\Gamma_{n+1}(f).\end{aligned}$$

The inequalities in (4.2) are sharp.

Proof. Let $m := n+1$. Then m is an even integer. Consider the mapping $F:[a,b]\to\mathbb{R}$, defined via $F(x) := \int_a^x f(t)\,dt$. Then we clearly have that $F \in C^m[a,b]$ and $F^{(m+1)}$ is integrable and bounded on (a,b). Now, applying inequality (3.6) in Theorem 3.2 to F by choosing $\rho = 1$, we readily get

$$\begin{aligned}\frac{2(b-a)^{m+1}}{(m+2)!\,(m+1)}\gamma_{m+1}(F) &\le -\frac{(m-1)}{(m+1)}(F(b)-F(a))\\ &\quad + \sum_{k=1}^{m-1}\frac{(m-k)}{(m+1)}\frac{(-1)^{k+1}F^{(k)}(b)+F^{(k)}(a)}{(k+1)!}(b-a)^k\\ &\le \frac{2(b-a)^{m+1}}{(m+2)!\,(m+1)}\Gamma_{m+1}(F),\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\frac{2\left(b-a\right)^{m+1}}{\left(m+2\right)!\left(m-1\right)}\gamma_m\left(f\right)\\
&\leq -\frac{m-1}{m+1}\int_a^b f\left(x\right)dx+\sum_{k=1}^{m-1}\frac{\left(m-k\right)}{m+1}\frac{\left(\left(-1\right)^{k+1}f^{(k-1)}\left(b\right)+f^{(k-1)}\left(a\right)\right)\left(b-a\right)^k}{\left(k+1\right)!}\\
&\leq \frac{2\left(b-a\right)^{m+1}}{\left(m+2\right)!\left(m+1\right)}\Gamma_m\left(f\right).
\end{aligned}$$

Multiplying the previous inequality by $\frac{m+1}{(m-1)(b-a)}$, and then using $m=n+1$ we have

$$\begin{aligned}
&\frac{2\left(b-a\right)^{n+1}}{\left(n+3\right)!n}\gamma_{n+1}\left(f\right)\\
&\leq -\frac{1}{b-a}\int_a^b f\left(x\right)dx+\sum_{k=1}^{n}\frac{\left(n+1-k\right)}{n}\frac{\left(\left(-1\right)^{k+1}f^{(k-1)}\left(b\right)+f^{(k-1)}\left(a\right)\right)\left(b-a\right)^{k-1}}{\left(k+1\right)!}\\
&\leq \frac{2\left(b-a\right)^{n+1}}{\left(n+3\right)!n}\Gamma_{n+1}\left(f\right),
\end{aligned}$$

and replacing k by $k+1$we get (4.2). Moreover, choosing $f\left(x\right)=x^{n+1}$ in (4.2), the equality holds. So, the inequalities in (4.2) are sharp. □

Remark 4.3. Applying Theorem 4.2 for $n=1$ we again obtain the classical trapezoid inequality (3.9) in Remark 3.3.

Remark 4.4. A simple calculation yields $\frac{2}{n(n+3)!}<\frac{1}{(n+2)!(n+1)}$ for any $n>1$. Thus inequality (4.2) in Theorem 4.2 is better than (4.1) in Theorem 4.1. Nevertheless inequality (4.1) is useful, because suitable combinations of (4.1), (4.2) lead to some interesting results, as for example in the following theorem.

Theorem 4.5. *Let n be an odd integer such that $n\geq 3$. Let $f\in C^{n-2}\left[a,b\right]$ such that $f^{(n-1)}$ is integrable and bounded on $\left(a,b\right)$. Then the following inequalities hold*

$$\begin{aligned}
&\frac{12\left(b-a\right)^n}{\left(n+3\right)!\left(n-2\right)\left(n-1\right)}\left(2\left(n+1\right)\gamma_{n-1}\left(f\right)-n\left(n+3\right)\Gamma_{n-1}\left(f\right)\right)\\
&\leq\int_a^b f\left(x\right)dx-\frac{12n\left(n+1\right)}{\left(n-2\right)\left(n-1\right)}\sum_{k=0}^{n-3}\frac{n\left(k+2\right)-2}{2n\left(n+1\right)}\\
&\qquad\times\frac{\left(n-k-2\right)\left(\left(-1\right)^k f^{(k)}\left(b\right)+f^{(k)}\left(a\right)\right)\left(b-a\right)^{k+1}}{\left(k+4\right)!}\\
&\leq\frac{12\left(b-a\right)^n}{\left(n+3\right)!\left(n-2\right)\left(n-1\right)}\left(2\left(n+1\right)\Gamma_{n-1}\left(f\right)-n\left(n+3\right)\gamma_{n-1}\left(f\right)\right).
\end{aligned}\tag{4.3}$$

The inequalities in (4.3) are sharp.

Proof. We set the mapping $F:\left[a,b\right]\to\mathbb{R}$ by

$$F\left(x\right):=\int_a^x\int_a^t f\left(s\right)dsdt.\tag{4.4}$$

Then we have that $F \in C^n[a,b]$ and $F^{(n+1)}$ is bounded and integrable on (a,b). Applying the inequalities (4.2) in Theorem 4.2 and (4.1) in Theorem 4.1 to F we respectively get the following inequalities:

$$\begin{aligned}\frac{2(b-a)^{n+1}}{(n+3)!n}\gamma_{n+1}(F) &\le -\frac{1}{b-a}\int_a^b F(x)\,dx + \frac{F(a)+F(b)}{2}\\ &\quad + \sum_{k=1}^{n-1}\frac{(n-k)}{n}\frac{\left((-1)^k F^{(k)}(b)+F^{(k)}(a)\right)(b-a)^k}{(k+2)!}\\ &\le \frac{2(b-a)^{n+1}}{(n+3)!n}\Gamma_{n+1}(F),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\frac{(b-a)^{n+1}}{(n+2)!(n+1)}\gamma_{n+1}(F)\\ &\le -\frac{1}{b-a}\int_a^b F(x)\,dx + \frac{F(b)+F(a)}{2}\\ &\qquad + \sum_{k=1}^{n-1}\frac{(n-k)}{2(n+1)}\cdot\frac{(b-a)^k\left(F^{(k)}(a)+(-1)^k F^{(k)}(b)\right)}{(k+1)!}\\ &\le \frac{(b-a)^{n+1}}{(n+2)!(n+1)}\Gamma_{n+1}(F).\end{aligned} \tag{4.6}$$

Multiplying (4.6) by (-1) and adding the resulting estimation with (4.5), we get

$$\begin{aligned}&\frac{(b-a)^{n+1}}{(n+2)!}\left(\frac{2}{n(n+3)}\gamma_{n+1}(F)-\frac{1}{n+1}\Gamma_{n+1}(F)\right)\\ &\le -\sum_{k=1}^{n-1}\frac{nk-2}{2n(n+1)}\frac{(n-k)\left((-1)^k F^{(k)}(b)+F^{(k)}(a)\right)(b-a)^{k-1}}{(k+2)!}\\ &\le \frac{(b-a)^{n+1}}{(n+2)!}\left(\frac{2}{n(n+3)}\Gamma_{n+1}(F)-\frac{1}{n+1}\gamma_{n+1}(F)\right).\end{aligned} \tag{4.7}$$

Dividing the last estimation with $(b-a)$ and splitting the first term of the sum we have

$$\begin{aligned}&\frac{(b-a)^{n}}{(n+2)!}\left(\frac{2}{n(n+3)}\gamma_{n+1}(F)-\frac{1}{n+1}\Gamma_{n+1}(F)\right)\\ &\le \frac{(n-2)(n-1)(F'(b)-F'(a))}{12n(n+1)}\\ &\qquad -\sum_{k=2}^{n-1}\frac{nk-2}{2n(n+1)}\frac{(n-k)\left((-1)^k F^{(k)}(b)+F^{(k)}(a)\right)(b-a)^{k-1}}{(k+2)!}\\ &\le \frac{(b-a)^{n}}{(n+2)!}\left(\frac{2}{n(n+3)}\Gamma_{n+1}(F)-\frac{1}{n+1}\gamma_{n+1}(F)\right).\end{aligned} \tag{4.8}$$

Finally, setting (4.4) in (4.7) and multiplying the resulting estimation by $\frac{12n(n+1)}{(n-2)(n-1)}$ we get

$$\frac{12\left(b-a\right)^{n}}{\left(n+3\right)!\left(n-2\right)\left(n-1\right)}\left(2\left(n+1\right)\gamma_{n-1}\left(f\right)-n\left(n+3\right)\Gamma_{n-1}\left(f\right)\right)$$
$$\leq\int_{a}^{b}f\left(x\right)dx-\frac{12n\left(n+1\right)}{\left(n-2\right)\left(n-1\right)}$$
$$\times\sum_{k=2}^{n-1}\frac{nk-2}{2n\left(n+1\right)}\frac{\left(n-k\right)\left(\left(-1\right)^{k}f^{\left(k-2\right)}\left(b\right)+f^{\left(k-2\right)}\left(a\right)\right)\left(b-a\right)^{k-1}}{\left(k+2\right)!}$$
$$\leq\frac{12\left(b-a\right)^{n}}{\left(n+3\right)!\left(n-2\right)\left(n-1\right)}\left(2\left(n+1\right)\Gamma_{n-1}\left(f\right)-n\left(n+3\right)\gamma_{n-1}\left(f\right)\right),$$

and replacing k by $k+2$ the inequalities in (4.3) are obtained.

Moreover, choosing $f\left(x\right)=x^{n-1}$ in (4.3), the equality holds. So, the inequalities in (4.3) are sharp. □

Applying Theorem 4.5 for $n=3$ we immediately obtain the following result:

Corollary 4.6. *Let $f\in C^{1}\left[a,b\right]$ such that f'' is integrable and bounded on $\left(a,b\right)$. Then,*

$$\frac{\left(b-a\right)^{2}}{60}\left(4\gamma_{2}\left(f\right)-9\Gamma_{2}\left(f\right)\right)\leq\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-\frac{f\left(a\right)+f\left(b\right)}{2}$$
$$\leq\frac{\left(b-a\right)^{2}}{60}\left(4\Gamma_{2}\left(f\right)-9\gamma_{2}\left(f\right)\right).\tag{4.9}$$

Remark 4.7. Let f be as in Corollary 4.6. If $\gamma_{2}\left(f\right)>\frac{4}{9}\Gamma_{2}\left(f\right)$ then from (4.8) we get the following inequality:

$$\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx<\frac{f\left(a\right)+f\left(b\right)}{2}.$$

5. GENERALIZED PERTURBED TRAPEZOID INEQUALITIES

In this section, using the results of the two previous sections, several perturbed trapezoid inequalities are obtained involving $n-$times differentiable functions.

Theorem 5.1. *Let $f\in C^{n}\left[a,b\right]$ such that $f^{\left(n+1\right)}$ is integrable and bounded on $\left(a,b\right)$. Then the following estimations are valid:*

$$\frac{\left(b-a\right)^{n+1}}{\left(n+2\right)!\left(n+1\right)}\gamma_{n+1}\left(f\right)\leq\frac{\left(-1\right)^{n}}{\left(b-a\right)}\int_{a}^{b}f\left(x\right)dx+\frac{\left(-1\right)^{n+1}\left(f\left(a\right)+nf\left(b\right)\right)}{\left(n+1\right)}$$
$$+\sum_{k=1}^{n-1}\frac{\left(n-k\right)}{\left(n+1\right)}\frac{\left(-1\right)^{n+k+1}f^{\left(k\right)}\left(b\right)}{\left(k+1\right)!}\left(b-a\right)^{k}$$
$$\leq\frac{\left(b-a\right)^{n+1}}{\left(n+2\right)!\left(n+1\right)}\Gamma_{n+1}\left(f\right),\tag{5.1}$$

$$\frac{(b-a)^{n+1}}{(n+2)!\,(n+1)}\gamma_{n+1}(f) \leq -\frac{1}{(b-a)}\int_a^b f(x)\,dx + \frac{nf(a)+f(b)}{n+1} + \sum_{k=1}^{n-1}\frac{(n-k)}{(n+1)}\frac{f^{(k)}(a)}{(k+1)!}(b-a)^k \leq \frac{(b-a)^{n+1}}{(n+2)!\,(n+1)}\Gamma_{n+1}(f). \tag{5.2}$$

Further, if n is an even positive integer, then

$$\left|-\frac{1}{(b-a)}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1}\frac{(n-k)}{2(n+1)}\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{(k+1)!}(b-a)^k\right| \leq \frac{(b-a)^{n+1}}{2(n+2)!\,(n+1)}\left(\Gamma_{n+1}(f)-\gamma_{n+1}(f)\right). \tag{5.3}$$

The inequalities in (5.1) *and* (5.2) *are sharp.*

Proof. Taking the limit of (3.6) in Theorem 3.2 as $\rho \to 0$ we obtain (5.1). Further for $\rho > 1$, dividing (3.6) by $\left(\rho + (-1)^{n+1}\right)$ and then obtaining the limit from the resulting estimation as $\rho \to \infty$ we get (5.2). Now, let n be an even integer. Then multiplying (5.2) by (-1), adding the resulting inequality with (5.1) and finally multiplying the obtained estimation by $(-\frac{1}{2})$ we easily get (5.3). □

Remark 5.2. Applying Theorem 5.1 for $n = 2$ we obtain the following inequalities:

$$\frac{(b-a)^3}{72}\gamma_3(f) \leq \frac{1}{(b-a)}\int_a^b f(x)\,dx - \frac{f(a)+2f(b)}{3} + \frac{f'(b)}{6}(b-a) \leq \frac{(b-a)^3}{72}\Gamma_3(f),$$

$$\frac{(b-a)^3}{72}\gamma_3(f) \leq -\frac{1}{(b-a)}\int_a^b f(x)\,dx + \frac{2f(a)+f(b)}{3} + \frac{f'(a)}{6}(b-a) \leq \frac{(b-a)^3}{72}\Gamma_3(f),$$

$$\left|\frac{1}{(b-a)}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} + \frac{f'(b)-f'(a)}{12}(b-a)\right| \leq \frac{(b-a)^3}{144}\left(\Gamma_3(f)-\gamma_3(f)\right), \tag{5.4}$$

where $f \in C^2[a,b]$ and is such that f''' is bounded and integrable on $[a,b]$. Therefore, inequality (5.4) can be regarded as a Grüss type generalization of the perturbed trapezoid inequality.

Theorem 5.3. *Let $f \in C^n[a,b]$ such that $f^{(n+1)}$ is integrable and bounded on (a,b). Suppose n is odd and greater than 1. Then the following estimation holds:*

$$\begin{aligned}(5.5)\qquad & \frac{2(b-a)^{n+1}\left((n+3)\gamma_{n+1}(f)-(n+1)\Gamma_{n+1}(f)\right)}{(n+3)!\,(n+1)(n-2)} \\ & \le \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(b)+f(a)}{2} \\ & \quad - \sum_{k=1}^{n-2}\frac{(n-k)(n-1-k)}{(n-2)(n+1)}\frac{\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)(b-a)^k}{(k+2)!} \\ & \le \frac{2(b-a)^{n+1}\left((n+3)\Gamma_{n+1}(f)-(n+1)\gamma_{n+1}(f)\right)}{(n+3)!\,(n+1)(n-2)}.\end{aligned}$$

The inequalities in (5.5) are sharp.

Proof. Multiplying (4.2) in Theorem 4.2 by $\frac{n}{n-2}$ and (4.1) in Theorem 4.1 by $-\frac{2}{n-2}$ and then adding the resulting estimations we see that the last term of the sum in the intermediate part of the obtained inequality is vanishing, and so, after some algebra, we get (5.5). Finally, choosing $f(x) := x^{n+1}$ in (5.5), a simple calculation verifies that the equalities hold. Therefore, the inequalities in (5.5) are sharp. □

Applying Theorem 5.3 for $n = 3$ we get immediately the following result.

Corollary 5.4. *Let $f \in C^3[a,b]$ such that $f^{(4)}$ is integrable and bounded on (a,b). Then the following estimation holds:*

$$\begin{aligned}(5.6)\qquad & \frac{1}{720}(b-a)^4\left(3\gamma_4(f)-2\Gamma_4(f)\right) \\ & \le \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(b)+f(a)}{2} + \frac{(f'(b)-f'(a))(b-a)}{12} \\ & \le \frac{1}{720}(b-a)^4\left(3\Gamma_4(f)-2\gamma_4(f)\right).\end{aligned}$$

The inequalities in (5.6) are sharp.

REFERENCES

[1] N.S. BARNETT AND S.S. DRAGOMIR, On the perturbed trapezoid formula, *Tamkang J. Math.*, **33**(2) (2002), 119–128.

[2] N.S. BARNETT AND S.S. DRAGOMIR, A perturbed trapezoid inequality in terms of the third derivative and applications, *RGMIA Research Report Collection*, **4**(2) (2001), Art. 6.

[3] N.S. BARNETT AND S.S. DRAGOMIR, Applications of Ostrowski's version of the Grüss inequality for trapezoid type rules, *Tamkang J. Math.*, **37**(2) (2006), 163–173.

[4] C. BUSE, S.S. DRAGOMIR, J. ROUMELIOTIS AND A. SOFO, Generalized trapezoid type inequalities for vector-valued functions and applications, *Math. Ineq. & Appl.*, **5**(3) (2002), 435–450.

[5] P. CERONE AND S.S. DRAGOMIR, Three point identities and inequalities for $n-$time differentiable functions, *SUT Journal of Mathematics*, **36**(2) (2000), 351–383.

[6] P. CERONE AND S.S. DRAGOMIR, Trapezoidal type rules from an inequalities point of view, *Analytic Computational Methods in Applied Mathematics*, G. Anastassiou (Ed.), CRC press, N.Y., 2000, 65–134.

[7] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS AND J. SUNDE, A new generalization of the trapezoid formula for $n-$time differentiable mappings and applications, *Demonstratio Mathematica*, **33**(4) (2000), 719–736.

[8] XIAO-LIANG CHENG AND JIE SUN, A note on the perturbed trapezoid inequality, *J. Inequal. Pure and Appl. Math.*, **3**(2) (2002), Art. 29. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=181`].

[9] S.S. DRAGOMIR, A generalised trapezoid type inequality for convex functions, *East Asian J. Math.*, **20**(1) (2004), 27–40.

[10] S.S. DRAGOMIR, P. CERONE AND A. SOFO, Some remarks on the trapezoid rule in numerical integration, *Indian J. Pure Appl. Math.*, **31**(5) (2000), 475–494.

[11] S.S. DRAGOMIR AND A. MCANDREW, On trapezoid inequality via a Grüss type result and applications, *Tamkang J. Math.*, **31**(3) (2000), 193–201.

[12] S. S. DRAGOMIR AND Th.M. RASSIAS, *Ostrowski Inequalities and Applications in Numerical Integration*, Kluwer Academic, Dordrecht, 2002.

[13] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1994.

[14] N. UJEVIĆ, On perturbed mid-point and trapezoid inequalities and applications, *Kyungpook Math. J.*, **43** (2003), 327–334.

[15] N. UJEVIĆ, A generalization of Ostrowski's inequality and applications in numerical integration, *Appl. Math. Lett.*, **17** (2004), 133–137.

[16] N. UJEVIĆ, Error inequalities for a generalized trapezoid rule, *Appl. Math. Lett.*, **19** (2006), 32–37.