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A SUFFICIENT CONDITION FOR STARLIKENESS OF ANALYTIC FUNCTIONS OF KOEBE TYPE

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ABSTRACT. By making use of Jack's Lemma as well as several differential and other inequalities (and parametric constraints), the authors derive sufficient conditions for starlikeness of a certain class of n-fold symmetric analytic functions of Koebe type. Relevant connections of the results presented here with those given in earlier works are also indicated.

Key words and phrases: Differential inequalities, n-fold symmetric functions, analytic functions of Koebe type, starlike functions, strongly starlike functions, Jack's Lemma.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions f which are analytic in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and *normalized* by

$$f(0) = f'(0) - 1 = 0.$$

Also, as usual, let

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}) \right\} \tag{1.1}$$

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and

$$\tilde{\mathcal{S}}^*(\alpha) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \right\} \tag{1.2}$$

be the familiar classes of *starlike functions* in $\mathbb{U}$ and *strongly starlike functions of order* α in $\mathbb{U}$ $(0 < \alpha \leqq 1)$, respectively. We note that

$$\tilde{\mathcal{S}}^*(\alpha) \subset \mathcal{S}^* \quad (0 < \alpha < 1) \qquad \text{and} \qquad \tilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*.$$

We denote by $\mathcal{H}(\alpha)$ the class of functions $f \in \mathcal{A}$ defined by

$$\begin{aligned} \mathcal{H}(\alpha) := \Bigg\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re \left(\alpha z^2 \frac{f''(z)}{f(z)} + z \frac{f'(z)}{f(z)} \right) > 0 \\ \left(\frac{f(z)}{z} \neq 0;\ z \in \mathbb{U};\ \alpha \geqq 0 \right) \Bigg\}, \end{aligned} \tag{1.3}$$

so that, as already observed by Ramesha *et al*. [6], we have the following inclusion relationships (*cf*. [6]):

$$\mathcal{H}(\alpha) \subset \mathcal{S}^* \qquad \text{and} \qquad \mathcal{H}(1) \subset \tilde{\mathcal{S}}^* \left(\frac{1}{2} \right). \tag{1.4}$$

In fact, a sharper inclusion relationship than the second one in (1.4) was given subsequently by Nunokawa *et al*. [4] as follows:

$$\mathcal{H}(1) \subset \tilde{\mathcal{S}}^*(\beta) \qquad \left(\beta < \frac{1}{2} \right). \tag{1.5}$$

Obradović and Joshi [5], on the other hand, made use of the method of differential inequalities in order to derive several other related results for classes of strongly starlike functions in $\mathbb{U}$.

Motivated essentially by the aforementioned earlier works, we aim here at deriving sufficient conditions for starlikeness of an n-fold symmetric function $f_b(z)$ of Koebe type, defined by

$$f_b(z) := \frac{z}{(1 - z^n)^b} \qquad (b \geqq 0;\ n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.6}$$

which obviously corresponds to the familiar Koebe function when

$$n = 1 \qquad \text{and} \qquad b = 2.$$

The following result (popularly known as *Jack's Lemma*) will also be required in the derivation of our main result (Theorem 1 below).

Lemma 1 (Jack [2])**.** *Let the* (*nonconstant*) *function* $w(z)$ *be analytic in* $|z| < \rho$ *with* $w(0) = 0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < \rho$ *at a point* z_0, *then*

$$z_0\, w'(z_0) = k w(z_0),$$

where k *is a real number and* $k \geqq 1$.

2. THE MAIN RESULT AND ITS CONSEQUENCES

We begin by proving a stronger result than what we indicated in the preceding section.

Theorem 1. *Let the* n*-fold symmetric function* $f_b(z)$, *defined by* (1.6), *be analytic in* $\mathbb{U}$ *with*

$$\frac{f_b(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

(i) *If $f_b(z)$ satisfies the inequality*:

$$\Re\left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + \frac{z f_b'(z)}{f_b(z)}\right) > -\frac{\alpha nb}{4} + \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha nb}{2}\right) \qquad (z \in \mathbb{U}), \tag{2.1}$$

then $f_b(z)$ is starlike in $\mathbb{U}$ for

$$\alpha > 0 \qquad \textit{and} \qquad \frac{3\alpha + 2 - \sqrt{\Delta}}{2\alpha} \leqq nb \leqq \frac{3\alpha + 2 + \sqrt{\Delta}}{2\alpha}$$
$$\left(\Delta := 9\alpha^2 - 4\alpha + 4\right).$$

(ii) *If $f_b(z)$ satisfies the inequality (2.1) with $\alpha = 0$, that is, if*

$$\Re\left(\frac{z f_b'(z)}{f_b(z)}\right) > 1 - \frac{nb}{2} \qquad (z \in \mathbb{U}), \tag{2.2}$$

then $f_b(z)$ is starlike in $\mathbb{U}$ for $0 \leqq nb \leqq 2$.

Proof. (i) Let $\alpha > 0$ and $f_b(z)$ satisfy the hypotheses of Theorem 1. We put

$$\frac{z f_b'(z)}{f_b(z)} = \frac{1 + (nb-1)w(z)}{1 - w(z)},$$

where $w(z)$ is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad w(z) \neq 1 \qquad (z \in \mathbb{U}).$$

Then we have

$$\frac{\{f_b'(z) + z f_b''(z)\} f_b(z) - z \{f_b'(z)\}^2}{\{f_b(z)\}^2}$$
$$= \frac{(nb-1)\, w'(z)\{1 - w(z)\} + w'(z)\{1 + (nb-1)w(z)\}}{\{1 - w(z)\}^2},$$

which implies that

$$z\, \frac{f_b''(z)}{f_b(z)} + \frac{f_b'(z)}{f_b(z)} - z\left(\frac{f_b'(z)}{f_b(z)}\right)^2 = \frac{nbw'(z)}{\{1 - w(z)\}^2}. \tag{2.3}$$

On the other hand, we can write

$$z^2\, \frac{f_b''(z)}{f_b(z)} = \frac{nbzw'(z)}{\{1 - w(z)\}^2} - \frac{1 + (nb-1)w(z)}{1 - w(z)} + \left(\frac{1 + (nb-1)w(z)}{1 - w(z)}\right)^2,$$

that is,

$$\alpha z^2\, \frac{f_b''(z)}{f_b(z)} = \alpha\left[\frac{nbzw'(z)}{\{1 - w(z)\}^2} + \left(\frac{1 + (nb-1)w(z)}{1 - w(z)}\right)^2\right] - \alpha \cdot \frac{1 + (nb-1)w(z)}{1 - w(z)},$$

which, in turn, implies that

$$\alpha z^2\, \frac{f_b''(z)}{f_b(z)} + z\, \frac{f_b'(z)}{f_b(z)} = \alpha\left[\frac{nbzw'(z)}{\{1 - w(z)\}^2} + \left(\frac{1 + (nb-1)w(z)}{1 - w(z)}\right)^2\right] + (1 - \alpha)\frac{1 + (nb-1)w(z)}{1 - w(z)}. \tag{2.4}$$

Now we claim that $|w(z)| < 1$ $(z \in \mathbb{U})$. If there exists a z_0 in $\mathbb{U}$ such that $|w(z_0)| = 1$, then (by Jack's Lemma) we have

$$z_0 w'(z_0) = k w(z_0) \qquad (k \geqq 1)\,.$$

By setting

$$w(z_0) = e^{i\theta} \qquad (0 \leqq \theta < 2\pi)\,,$$

we thus find that

$$\begin{aligned}
&\Re\left(\alpha z_0^2\,\frac{f_b''(z_0)}{f_b(z_0)} + z_0\,\frac{f_b'(z_0)}{f_b(z_0)}\right)\\
&\quad = \Re\left(\alpha\left[\frac{nbz_0w'(z_0)}{(1-w(z_0))^2} + \left(\frac{1+(nb-1)w(z_0)}{1-w(z_0)}\right)^2\right] + (1-\alpha)\frac{1+(nb-1)w(z_0)}{1-w(z_0)}\right)\\
&\quad = \Re\left(\alpha\left[\frac{nbke^{i\theta}}{(1-e^{i\theta})^2} + \left(\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)^2\right] + (1-\alpha)\frac{1+(nb-1)e^{i\theta}}{1-e^{i\theta}}\right)\\
&\quad = \alpha\left[\frac{-nbk}{4\sin^2\left(\frac{\theta}{2}\right)} + \left(1-\frac{nb}{2}\right)^2 - \frac{n^2b^2}{4}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right] + (1-\alpha)\left(1-\frac{nb}{2}\right)\\
&\quad = -\frac{\alpha nb}{4}\left(\frac{k+nb\cos^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\right) + \left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\\
&\quad \leqq -\frac{\alpha nb}{4} + \left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right) \qquad (z \in \mathbb{U})\,,
\end{aligned}$$

since $k \geqq 1$.

If we let

$$\begin{aligned}
\Re\left(\alpha z_0^2\,\frac{f_b''(z_0)}{f_b(z_0)} + z_0\,\frac{f_b'(z_0)}{f_b(z_0)}\right) &\leqq -\frac{\alpha nb}{4} + \left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\\
&= \frac{1}{4}\left[\alpha(nb)^2 - (3\alpha+2)(nb) + 4\right]\\
&=: \vartheta(nb) \qquad (z \in \mathbb{U})\,,
\end{aligned} \tag{2.5}$$

then

$$\vartheta(nb) \leqq 0 \qquad \left(\frac{3\alpha+2-\sqrt{\Delta}}{2\alpha} \leqq nb \leqq \frac{3\alpha+2+\sqrt{\Delta}}{2\alpha};\ \Delta := 9\alpha^2 - 4\alpha + 4\right).$$

Thus we have

$$\Re\left(\alpha z_0^2\,\frac{f_b''(z_0)}{f_b(z_0)} + z_0\,\frac{f_b'(z_0)}{f_b(z_0)}\right) \leqq 0 \qquad (z \in \mathbb{U}) \tag{2.6}$$

$$\left(\frac{3\alpha+2-\sqrt{\Delta}}{2\alpha} \leqq nb \leqq \frac{3\alpha+2+\sqrt{\Delta}}{2\alpha};\ \Delta := 9\alpha^2 - 4\alpha + 4\right),$$

which is a contradiction to the hypotheses of Theorem 2.

Therefore, $|w(z)| < 1$ for all z in $\mathbb{U}$. Hence $f_b(z)$ is starlike in $\mathbb{U}$, thereby proving the assertion (i) of Theorem 1.

(ii) The proof of the assertion (ii) of Theorem 1 was given by Fukui *et al.* [1], and so we omit the details here. □

Corollary 1. *The following inclusion relationship holds true*:

$$\mathcal{H}_b(\alpha) := \left\{ f_b : f_b \in \mathcal{A} \quad \textit{and} \quad \Re\left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + z\,\frac{f_b'(z)}{f_b(z)} \right) > 0 \right.$$
$$\left. \left(\frac{f_b(z)}{z} \neq 0;\ z \in \mathbb{U};\ \alpha \geqq 0 \right) \right\} \subset \mathcal{S}^*$$

for the n-fold symmetric function $f_b(z)$ defined by (1.6).

3. APPLICATIONS OF DIFFERENTIAL INEQUALITIES

In this section, we apply the following known result involving differential inequalities with a view to deriving several further sufficient conditions for starlikeness of the n-fold symmetric function $f_b(z)$ defined by (1.6).

Lemma 2 (Miller and Mocanu [3])**.** *Let $\Theta(u,v)$ be a complex-valued function such that*

$$\Theta : \mathbb{D} \to \mathbb{C} \qquad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}),$$

$\mathbb{C}$ *being* $($*as usual*$)$ *the complex plane*, *and let*

$$u = u_1 + iu_2 \qquad \textit{and} \qquad v = v_1 + iv_2.$$

Suppose that the function $\Theta(u,v)$ satisfies each of the following conditions:

(i) $\Theta(u,v)$ *is continuous in* $\mathbb{D}$;
(ii) $(1,0) \in \mathbb{D}$ *and* $\Re(\Theta(1,0)) > 0$;
(iii) $\Re(\Theta(iu_2, v_1)) \leqq 0$ *for all* $(iu_2, v_1) \in \mathbb{D}$ *such that*

$$v_1 \leqq -\frac{1}{2}\left(1 + u_2^2\right).$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

be analytic $($*regular*$)$ *in* $\mathbb{U}$ *such that*

$$\big(p(z), zp'(z)\big) \in \mathbb{D} \qquad (z \in \mathbb{U}).$$

If

$$\Re\big(\Theta(p(z), zp'(z))\big) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Let us now consider the following implication:

$$\Re\left(\alpha z^2 \frac{f_b''(z)}{f_b(z)} + z\,\frac{f_b'(z)}{f_b(z)} \right) > -\frac{\alpha n b}{4} + \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha n b}{2}\right)$$
$$\Rightarrow \Re\left(\left(z\,\frac{f_b'(z)}{f_b(z)} \right)^{\mu} \right) > 0 \tag{3.1}$$
$$\left(z \in \mathbb{U};\ -\frac{\alpha n b}{4} + \left(1 - \frac{nb}{2}\right)\left(1 - \frac{\alpha n b}{2}\right) < 1;\ \alpha \geqq 0;\ \mu \geqq 1 \right).$$

If we put

$$p(z)=\left(z\,\frac{f_b'(z)}{f_b(z)}\right)^{\mu},$$

then (3.1) is equivalent to

$$\begin{aligned}\text{(3.2)}\quad \Re\Bigg(\frac{\alpha}{\mu}\left\{p\left(z\right)\right\}^{(1-\mu)/\mu}\; zp'(z)+\alpha\left\{p\left(z\right)\right\}^{2/\mu}\\ +\left(1-\alpha\right)\left\{p\left(z\right)\right\}^{1/\mu}+\frac{\alpha nb}{4}-\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\Bigg)>0\\ \Rightarrow \Re\big(p\left(z\right)\big)>0 \qquad (z\in\mathbb{U})\,.\end{aligned}$$

By setting

$$p\left(z\right)=u \qquad \text{and} \qquad zp'\left(z\right)=v,$$

and letting

$$\Theta(u,v)=\frac{\alpha}{\mu}u^{(1-\mu)/\mu}\;v+\alpha u^{2/\mu}+(1-\alpha)u^{1/\mu}+\frac{\alpha nb}{4}-\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right),$$

it is easy to show that, for

$$\alpha\geqq 0 \qquad \text{and} \qquad \mu\geqq 1,$$

we have

(i) $\Theta(u,v)$ is continuous in $\mathbb{D}=(\mathbb{C}\setminus\{0\})\times\mathbb{C}$;

(ii) $(1,0)\in\mathbb{D}$ and

$$\Re\big(\Theta(1,0)\big)=\frac{3\alpha nb}{4}+\frac{nb}{2}-\frac{\alpha n^2b^2}{4}>0,$$

since

$$-\frac{\alpha nb}{4}+\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)<1.$$

Thus the conditions (i) and (ii) of Lemma 2 are satisfied. Moreover, for

$$(iu_2,v_1)\in\mathbb{D} \qquad \text{such that} \qquad v_1\leqq-\frac{1}{2}\left(1+u_2^2\right),$$

we obtain

$$\begin{aligned}\Re\big(\theta\left(iu_2,v_1\right)\big)&=\frac{\alpha}{\mu}\left|u_2\right|^{(1-\mu)/\mu}\;v_1\cos\left(\frac{\left(1-\mu\right)\pi}{2\mu}\right)+\alpha\left|u_2\right|^{2/\mu}\cos\left(\frac{\pi}{\mu}\right)\\ &\quad+\left(1-\alpha\right)\left|u_2\right|^{1/\mu}\;\cos\left(\frac{\pi}{2\mu}\right)+\frac{\alpha nb}{4}-\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right)\\ &\leqq-\frac{\alpha}{2\mu}(1+u_2^2)\left|u_2\right|^{(1-\mu)/\mu}\;\sin\left(\frac{\pi}{2\mu}\right)+\alpha\left|u_2\right|^{2/\mu}\;\cos\left(\frac{\pi}{\mu}\right)\\ &\quad+\left(1-\alpha\right)\left|u_2\right|^{1/\mu}\;\cos\left(\frac{\pi}{2\mu}\right)+\frac{\alpha nb}{4}-\left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right),\end{aligned}$$

which, upon putting $|u_2|=s$ $(s>0)$, yields

$$\text{(3.3)}\qquad \Re\big(\Theta\left(iu_2,v_1\right)\big)\leqq\Phi(s),$$

where

$$\Phi(s) := -\frac{\alpha}{2\mu}(1+s^2)s^{(1-\mu)/\mu}\ \sin\left(\frac{\pi}{2\mu}\right) + \alpha s^{2/\mu}\ \cos\left(\frac{\pi}{\mu}\right) + (1-\alpha)s^{1/\mu}\ \cos\left(\frac{\pi}{2\mu}\right) + \frac{\alpha nb}{4} - \left(1-\frac{nb}{2}\right)\left(1-\frac{\alpha nb}{2}\right). \tag{3.4}$$

Remark. *If, for some choices of the parameters $\alpha, \mu,$ and nb, we find that*

$$\Phi\left(s\right) \leqq 0 \qquad \left(s > 0\right),$$

then we can conclude from (3.3) and Lemma 2 that the corresponding implication (3.1) holds true.

First of all, for the choice:

$$\mu = 1 \qquad \text{and} \qquad nb = 2,$$

we obtain

Theorem 2. *If the n-fold symmetric function $f_b\left(z\right)$, defined by (1.6) and analytic in $\mathbb{U}$ with*

$$\frac{f_b\left(z\right)}{z} \neq 0 \qquad \left(z \in \mathbb{U}\right),$$

satisfies the following inequality:

$$\Re\left(\alpha z^2\ \frac{f_b''(z)}{f_b(z)} + z\ \frac{f_b'(z)}{f_b(z)}\right) > -\ \frac{\alpha}{2} \qquad \left(z \in \mathbb{U}\right), \tag{3.5}$$

then $f_b \in \mathcal{S}^*$ for any real $\alpha \geqq 0$.

Proof. For $\mu = 1$ and $nb = 2$, we find from (3.4) that

$$\Phi(s) = -\ \frac{3}{2}\alpha s^2 \leqq 0 \qquad \left(s \in \mathbb{R}\right),$$

which implies Theorem 2 in view of the above remark. □

Next, for

$$\alpha = \frac{2}{3}, \qquad nb = 3 \pm \sqrt{3}, \qquad \text{and} \qquad \mu = 2,$$

we get

Theorem 3. *If the n-fold symmetric function $f_b\left(z\right)$, defined by (1.6) and analytic in $\mathbb{U}$ with*

$$\frac{f_b\left(z\right)}{z} \neq 0 \qquad \left(z \in \mathbb{U}\right),$$

satisfies the following inequality:

$$\Re\left(\frac{2}{3}\ z^2\ \frac{f_b''(z)}{f_b(z)} + z\ \frac{f_b'(z)}{f_b(z)}\right) > 0 \qquad \left(z \in \mathbb{U}\right), \tag{3.6}$$

then

$$\left|\arg\left(\frac{z f_b'(z)}{f_b(z)}\right)\right| < \frac{\pi}{4} \qquad \left(z \in \mathbb{U}\right)$$

or, equivalently,

$$\mathcal{H}_b\left(\frac{2}{3}\right) \subset \tilde{\mathcal{S}}^*\left(\frac{1}{2}\right).$$

Proof. By setting

$$\alpha = \frac{2}{3}, \qquad nb = 3 \pm \sqrt{3}, \qquad \text{and} \qquad \mu = 2$$

in (3.4), we have

$$\Phi(s) = -\frac{(1-s)^2}{6\sqrt{2s}} \leqq 0 \qquad (s > 0)\,,$$

which leads us to Theorem 3 just as in the proof of Theorem 2. □

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