

A NOTE ON SOME NEW REFINEMENTS OF JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS

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Abstract: In this note, we obtain two new refinements of Jensen's inequality for convex functions.

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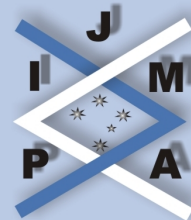
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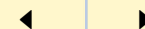
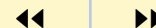
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1. Introduction

Let X be a real linear space and $I \subseteq X$ be a non-empty convex set. $f : I \rightarrow \mathbb{R}$ is called a convex function, if for every $x, y \in I$ and any $t \in (0, 1)$, we have (see [1])

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

Let f be a convex function on I . For a given positive integer $n > 2$ and any $x_i \in I$ ($i = 1, 2, \dots, n$), it is well-known that the following Jensen's inequality holds

$$(1.1) \quad f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \frac{1}{n} \sum_{i=1}^n f(x_i).$$

The classical inequality (1.1) has many applications and there are many extensive works devoted to generalizing or improving Jensen's inequality. In this respect, we refer the reader to [1] – [10] and the references cited therein for updated results.

In this paper, we assume that $x_{n+r} = x_r$ ($r = 1, 2, \dots, n - 2; n > 2$).

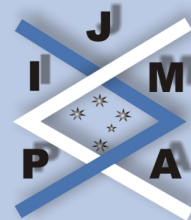
Using (1.1), L. Bougoffa in [11] proved the following two inequalities

$$(1.2) \quad \frac{n-1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \sum_{i=1}^n f(x_i)$$

and

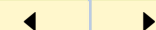
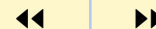
$$(1.3) \quad \frac{n-1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+n-2}}{n-1}\right) + f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \sum_{i=1}^n f(x_i).$$

In this paper, we generalize (1.2) and (1.3), obtain refinements of (1.1).



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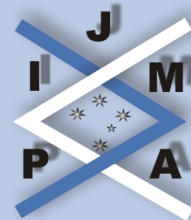
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2. Main Results

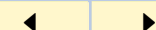
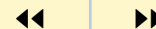
Theorem 2.1. Let f be a convex function on I and $n(> 2)$ be a given positive integer. For any $x_i \in I$ ($i = 1, 2, \dots, n$), $m = 2, 3, \dots$, $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots, n - 2$, then we have the following refinements of (1.1)

$$\begin{aligned} (2.1) \quad & f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \dots \\ & \leq \frac{1}{m+1} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{m}{m+1} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ & \leq \frac{1}{m} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ & \leq \dots \leq \frac{1}{3} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{2}{3} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ & \leq \frac{1}{2} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{1}{2} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ & \leq \frac{n-1}{n} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{1}{n} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ & \leq \frac{n}{n+1} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{1}{n+1} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ & \leq \dots \leq \frac{n+k-1}{n+k} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \dots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \end{aligned}$$



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$$\begin{aligned} &\leq \frac{n+k}{n+k+1} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k+1} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ &\leq \cdots \leq \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \leq \frac{1}{n} \sum_{i=1}^n f(x_i). \end{aligned}$$

Remark 1. It is easy to see that (1.2) and (1.3) are parts of (2.1) for $r = 1$ and $r = n - 2$, respectively.

Theorem 2.2. Let f , m , k and n be defined as in Theorem 2.1. For any $x_i \in I$ ($i = 1, 2, \dots, n$) and $r = 1, 2, \dots, n - 2$, we have the following refinements of (1.1)

$$\begin{aligned} (2.2) \quad \frac{1}{n} \sum_{i=1}^n f(x_i) &\geq \frac{m-1}{m} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{m} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ &\geq \left(\frac{n+k-1}{n+k} - \frac{1}{m}\right) \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\ &\quad + \left(\frac{1}{n+k} + \frac{1}{m}\right) f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ &\geq \left| \left(\frac{n+k-1}{n+k} - \frac{1}{m}\right) \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \right| \\ &\quad - \left| \left(\frac{m-1}{m} - \frac{1}{n+k}\right) f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right| + f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\ &\geq f\left(\frac{1}{n} \sum_{i=1}^n x_i\right). \end{aligned}$$

3. Proof of Theorems

Proof of Theorem 2.1. From (1.1), we have

$$(3.1) \quad \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \geq f \left(\frac{1}{n} \sum_{i=1}^n \frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\ = f \left(\frac{1}{n} \sum_{i=1}^n x_i \right).$$

For $m = 2, 3, \dots$, by (3.1) we can get

$$(3.2) \quad f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\ = \frac{1}{m+1} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) + \frac{m}{m+1} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\ \leq \frac{1}{m+1} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{m}{m+1} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\ = \frac{1}{m+1} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\ + \left(\frac{1}{m(m+1)} + \frac{m-1}{m} \right) f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\ \leq \left(\frac{1}{m+1} + \frac{1}{m(m+1)} \right) \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right)$$



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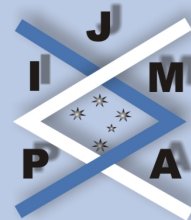
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$$\begin{aligned}
 & + \frac{m-1}{m} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\
 = & \frac{1}{m} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right).
 \end{aligned}$$

The inequality (3.1) yields

$$\begin{aligned}
 (3.3) \quad & \frac{1}{2} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{2} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\
 & = \left(\frac{n-1}{n} - \frac{n-2}{2n}\right) \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{2} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\
 & \leq \frac{n-1}{n} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
 & \quad - \frac{n-2}{2n} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) + \frac{1}{2} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \\
 & = \frac{n-1}{n} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right).
 \end{aligned}$$

For $k = 0, 1, 2, \dots$, using inequality (3.1), we obtain

$$(3.4) \quad \frac{n+k-1}{n+k} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

$$\begin{aligned}
&= \left(\frac{n+k}{n+k+1} - \frac{1}{(n+k+1)(n+k)} \right) \\
&\quad \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{1}{n+k} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\
&\leq \frac{n+k}{n+k+1} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\
&\quad - \frac{1}{(n+k+1)(n+k)} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) + \frac{1}{n+k} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\
&= \frac{n+k}{n+k+1} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\
&\quad + \frac{1}{n+k+1} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\
&\leq \frac{n+k}{n+k+1} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\
&\quad + \frac{1}{n+k+1} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\
&= \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right).
\end{aligned}$$



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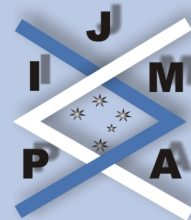
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From (1.1), we have

$$(3.5) \quad \begin{aligned} & \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\ & \leq \frac{1}{n} \sum_{i=1}^n \frac{f(x_i) + f(x_{i+1}) + \cdots + f(x_{i+r})}{r+1} \\ & = \frac{1}{n} \sum_{i=1}^n f(x_i). \end{aligned}$$

Combination of (3.2) – (3.5) yields (2.1).

The proof of Theorem 2.1 is completed. \square

Proof of Theorem 2.2. For $k = 0, 1, 2, \dots$ and $m = 2, 3, \dots$, from (2.1), we obtain

$$(3.6) \quad \begin{aligned} & \frac{1}{n} \sum_{i=1}^n f(x_i) - f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\ & \geq \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \\ & \quad - \left(\frac{1}{m} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{m-1}{m} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \right) \\ & \geq \frac{n+k-1}{n+k} \cdot \frac{1}{n} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{1}{n+k} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \\ & \quad - \left(\frac{1}{m} \sum_{i=1}^n f \left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{m-1}{m} f \left(\frac{1}{n} \sum_{i=1}^n x_i \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{n+k-1}{n+k} \cdot \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right. \\
&\quad \left. - \left(\frac{1}{m} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right) \right| \\
&\geq \left| \left(\frac{n+k-1}{n+k} - \frac{1}{m} \right) \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \right| \\
&\quad \left| \left(\frac{m-1}{m} - \frac{1}{n+k} \right) f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right| \geq 0.
\end{aligned}$$

Expression (3.6) plus

$$f\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

yields (2.2).

The proof of Theorem 2.2 is completed. □

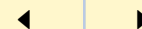
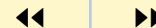


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References

- [1] S.S. DRAGOMIR, Some refinements of Jensen's inequality, *J. Math. Anal. Appl.*, **168** (1992), 518–522.
- [2] P.M. VASIĆ AND Z. MIJALKOVIĆ, On an idex set function connected with Jensen inequality, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, **544-576** (1976), 110–112.
- [3] C.L. WANG, Inequalities of the Rado-Popoviciu type for functions and their applications, *J. Math. Anal. Appl.*, **100** (1984), 436–446.
- [4] J. PEČARIĆ AND V. VOLENEC, Interpolation of the Jensen inequality with some applications, *Österreich. Akad. Wiss. Math.-Natur. Kl Sonderdruck Sitzungsber*, **197** (1988), 463–476.
- [5] J. PEČARIĆ AND D. SVRTAN, New refinements of the Jensen inequalities based on samples with repetitions, *J. Math. Anal. Appl.*, **222** (1998), 365–373.
- [6] L.C. WANG, *Convex Functions and Their Inequalities*, Sichuan University Press, Chengdu, China, 2001. (Chinese).
- [7] L.C. WANG, On a chains of Jensen inequalities for convex functions, *Math. In Practice and Theory*, **31**(6), (2001), 719–724. (Chinese).
- [8] J.C. KUANG, *Applied Inequalities*, Shandong Science and Technology Press, Jinan, China, 2004. (Chinese).
- [9] L.C. WANG AND X. ZHANG, Generated by chains of Jensen inequalities for convex functions, *Kodai. Math. J.*, **27**(2) (2004), 112–133.



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- [10] L.C. WANG, Chain of Jensen's inequalities with two parameters for convex functions, *Math. In Practice and Theory*, **35**(10) (2005), 195–199. (Chinese).
- [11] L. BOUGOFFA, New inequalities about convex functions, *J. Inequal. Pure Appl. Math.*, **7**(4) (2006), Art. 148. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=766>]



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