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A NOTE ON SOME NEW REFINEMENTS OF JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this note, we obtain two new refinements of Jensen's inequality for convex functions.

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1. INTRODUCTION

Let X be a real linear space and $I \subseteq X$ be a non-empty convex set. $f : I \to \mathbb{R}$ is called a convex function, if for every $x, y \in I$ and any $t \in (0,1)$, we have (see [1])

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Let f be a convex function on I. For a given positive integer $n > 2$ and any $x_i \in I$ $(i = 1, 2, \dots, n)$, it is well-known that the following Jensen's inequality holds

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{1.1}$$

The classical inequality (1.1) has many applications and there are many extensive works devoted to generalizing or improving Jensen's inequality. In this respect, we refer the reader to [1] – [10] and the references cited therein for updated results.

In this paper, we assume that $x_{n+r} = x_r$ $(r = 1, 2, \dots, n-2; n > 2)$.

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Using (1.1), L. Bougoffa in [11] proved the following two inequalities

$$\frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}}{2}\right)+f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\le\sum_{i=1}^{n} f(x_i) \tag{1.2}$$

and

$$\frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+n-2}}{n-1}\right)+f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\le\sum_{i=1}^{n} f(x_i). \tag{1.3}$$

In this paper, we generalize (1.2) and (1.3), obtain refinements of (1.1).

2. MAIN RESULTS

Theorem 2.1. *Let f be a convex function on I and $n(>2)$ be a given positive integer. For any $x_i \in I$ $(i=1,2,\dots,n)$, $m=2,3,\dots$, $k=0,1,2,\dots$ and $r=1,2,\dots,n-2$, then we have the following refinements of (1.1)*

$$\begin{aligned}
&f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\le\cdots\\
&\le\frac{1}{m+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{m}{m+1}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\frac{1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{m-1}{m}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\cdots\le\frac{1}{3}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{2}{3}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\frac{1}{2}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{2}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\frac{n-1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\frac{n}{n+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n+1}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\cdots\le\frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n+k}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\frac{n+k}{n+k+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n+k+1}f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le\cdots\le\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)\le\frac{1}{n}\sum_{i=1}^{n} f(x_i).
\end{aligned} \tag{2.1}$$

Remark 1. It is easy to see that (1.2) and (1.3) are parts of (2.1) for $r=1$ and $r=n-2$, respectively.

Theorem 2.2. *Let f, m, k and n be defined as in Theorem 2.1. For any $x_i \in I$ $(i = 1, 2, \ldots, n)$ and $r = 1, 2, \ldots, n-2$, we have the following refinements of (1.1)*

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} f(x_i) &\geq \frac{m-1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\geq \left(\frac{n+k-1}{n+k} - \frac{1}{m}\right)\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&\qquad + \left(\frac{1}{n+k} + \frac{1}{m}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\geq \left|\left(\frac{n+k-1}{n+k} - \frac{1}{m}\right)\frac{1}{n}\left|\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right)\right|\right. \\
&\qquad \left. - \left(\frac{m-1}{m} - \frac{1}{n+k}\right)\left| f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right|\right| + f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\geq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).
\end{aligned} \tag{2.2}$$

3. PROOF OF THEOREMS

Proof of Theorem 2.1. From (1.1), we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) &\geq f\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&= f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).
\end{aligned} \tag{3.1}$$

For $m = 2, 3, \ldots$, by (3.1) we can get

$$\begin{aligned}
& f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{m}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\leq \frac{1}{m+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{m+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&\qquad + \left(\frac{1}{m(m+1)} + \frac{m-1}{m}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\le \left(\frac{1}{m+1}+\frac{1}{m(m+1)}\right)\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)\\
&\qquad +\frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&=\frac{1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).
\end{aligned}$$

The inequality (3.1) yields

$$\begin{aligned}
(3.3)\qquad &\frac{1}{2}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&=\left(\frac{n-1}{n}-\frac{n-2}{2n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le \frac{n-1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)\\
&\qquad -\frac{n-2}{2n} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)+\frac{1}{2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&=\frac{n-1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).
\end{aligned}$$

For $k=0,1,2,\dots$, using inequality (3.1), we obtain

$$\begin{aligned}
(3.4)\qquad &\frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&=\left(\frac{n+k}{n+k+1}-\frac{1}{(n+k+1)(n+k)}\right)\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)\\
&\qquad +\frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le \frac{n+k}{n+k+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)\\
&\qquad -\frac{1}{(n+k+1)(n+k)} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)+\frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&=\frac{n+k}{n+k+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n+k+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\\
&\le \frac{n+k}{n+k+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)\\
&\qquad +\frac{1}{n+k+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)
\end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right).$$

From (1.1), we have

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \le \frac{1}{n}\sum_{i=1}^{n} \frac{f(x_i) + f(x_{i+1}) + \cdots + f(x_{i+r})}{r+1} = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{3.5}$$

Combination of (3.2) – (3.5) yields (2.1).

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. For $k = 0, 1, 2, \dots$ and $m = 2, 3, \dots$, from (2.1), we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\ge \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&\qquad - \left(\frac{1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \\
&\ge \frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\qquad - \left(\frac{1}{m}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \\
&= \Bigg|\frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\qquad - \left(\frac{1}{m}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right)\Bigg| \\
&\ge \Bigg|\left(\frac{n+k-1}{n+k} - \frac{1}{m}\right)\frac{1}{n}\left|\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right)\right| \\
&\qquad - \left(\frac{m-1}{m} - \frac{1}{n+k}\right)\left|f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right|\Bigg| \ge 0.
\end{aligned} \tag{3.6}$$

Expression (3.6) plus

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

yields (2.2).

The proof of Theorem 2.2 is completed. □

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