A NOTE ON SOME NEW REFINEMENTS OF JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS

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Abstract: In this note, we obtain two new refinements of Jensen's inequality for convex functions.

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Contents

New Refinements of Jensen's Inequality
Liang-Cheng Wang, Xiu-Fen Ma and Li-Hong Liu
vol. 10, iss. 2, art. 48, 2009

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1. Introduction

Let X be a real linear space and $I \subseteq X$ be a non-empty convex set. $f : I \to \mathbb{R}$ is called a convex function, if for every $x, y \in I$ and any $t \in (0,1)$, we have (see [1])

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let f be a convex function on I. For a given positive integer $n > 2$ and any $x_i \in I$ ($i = 1, 2, \ldots, n$), it is well-known that the following Jensen's inequality holds

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{1.1}$$

The classical inequality (1.1) has many applications and there are many extensive works devoted to generalizing or improving Jensen's inequality. In this respect, we refer the reader to [1] – [10] and the references cited therein for updated results.

In this paper, we assume that $x_{n+r} = x_r$ ($r = 1, 2, \ldots, n-2; n > 2$).

Using (1.1), L. Bougoffa in [11] proved the following two inequalities

$$\frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} f(x_i) \tag{1.2}$$

and

$$\frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+n-2}}{n-1}\right) + f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} f(x_i). \tag{1.3}$$

In this paper, we generalize (1.2) and (1.3), obtain refinements of (1.1).

vol. 10, iss. 2, art. 48, 2009

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issn: 1443-5756

2. Main Results

Theorem 2.1. *Let f be a convex function on I and $n(>2)$ be a given positive integer. For any $x_i \in I$ $(i = 1, 2, \ldots, n)$, $m = 2, 3, \ldots$, $k = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots, n-2$, then we have the following refinements of (1.1)*

$$
(2.1) \qquad f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \cdots
$$

$$
\le \frac{1}{m+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

$$
\le \frac{1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

$$
\le \cdots \le \frac{1}{3}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{2}{3} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

$$
\le \frac{1}{2}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

$$
\le \frac{n-1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

$$
\le \frac{n}{n+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

$$
\le \cdots \le \frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)
$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$\leq \frac{n+k}{n+k+1} \cdot \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k+1} f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)$$

$$\leq \cdots \leq \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i).$$

Remark 1. It is easy to see that (1.2) and (1.3) are parts of (2.1) for $r = 1$ and $r = n-2$, respectively.

Theorem 2.2. *Let f, m, k and n be defined as in Theorem 2.1. For any $x_i \in I$ $(i = 1, 2, \ldots, n)$ and $r = 1, 2, \ldots, n-2$, we have the following refinements of (1.1)*

$$\begin{aligned}(2.2)\quad \frac{1}{n} \sum_{i=1}^{n} f(x_i) &\geq \frac{m-1}{m} \cdot \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{m} f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \\ &\geq \left(\frac{n+k-1}{n+k} - \frac{1}{m}\right) \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\ &\quad + \left(\frac{1}{n+k} + \frac{1}{m}\right) f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \\ &\geq \left| \left(\frac{n+k-1}{n+k} - \frac{1}{m}\right) \frac{1}{n} \left| \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \right| \right. \\ &\quad \left. - \left(\frac{m-1}{m} - \frac{1}{n+k}\right) \left| f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \right| \right| + f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \\ &\geq f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right).\end{aligned}$$

journal of inequalities in pure and applied mathematics
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3. Proof of Theorems

Proof of Theorem 2.1. From (1.1), we have

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \geq f\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) = f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right). \tag{3.1}$$

For $m = 2, 3, \ldots$, by (3.1) we can get

$$\begin{aligned} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) &= \frac{1}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{m}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &\leq \frac{1}{m+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m}{m+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &= \frac{1}{m+1}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\ &\quad + \left(\frac{1}{m(m+1)} + \frac{m-1}{m}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &\leq \left(\frac{1}{m+1} + \frac{1}{m(m+1)}\right)\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \end{aligned} \tag{3.2}$$

New Refinements of Jensen's Inequality
Liang-Cheng Wang, Xiu-Fen Ma and Li-Hong Liu
vol. 10, iss. 2, art. 48, 2009
journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$+\frac{m-1}{m}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$=\frac{1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{m-1}{m}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right).$$

The inequality (3.1) yields

$$\frac{1}{2}\cdot\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{2}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \tag{3.3}$$

$$=\left(\frac{n-1}{n}-\frac{n-2}{2n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{2}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$\le\frac{n-1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)$$

$$-\frac{n-2}{2n}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)+\frac{1}{2}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$$

$$=\frac{n-1}{n}\cdot\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right).$$

For $k=0,1,2,\ldots$, using inequality (3.1), we obtain

$$\frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{x_i+x_{i+1}+\cdots+x_{i+r}}{r+1}\right)+\frac{1}{n+k}f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \tag{3.4}$$

vol. 10, iss. 2, art. 48, 2009

journal of inequalities in pure and applied mathematics
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$$
\begin{aligned}
&= \left(\frac{n+k}{n+k+1} - \frac{1}{(n+k+1)(n+k)}\right) \\
&\qquad \cdot \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\le \frac{n+k}{n+k+1} \cdot \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&\qquad - \frac{1}{(n+k+1)(n+k)} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{n+k}{n+k+1} \cdot \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&\qquad + \frac{1}{n+k+1} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&\le \frac{n+k}{n+k+1} \cdot \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&\qquad + \frac{1}{n+k+1} \cdot \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right).
\end{aligned}
$$

New Refinements of Jensen's Inequality
Liang-Cheng Wang, Xiu-Fen Ma and Li-Hong Liu
vol. 10, iss. 2, art. 48, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

From (1.1), we have

$$\begin{aligned}(3.5)\qquad \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) &\le \frac{1}{n}\sum_{i=1}^{n} \frac{f(x_i) + f(x_{i+1}) + \cdots + f(x_{i+r})}{r+1} \\ &= \frac{1}{n}\sum_{i=1}^{n} f(x_i).\end{aligned}$$

Combination of (3.2) – (3.5) yields (2.1).

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. For $k = 0, 1, 2, \ldots$ and $m = 2, 3, \ldots$, from (2.1), we obtain

$$\begin{aligned}(3.6)\qquad &\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &\ge \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) \\ &\quad - \left(\frac{1}{m}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \\ &\ge \frac{n+k-1}{n+k}\cdot\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{1}{n+k} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\ &\quad - \left(\frac{1}{m}\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1}\right) + \frac{m-1}{m} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right)\end{aligned}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= \left| \frac{n+k-1}{n+k} \cdot \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{1}{n+k} f\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \right.$$
$$\left. - \left(\frac{1}{m} \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) + \frac{m-1}{m} f\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \right) \right|$$
$$\geq \left| \left(\frac{n+k-1}{n+k} - \frac{1}{m} \right) \frac{1}{n} \left| \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1} + \cdots + x_{i+r}}{r+1} \right) \right| \right.$$
$$\left. - \left(\frac{m-1}{m} - \frac{1}{n+k} \right) \left| f\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \right| \right| \geq 0.$$

Expression (3.6) plus

$$f\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)$$

yields (2.2).

The proof of Theorem 2.2 is completed. □

New Refinements of Jensen's Inequality
Liang-Cheng Wang, Xiu-Fen Ma and Li-Hong Liu
vol. 10, iss. 2, art. 48, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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New Refinements of Jensen's Inequality
Liang-Cheng Wang, Xiu-Fen Ma and Li-Hong Liu

vol. 10, iss. 2, art. 48, 2009

journal of inequalities in pure and applied mathematics

issn: 1443-5756

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